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A Bijective Proof of the Hook Formula for the Number of Column Strict Tableaux with Bounded Entries

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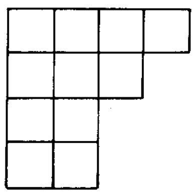
A formula for the generating function for the number of column strict tableaux of a fixed shape λ with entries bounded by some positive number a was proved by Littlewood and given an expression in terms of the hook and content numbers of the Ferrers diagram of shape λ by Stanley. We give a bijective proof of this formula and in the process give a combinatorial explanation for the appearance of the hook and content numbers in the formula.

0. INTRODUCTION

Let $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k > 0)$ be a partition. The *Ferrers* diagram of shape λ is a set of left justified rows of squares or cells with λ_i cells in the i-th row for $i = 1, \ldots, k$. Thus

$F_{(4,3,2,2)} =$ 

We let (i, j) denote the cell in the i-th row and j-th column of F_λ. The *hook number* of cell (i, j), denoted by $h_{i,j}$, equals 1 plus the number of cells to the right and below cell (i, j) and the *content* of cell (i, j), denoted by $c_{i,j}$, equals $j - i$. For example, the hook numbers and contents of $F_{(4,3,2,2)}$ are pictured below:

7	6	3	1
5	4	1	
3	2		
2	1		

hooks

0	1	2	3
−1	0	1	
−2	−1		
−3	−2		

contents

A *tabloid A of shape* λ is a filling of the cells of F_λ with nonnegative integers $a_{i,j}$. A *column strict tableau T of shape* λ is a filling of the cells with positive integers $t_{i,j}$ such that the numbers $t_{i,j}$ are weakly increasing in rows and strictly increasing in columns. The *weight* of a tabloid or column strict tableau T, denoted by $\omega(T)$, is just the sum of the entries of T, i.e. $\omega(T) = \sum_{(i,j)\in F_\lambda} t_{i,j}$.

Let $\mathcal{T}^a_\lambda$ denote the set of all column strict tableaux T of shape λ such that $t_{i,j} \leqslant a$ for all $(i, j) \in F_\lambda$. An explicit formula for the generating function, $\sum_{T\in\mathcal{T}^a_\lambda} q^{\omega(T)}$, was first derived by Littlewood [6] and Stanley [13] transformed Littlewood's formula to give an explicit expression for the generating function in terms of contents and hook numbers yielding the following.

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THEOREM 0.1.

$$\sum_{T\in\mathcal{T}_\lambda^a} q^{\omega(T)} = q^{p_\lambda} \prod_{(i,j)\in F_\lambda} [a+c_{i,j}] \Big/ \prod_{(i,j)\in F_\lambda} [h_{i,j}], \tag{0.1}$$

where

$$p_\lambda = \sum_{i=1}^{k} i\lambda_i \quad \textit{and} \quad [n] = \left(\frac{1-q^n}{1-q}\right) = 1+q+\cdots+q^{n-1}.$$

The main purpose of this paper is to give the first completely combinatorial proof of Theorem 0.1. First, we transform expression (0.1) by multiplying both sides by the product of the hooks to get

$$\left(\prod_{(i,j)\in F_\lambda} [h_{i,j}]\right)\left(\sum_{T\in\mathcal{T}_\lambda^a} q^{\omega(T)}\right) = q^{p_\lambda} \prod_{(i,j)\in F_\lambda} [a+c_{i,j}]. \tag{0.2}$$

Next, we let $\mathcal{H}_\lambda$ denote the set of all tabloids A of shape λ such that $0 \leq a_{i,j} < h_{i,j}$ for all $(i,j)\in F_\lambda$ and $\mathcal{C}_\lambda^a$ denote the set of all tabloids B of shape λ such that $0\leq b_{i,j} < a+c_{i,j}$ for all $(i,j)\in F_\lambda$. It is then easy to see that

$$\sum_{A\in\mathcal{H}_\lambda} q^{\omega(A)} = \prod_{(i,j)\in F_\lambda} [h_{i,j}] \tag{0.3}$$

and

$$\sum_{B\in\mathcal{C}_\lambda^a} q^{\omega(B)} = \prod_{(i,j)\in F_\lambda} [a+c_{i,j}]. \tag{0.4}$$

It now follows that we can prove expression (0.2) and hence Theorem 0.1 by constructing a bijection $\theta: \mathcal{H}_\lambda \times \mathcal{T}_\lambda^a \to \mathcal{C}_\lambda^a$ which is weight preserving in the sense that for all pairs $(H, T)\in \mathcal{H}_\lambda \times \mathcal{T}_\lambda^a$, we have

$$\omega(H)+\omega(T) = p_\lambda + \omega(\theta(H,T)). \tag{0.5}$$

The construction of our bijection θ is based on several fundamental ideas. The basic idea is to take the algebraic proof and replace each algebraic step by a bijective one. This idea is not, in practice, always so simple to carry out as sometimes rather trivial algebraic steps become fairly complicated to do bijectively. Nevertheless, we often gain more information and insight into the problem for such an effort. For example, in this case we shall get explicit combinatorial interpretations of the numbers $[h_{i,j}]$ and $[a+c_{i,j}]$ that appear in the right-hand side of expression (0.1). To carry out the transformation of the algebraic proof into a bijective proof, we use two other ideas from separate sources. Firstly, we use a lattice path interpretation of tableaux and certain determinants which count tableaux developed by Gessel [2] and Gessel and Viennot [3, 4, 5]. Secondly, we use an important new method due to Garsia and Milne [1] for constructing bijections out of certain pairs of involutions.

The outline of this paper is as follows. In section 1 we shall present the ideas from [3] and [1] in the form in which we shall use them. In section 2, we shall define θ and give some examples.

Finally the authors gratefully acknowledge A. M. Garsia for several helpful conversations and for his suggestion that the involution principle (see Theorem 1.3) should prove useful in this area.

1. LATTICE PATHS AND THE INVOLUTION PRINCIPLE

We begin this section with a brief introduction to the theory of interpreting tableaux and certain determinants which count tableaux developed in [2, 3, 4, 5]. Since our main interest in this paper is in the class of column strict tableaux with bounded entries, we shall start with a lattice path interpretation designed specifically for interpreting such tableaux found in [3]. However, the basic ideas can be used to interpret many other classes of tableaux. (See [2, 4, 5].)

For pictorial purposes we shall consider a grid of lattice points in $\mathbb{R}^2$ which lie below the main diagonal in the first quadrant, i.e. the set of (i, j) with $0 \leqslant j \leqslant i$ where i and j are natural numbers. We shall always picture this grid rotated 90° clockwise, see Figure 1.1.

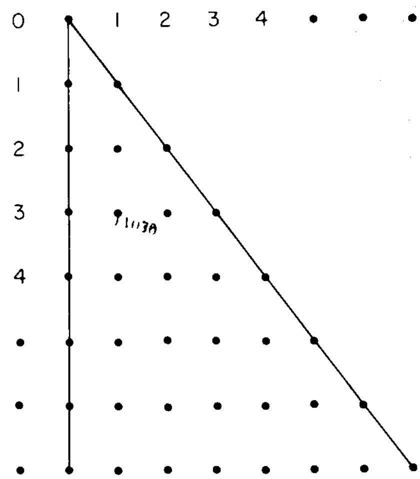


FIGURE 1.1. Basic grid.

For any integers $a, b \geqslant 0$, P_b^a denotes the set of all directed paths from $(a, 0)$ to (b, b) which can be formed from two types of directed segments or vectors, namely, horizontal segments which go from (i, j) to $(i, j+1)$ for some i and j and vertical segments which go from (i, j) to $(i-1, j)$. Thus in our basic grid we can go either right or up to one unit so that P_b^a is empty unless $b \leqslant a$. Assume $b_1, \ldots, b_k$ and $a_1, \ldots, a_k$ are sets of nonnegative integers such that $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_k$, then we let $P_{b_1 \cdots b_k}^{a_1 \cdots a_k}$ denote the k-fold product $\mathsf{X}_{i=1}^{k} P_{b_i}^{a_i}$. Figure 1.2 gives a geometrical representation of an element of P_{043}^{156}.

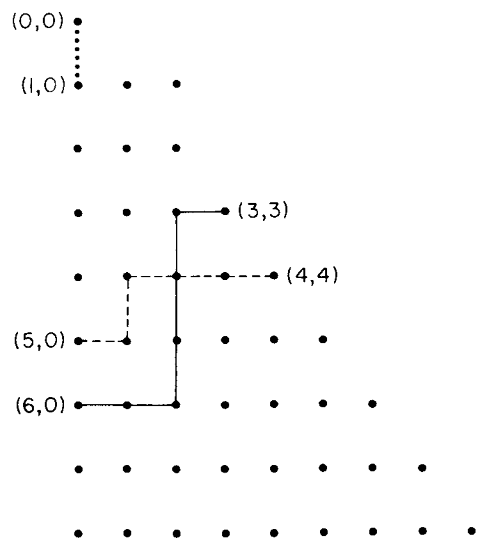


FIGURE 1.2.

A k-tuple of paths $(p_1, \ldots, p_k)$ is said to be *intersecting* if at least one pair of paths p_i and p_j have some point in common. A k-tuple of paths is said to be *nonintersecting* if it is not intersecting. We let $NP^{a_1 \cdots a_k}_{b_1 \cdots b_k}$ denote the nonintersecting k-tuples of paths in $P^{a_1, \ldots, a_k}_{b_1, \ldots, b_k}$. Notice that $NP^{a_1 \cdots a_k}_{b_1 \cdots b_k}$ is empty unless $a_1 < \cdots < a_k$, $b_1 < \cdots < b_k$, and $b_i \leq a_i$ for $i = 1, \ldots, k$.

Given any sequence of nonnegative integers $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$, we let $\mathrm{D}\begin{pmatrix} a_1 & \cdots & a_k \\ b_1 & \cdots & b_k \end{pmatrix}$ denote the $k \times k$ determinant of the matrix whose entry in the i-th row and j-th column is the binomial coefficient $\binom{a_i}{b_j}$ where $\binom{a_i}{b_j}$ is set equal to 0 if $a_i < b_j$. A basic result of [3] is that the determinant $\mathrm{D}\begin{pmatrix} a_1 & \cdots & a_k \\ b_1 & \cdots & b_k \end{pmatrix}$ counts the number of nonintersecting paths $|NP^{a_1 \cdots a_k}_{b_1 \cdots b_k}|$ when the a_i and b_i are increasing sequences of nonnegative integers. The proof of the following result uses an involution which will be important in our construction of the main bijection θ so we shall give the details of this proof.

THEOREM 1.1. (*Gessel and Viennot* [3]). *If* $0 \leq a_1 < \cdots < a_k$ *and* $0 \leq b_1 < \cdots < b_k$, *then*

$$|NP^{a_1 \cdots a_k}_{b_1 \cdots b_k}| = \mathrm{D}\begin{pmatrix} a_1 & \cdots & a_k \\ b_1 & \cdots & b_k \end{pmatrix}. \tag{1.1}$$

PROOF. Let $\mathscr{S}_k$ denote the set of all permutations of $1, \ldots, k$. Notice that for any a and b, a path $p \in P^a_b$ must have b horizontal segments and $a - b$ vertical segments so that $|P^a_b| = \binom{a}{b}$. Similarly $|P^{a_1 \cdots a_k}_{b_1 \cdots b_k}| = \binom{a_1}{b_1} \cdots \binom{a_k}{b_k}$ so that

$$\begin{aligned} \mathrm{D}\begin{pmatrix} a_1 & \cdots & a_k \\ b_1 & \cdots & b_k \end{pmatrix} &= \sum_{\sigma \in \mathscr{S}_k} \operatorname{sgn}(\sigma) \binom{a_1}{b_{\sigma_1}} \cdots \binom{a_k}{b_{\sigma_k}} \\ &= \sum_{\sigma \in \mathscr{S}_k} \operatorname{sgn}(\sigma) |P^{a_1 \cdots a_k}_{b_{\sigma_1} \cdots b_{\sigma_k}}|. \end{aligned} \tag{1.2}$$

Let $\mathscr{P}\begin{pmatrix} a_1 & \cdots & a_k \\ b_1 & \cdots & b_k \end{pmatrix}$ denote the set of signed objects $\sum_{\sigma \in \mathscr{S}_k} \operatorname{sgn}(\sigma) P^{a_1 \cdots a_k}_{b_{\sigma_1} \cdots b_{\sigma_k}}$. We can then prove equation (1.1) by constructing an involution $\varphi : \mathscr{P}\begin{pmatrix} a_1 & \cdots & a_k \\ b_1 & \cdots & b_k \end{pmatrix} \to \mathscr{P}\begin{pmatrix} a_1 & \cdots & a_k \\ b_1 & \cdots & b_k \end{pmatrix}$ such that φ leaves fixed all k-tuples in $NP^{a_1 \cdots a_k}_{b_1 \cdots b_k}$ and φ sends any intersecting k-tuple of paths $(p_1, \ldots, p_k)$ to a k-tuple of paths with an opposite sign. The involution φ is defined as follows. Suppose $(p_1, \ldots, p_k) \in P^{a_1 \cdots a_k}_{b_1 \cdots b_k}$ is intersecting. Find the lexicographically least lattice point (x, y) for which at least two paths p_i and p_j with $i, j \leq k$ go through (x, y). Then given (x, y), pick the lexicographically least pair (i, j) such that $i \neq j$ and paths p_i and p_j go through (x, y). Then φ sends $(p_1, \ldots, p_k)$ to $(p'_1, \ldots, p'_k)$ where $p'_e = p_e$ if $e \notin \{i, j\}$ and p'_j is the path obtained by following p_i to (x, y) and then following p_j and p'_j is the path obtained by following p_j to (x, y) and then following p_i. Thus $(p'_1, \ldots, p'_k)$ is obtained from $(p_1, \ldots, p_k)$ by simply interchanging the tails of p_i and p_j after (x, y). For example Figure 1.3 exhibits the image of Figure 1.2 under φ. It is easy to see that $(p'_1, \ldots, p'_k) \in P^{a_1 \cdots a_k}_{b_{\gamma_1} \cdots b_{\gamma_k}}$, where γ is the permutation obtained from σ by composing it with the permutation which interchanges i and j. Thus $\operatorname{sgn}(\sigma) = -\operatorname{sgn}(\gamma)$. Now if $(p_1, \ldots, p_k)$ is nonintersecting, then $\varphi((p_1, \ldots, p_k)) = (p_1, \ldots, p_k)$. It is easy to see that $\varphi^2 = \varphi$ so φ is the desired involution to prove equation (1.1).

Next we shall describe a bijection between tableaux and set of lattice paths due to Gessel and Viennot [3]. Given a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0)$, let $\lambda' =$

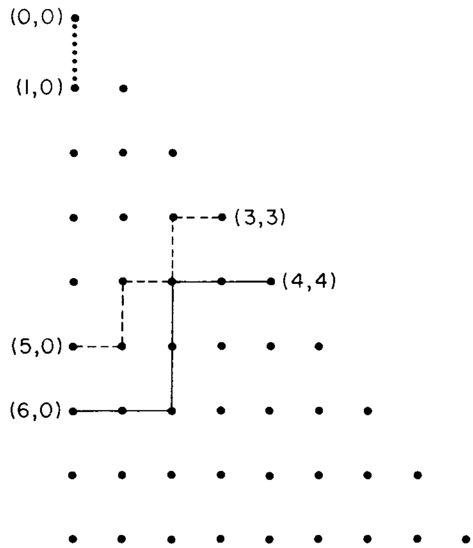


FIGURE 1.3.

$(\lambda_1' \geq \cdots \geq \lambda_{\lambda_1}' > 0)$ denote the conjugate partition of λ, i.e. the partition that arises from the column heights of F_λ. Let $\lambda^* = (\lambda_1^* \leq \cdots \leq \lambda_{\lambda_1}^*)$ denote λ' written in reverse order and $\hat{\lambda}^* = (\hat{\lambda}_1^* < \cdots < \hat{\lambda}_{\lambda_1}^*)$ where $\hat{\lambda}_j^* = \lambda_j^* + j - 1$. For example, if $\lambda = (4, 3, 2, 2)$ is the partition pictured in the introduction, then $\lambda' = (4, 4, 2, 1)$, $\lambda^* = (1, 2, 4, 4)$, and $\hat{\lambda}^* = (1, 3, 6, 7)$.

The bijection will associate to each λ_1-tuple of paths $(p_1, \ldots, p_{\lambda_1})$ in $NP^{a\ a+1\ \cdots\ a+\lambda_1-1}_{\hat{\lambda}_1^*\ \hat{\lambda}_2^*\ \cdots\ \hat{\lambda}_{\lambda_1}^*}$ for $a \geq 0$, a tableau $T(p_1, \ldots, p_{\lambda_1})$ in $\mathcal{T}_\lambda^a$. This bijection will also have certain weight preserving properties so before giving the correspondence, we shall describe the weight of a path. let $P^{a\ a+1\ \cdots\ a+\lambda_1-1}_{\hat{\lambda}_1^*\ \hat{\lambda}_2^*\ \cdots\ \hat{\lambda}_{\lambda_1}^*}$ and $NP^{a\ a+1\ \cdots\ a+\lambda_1-1}_{\hat{\lambda}_1^*\ \hat{\lambda}_2^*\ \cdots\ \hat{\lambda}_{\lambda_1}^*}$ be denoted by simply $P^a_{\hat{\lambda}^*}$ and $NP^a_{\hat{\lambda}^*}$ respectively. Then given $(q_1, \ldots, q_{\lambda_1}) \in P^a_{\hat{\lambda}^*}$, let (i) the weight $\omega(h)$ of a horizontal segment h between $(k, j-1)$ and (k, j) equal $a - k + j$, (ii) the weight $\omega(q)$ of a path q equal the sum of the weights of the horizontal segments on q, and (iii) the weight of a set of paths $(q_1, \ldots, q_{\lambda_1})$ be given by $(q_1, \ldots, q_{\lambda_1}) = \sum_{i=1}^{\lambda_1} \omega(q_i)$.

Now given $(p_1, \ldots, p_{\lambda_1}) \in NP^a_{\hat{\lambda}^*}$, it is easy to see that for each $i = 1, \ldots, \lambda_1$, the nonintersecting condition forces the first $i-1$ segments of p_i starting at $(a+i-1, 0)$ to be horizontal. Thus p_i passes through $(a+i-1, i-1)$. Now p_i has $a+i-1$ segments of which $\hat{\lambda}_i^* = \lambda_i^* + i - 1$ must be horizontal. Then starting at $(a+i-1, i-1)$ and reading the segments of p_i in order, suppose that the j_1, j_2-th, $\ldots, j_{\lambda_i^*}$-th segments are horizontal. Notice that λ_i^* is the column height of the $(\lambda_1 - i + 1)$-th column of F_λ. Thus we fill the $(\lambda_1 - i + 1)$-th column of F_λ from top to bottom with the numbers $j_1, \ldots, j_{\lambda_i^*}$. Proceeding in this way for each path p_i, we produce a filling of F_λ with the numbers between 1 and a, which we denote by $T(p_1, \ldots, p_{\lambda_1})$. (See Figure 1.4 for an example of the correspondence.) Now it is easy to see that our definition ensures that the columns of $T(p_1, \ldots, p_{\lambda_1})$ are strictly increasing. It is not difficult to show that the nonintersecting condition ensures that the rows of $T(p_1, \ldots, p_{\lambda_1})$ are weakly increasing. Now for each i, if the j_1-th, j_2-th, $\ldots, j_{\lambda_i^*}$-th segments on p_i after $(a+i-1, i-1)$ are horizontal, then for each $1 \leq l \leq \lambda_i^*$, the j_l-th segment lies between $(k, j-1)$ and (k, j) where $k = a + i - 1 + (l-1) - (j_l - 1)$ and $j = i - 1 + l$ so that the j_l-th segment contributes $a - k + j = j_l$ to the weight of $(p_1, \ldots, p_{\lambda_1})$ and to the weight of $T(p_1, \ldots, p_{\lambda_1})$. Thus the correspondence $(p_1, \ldots, p_{\lambda_1}) \to T(p_1, \ldots, p_{\lambda_1})$ is weight preserving except for the fact that we have neglected the weights

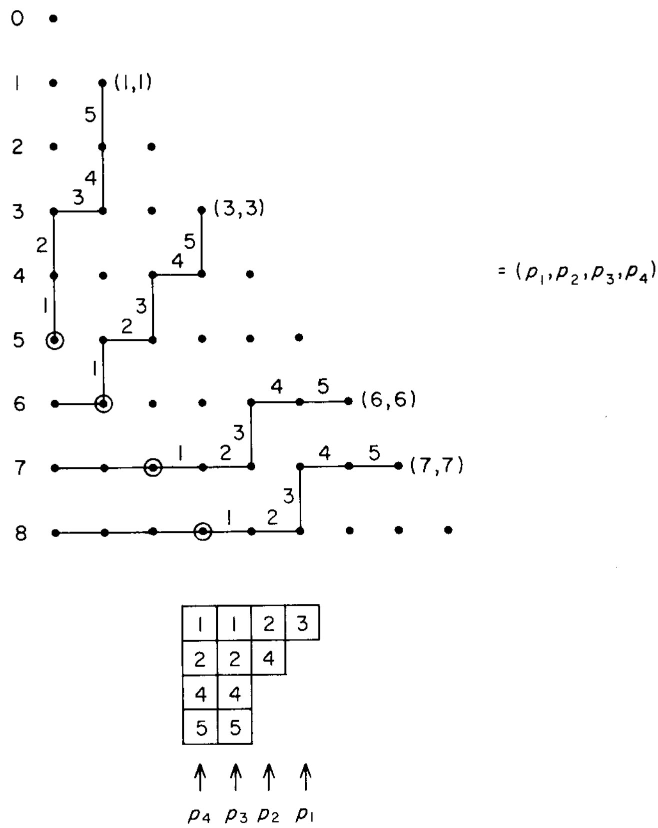


FIGURE 1.4. Here, $\lambda = (4, 3, 2, 2)$, $\lambda^* = (1, 2, 4, 4)$, $a = 5$, $\hat{\lambda}^* = (1, 3, 6, 7)$.

of the first $i-1$ horizontal segments on p_i for $i = 1, \ldots, \lambda_1$. It follows that

$$r + \omega(p_1, \ldots, p_{\lambda_1}) = \omega(T(p_1, \ldots, p_{\lambda_1})), \tag{1.3}$$

where $r = \sum_{i=1}^{\lambda_1} -\binom{i-1}{2}$.

Finally the correspondence is reversible so that the map $(p_1, \ldots, p_{\lambda_1}) \to T(p_1, \ldots, p_{\lambda_1})$ establishes the following. (We refer the reader to [3] for details.)

THEOREM 1.2. (Gessel and Viennot [3]). *Let* $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_k)$ *be a partition and a be a positive integer, then*

$$\sum_{T \in \mathcal{T}_\lambda^a} q^{\omega(T)} = q^r \sum_{(p_1, \ldots, p_{\lambda_1}) \in NP_{\hat{\lambda}^*}^a} q^{\omega(p_1, \ldots, p_{\lambda_1})}, \tag{1.4}$$

where

$$r = \sum_{i=1}^{\lambda_1} -\binom{i-1}{2}.$$

Next we outline a fundamental method for constructing bijections out of certain pairs of involutions discovered by Garsia and Milne [1] who used the method to give the first bijective proof of the famous Rogers–Ramanujan identities. Assume we have two disjoint finite spaces A and B and both A and B are further partitioned into 'positive' and 'negative' parts, $A = A^+ \cup A^-$ and $B = B^+ \cup B^-$. (For example, the set of signed paths

$\mathscr{P}\binom{a_1 \cdots a_k}{b_1 \cdots b_k}$ described in Theorem 1.1 is such a space.) Assume we have a sign preserving bijection f from A onto B, that is, $f(A^+)=B^+$ and $f(A^-)=B^-$. Next assume we have a pair of what we term 'sign reversing bijections' α from A onto A and β from B onto B with positive fixed points. That is, we assume $\alpha : A \to A$ and for all $a \in A$, either (i) $\alpha(a)=a$ and $a \in A^+$ or (ii) $\alpha(a) \neq a$ in which case $a \in A^+$ implies $\alpha(a) \in A^-$ and $a \in A^-$ implies $\alpha(a) \in A^+$ and similarly for β. (For example, if $A=\mathscr{P}\binom{a_1 \cdots a_k}{b_1 \cdots b_k}$ then the involution φ of Theorem 1.1 is such a sign reversing bijection.) We let F_α and F_β denote the fixed point sets of α and β respectively. Notice we immediately have that $|F_\alpha|=|F_\beta|$ since by α we have $|A^+|-|F_\alpha|=|A^-|$, by f we have $|A^-|=|B^-|$ and $|A^+|=|B^+|$, and by β we have $|B^-|=|B^+|-|F_\beta|$. The fundamental observation of Garsia and Milne is that a direct bijection between F_α and F_β can be constructed out of α, β, and f. Let $\alpha^*=f \circ \alpha$ and $\beta^*=f^{-1} \circ \beta$ so that α^* maps A onto B and β^* maps B onto A. Start with a fixed point $a \in F_\alpha$ and form a sequence $a=a_1, b_1, a_2, b_2, \ldots$ by first applying α^*, then β^*, then α^*, etc., that is, $a_{i+1}=\beta^*(b_i)$ and $b_i=\alpha^*(a_i)$ for all i. We call $a_1, b_1, \ldots$ the *iterated* (α, β)*-sequence* associated to a. There are two basic facts to establish:

(A) If $a \in F_\alpha$, then there is a least n, denoted by n_a, such that $b_n \in F_\beta$.

This fact is easily established by showing by induction that if there is no such n, then $a_1, b_1, a_2, b_2, \ldots$ are all pairwise distinct violating the finiteness of A and B. Having established (A), then we can show the following by appealing to the fact that α^* and β^* are one–one.

(B) If $a, a' \in F_\alpha$ with iterated (α,β)-sequences $a=a_1, b_1, \ldots, a_{n_a}, b_{n_a}$ and $a'=a'_1, b'_1, \ldots, b'_{n_{a'}}$ respectively, then $a \neq a'$ implies $b_{n_a} \neq b'_{n_{a'}}$.

Now given (A), we can define a map $I\langle \alpha, \beta, f\rangle : F_\alpha \to F_\beta$, which we shall call the *iterated* (α,β)*-map*, by $I\langle \alpha, \beta, f\rangle(a)=b_{n_a}$ for all $a \in F_\alpha$. By (B), it follows that $I\langle \alpha, \beta, f\rangle$ is one–one and by a symmetrical argument, it is easy to show that $I\langle \alpha, \beta, f\rangle$ is onto. Thus we have the following:

THEOREM 1.3. (Garsia–Milne [1]). *Let* $A=A^+ \cup A^-$, $B=B^+ \cup B^-$, $f : A \to B$, $\alpha : A \to A$, *and* $\beta : B \to B$ *be described as above, then the iterated* (α,β)*-map* $I\langle \alpha, \beta, f\rangle$ *is a bijection between* F_α *and* F_β.

REMARK. There are many applications of Theorem 1.3. See for example [1], [8], [9], and [10]. In all of these applications, α and β are involutions. In fact, in Garsia and Milne's original paper [1], they state theorem 1.3 in a slightly different setting and only for involutions although their proof remains unchanged for sign reversing bijections. Thus, we shall sometimes refer to an application of Theorem 1.3 as an application of the involution principle.

2. THE CONSTRUCTION OF THE BIJECTION θ

Let $\lambda=(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0)$ be a partition. Recall that $\mathscr{H}_\lambda$ is the set of tabloids B of shape λ such that $0 \leq b_{i,j} \leq h_{i,j}-1$ for all $(i,j) \in F_\lambda$ where $h_{i,j}$ is the hook number of the (i,j)-th cell of F_λ and $\mathscr{C}^a_\lambda$ is the set of all tabloids D of shape λ such that $0 \leq d_{i,j} \leq a+c_{i,j}-1$ for all $(i,j) \in F_\lambda$ where $c_{i,j}$ is the content number of the (i,j)-th cell of F_λ. In this section, we shall construct a bijection $\theta : \mathscr{H}_\lambda \times \mathscr{T}^a_\lambda \to \mathscr{C}^a_\lambda$ by induction on the number of rows in F_λ. Since the base step of the induction is essentially the same as the general inductive step, we shall deal only with the general inductive step. Our induction will yield a recursive procedure to find the image under θ of any pair $(A, T) \in \mathscr{H}_\lambda \times \mathscr{T}^a_\lambda$ and in the last part of this section we shall carry out this recursive procedure with some examples.

Let $\lambda-\lambda_1$ denote the partition $(\lambda_2 \geqslant \cdots \lambda_k > 0)$. Thus the Ferrers diagram $F_{\lambda-\lambda_1}$ of $\lambda-\lambda_1$ is obtained from F_λ by simply removing the first row of F_λ. Hence a cell $(i,j)\in F_\lambda$ with $i\geqslant 2$ will remain in $F_{\lambda-\lambda_1}$, however the cell (i,j) of F_λ is indexed by $(i-1,j)$ in $F_{\lambda-\lambda_1}$ since its row number has decreased by 1. This reindexing of cell $(i,j)\in F_\lambda$ relative to $F_{\lambda-\lambda_1}$ does not affect its hook number but it does affect its content number. That is, if for a partition λ we let $h^\lambda_{i,j}$ and $c^\lambda_{i,j}$ denote the hook number and content number respectively of the (i,j)-th cell in F_λ, then (i) $h^\lambda_{i,j}=h^{\lambda-\lambda_1}_{i-1,j}$ and (ii) $c^\lambda_{i,j}=c^{\lambda-\lambda_1}_{i-1,j}-1$. Thus, for example, if we take a tabloid $B\in\mathscr{H}_\lambda$ and remove its first row to obtain a tabloid $B\,|\,\lambda-\lambda_1$, then $B\,|\,\lambda-\lambda_1\in\mathscr{H}_{\lambda-\lambda_1}$. However, if we take a tabloid $D\in\mathscr{C}^a_\lambda$ and remove its first row to get a tabloid $D\,|\,\lambda-\lambda_1$, then $D\,|\,\lambda-\lambda_1\in\mathscr{C}^{a-1}_{\lambda-\lambda_1}$ since the condition for the (i,j)-th cell of D $0\leqslant d^\lambda_{i,j}<a+c^\lambda_{i,j}$ becomes the condition on the $(i-1,j)$-cell of $D\,|\,\lambda-\lambda_1$ that $0\leqslant d_{i-1,j}<a+(c^{\lambda-\lambda_1}_{i-1,j}-1)=(a-1)+c^{\lambda-\lambda_1}_{i-1,j}$. We also note that the tabloid obtained from D by just taking its first row, denoted by $D\,|\,\lambda_1$, is an element $\mathscr{C}^a_{\lambda_1}$ where λ_1 denotes the partition with one part equal to λ_1. Thus a $D\in\mathscr{C}^a_\lambda$ can be decomposed into two tabloids $D\,|\,\lambda_1\in\mathscr{C}^a_{\lambda_1}$ and $D\,|\,\lambda-\lambda_1\in\mathscr{C}^{a-1}_{\lambda-\lambda_1}$.

Our inductive step will proceed as follows. We start with a pair $(B,T)\in\mathscr{H}_\lambda\times\mathscr{T}^a_\lambda$. We shall take the first row of B, denoted by $B\,|\,\lambda_1$, and the tableaux T and produce a one row tabloid of length λ_1, $D\,|\,\lambda_1\in\mathscr{C}^a_{\lambda_1}$ and a column strict tableaux $T'\in\mathscr{C}^{a-1}_{\lambda-\lambda_1}$. Then by induction we assume we have a bijection $\theta':\mathscr{H}_{\lambda-\lambda_1}\times\mathscr{T}^{a-1}_{\lambda-\lambda_1}\to\mathscr{C}^{a-1}_{\lambda-\lambda_1}$ so that θ' will associate with the pair $(B\,|\,\lambda-\lambda_1,T')$, a tabloid in $\mathscr{C}^{a-1}_{\lambda-\lambda_1}$ which we shall denote by $D\,|\,\lambda-\lambda_1$. Then $D\,|\,\lambda_1\cup D\,|\,\lambda-\lambda_1=D\in\mathscr{C}^a_\lambda$ so that we let $\theta(B,T)=D$. Now for θ to have the desired weight preserving properties, we saw in the introduction that we must have

$$\omega(B)+\omega(T)=p+\omega(D)\quad\text{where } p=\sum_{i=1}^{k} i\lambda_i. \tag{2.1}$$

By induction, we assume that θ' has the required weight preserving properties so that

$$\omega(B\,|\,\lambda-\lambda_1)+\omega(T')=p'+\omega(D\,|\,\lambda-\lambda_1), \tag{2.2}$$

where $p'=\sum_{i=2}^k (i-1)\cdot\lambda_i$. Combining expression (2.1) and (2.2) and using the facts that $\omega(B)=\omega(B\,|\,\lambda_1)+\omega(B\,|\,\lambda-\lambda_1)$ and $\omega(D)=\omega(D\,|\,\lambda_1)+\omega(D\,|\,\lambda-\lambda_1)$, we see that it must be the case that

$$\omega(B\,|\,\lambda_1)+\omega(T)=n+\omega(D\,|\,\lambda_1)+\omega(T'), \tag{2.3}$$

where $n=\sum_{i=1}^k\lambda_i=$ no. of cells in F_λ.

Finally, to show that θ is a bijection, we must be able to reverse the whole process. Now clearly, we can decompose D into $D\,|\,\lambda_1\cup D\,|\,\lambda-\lambda_1$ and from θ' we can find the pair $(B\,|\,\lambda-\lambda_1,T')$. Thus we will be done if we show that from the pair $(T',D\,|\,\lambda_1)$, we can reconstruct T and $B\,|\,\lambda_1$. Thus we can prove by induction that θ exists if we can prove the following lemma.

LEMMA 2.1. *Let $\lambda=(\lambda_1\geqslant\cdots\geqslant\lambda_k>0)$ be a partition. There is a bijection $\chi:\mathscr{H}_\lambda\,|\,\lambda_1\times\mathscr{T}^a_\lambda\to\mathscr{C}^a_{\lambda_1}\times\mathscr{T}^{a-1}_{\lambda-\lambda_1}$ (where $\mathscr{H}_\lambda\,|\,\lambda_1$ is the set of a tabloids which result by taking the first row of tabloids $B\in\mathscr{H}_\lambda$) such that for all $(B\,|\,\lambda_1,T)\in\mathscr{H}_\lambda\,|\,\lambda_1\times\mathscr{T}^a_\lambda$, if $\chi(B\,|\,\lambda_1,T)=(D\,|\,\lambda_1,T')$, then equation* (2.3) *holds. (Notice that in the case where λ has just one part, $\mathscr{T}^{a-1}_{\lambda-\lambda_1}$ is empty and in such cases χ will simply be a bijection from $\mathscr{H}_\lambda\times\mathscr{T}^a_\lambda$ onto $\mathscr{C}^a_\lambda$.)*

PROOF. The definition of χ will involve all the ideas of section 1. First we shall interpret $\mathscr{H}_\lambda\,|\,\lambda_1\times\mathscr{T}^a_\lambda$ and $\mathscr{C}^a_{\lambda_1}\times\mathscr{T}^{a-1}_{\lambda-\lambda_1}$ in terms of lattice paths and then use the involution principle to construct χ.

First we shall interpret $\mathscr{H}_\lambda\,|\,\lambda_1\times\mathscr{T}^a_\lambda$ in terms of lattice paths. We saw in Theorem 1.2 how to associate to each tableau $T\in\mathscr{T}^a_\lambda$ a set of paths $(p_1,\ldots,p_{\lambda_1})\in NP^{a\;\;a+1\;\cdots\;a+\lambda_1-1}_{\hat\lambda^*_1\;\hat\lambda^*_2\;\cdots\;\hat\lambda^*_{\lambda_1}}$

such that $T(p_1, \ldots, p_{\lambda_1}) = T$ and $\omega(T) = r + \omega(p_1, \ldots, p_{\lambda_1})$ where $r = \sum_{i=1}^{\lambda_1} - \binom{i-1}{2}$. Now p_i is a path with $\hat{\lambda}_i^* = \lambda_i^* + i - 1$ horizontal segments and $a + i - 1 - \hat{\lambda}_i^*$ vertical segments. Notice that $\lambda_i^* + i - 1$ is the hook number of the $(1, \lambda_1 - i + 1)$-th cell in F_λ. Thus we consider λ_1-tuples of paths $(p_1, \ldots, p_{\lambda_1})$ where we mark one horizontal segment on each path p_i. We shall denote the collection of such 'marked' paths by $ml(NP^a_{\hat{\lambda}^*})$. (Here, ml stands for marked levels and we shall often refer to a horizontal segment as simply a *level*.) Then with each $(p_1, \ldots, p_{\lambda_1}) \in ml(NP^a_{\hat{\lambda}^*})$ we can associate a tableau $T(p_1, \ldots, p_{\lambda_1}) \in \mathscr{T}^a_\lambda$ as in Theorem 1.2 and a tabloid $B(p_1, \ldots, p_{\lambda_1})$ in $\mathscr{H}|\lambda_1$ where the $(1, \lambda_1 - i + 1)$-th entry of $B(p_1, \ldots, p_{\lambda_1})$ is $k_i - 1$ if the level marked on path p_i is the k_i-th level reading left to right. Notice that since $1 \leqslant k_i \leqslant \hat{\lambda}_i^* = h_{1,\lambda_1-i+1}$, we have $0 \leqslant b_{1,\lambda_1-i+1} = k_i - 1 \leqslant h_{1,\lambda_1-i+1} - 1$. Now if we define the weight of a marked k-tuple of paths $(p_1, \ldots, p_{\lambda_1})$, denoted by $\omega_{ml}(p_1, \ldots, p_k)$, to be equal to the old weight $\omega(p_1, \ldots, p_{\lambda_1})$ plus $\sum_{i=1}^{\lambda_1} k_i - 1$, then we have

$$r + \omega_{ml}(p_1, \ldots, p_{\lambda_1}) = \omega(T(p_1, \ldots, p_{\lambda_1})) + \omega(B(p_1, \ldots, p_{\lambda_1})). \tag{2.4}$$

See Figure 2.1, for an example of this correspondence where we indicate a marked level by placing an $\times$ on that level.

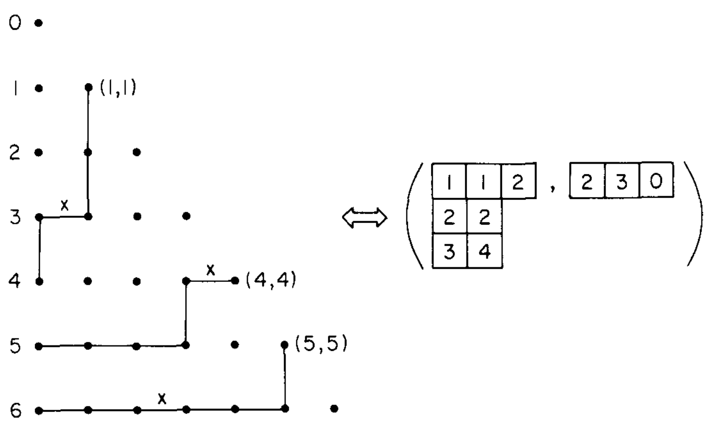


FIGURE 2.1. Here $\lambda = (3, 2, 2)$, $\lambda^* = (1, 3, 3)$, $\hat{\lambda}^* = (1, 4, 5)$, and $a = 4$.

Next we turn our attention to giving a lattice path interpretation to the pairs in $\mathscr{C}^a_{\lambda_1} \times \mathscr{T}^{a-1}_{\lambda-\lambda_1}$. Given a tableau $T' \in \mathscr{T}^{a-1}_{\lambda-\lambda_1}$, the coding used in Theorem 1.2 would associate T' to a λ_2-tuple of paths. However, for reasons which will become clear shortly, we shall use a λ_1-tuple of nonintersecting paths $(p'_1, \ldots, p'_{\lambda_1}) \in NP^{a-1\; a\; \cdots\; a+\lambda_1-2}_{\hat{\lambda}_1^*-1\; \hat{\lambda}_2^*-1\; \cdots\; \hat{\lambda}^*_{\lambda_1}-1}$ to code such a tableau T'. The idea is simple. If we start with the column heights of F_λ, $\lambda_1^* \leqslant \cdots \leqslant \lambda^*_{\lambda_1}$, and subtract 1 from each height to get $\lambda_1^* - 1, \ldots, \lambda^*_{\lambda_1} - 1$, we get the column heights of $F_{\lambda-\lambda_1}$. Of course, if $\lambda_1 > \lambda_2$, then the first $\lambda_1 - \lambda_2$ of these numbers are 0. Next if we consider the sequence $\hat{\lambda}_1^* - 1 < \hat{\lambda}_2^* - 1 < \cdots < \hat{\lambda}^*_{\lambda_1} - 1$, then in the case where $\lambda_1 > \lambda_2$, the first $\lambda_1 - \lambda_2$ of these numbers are $0, 1, \ldots, \lambda_1 - \lambda_2 - 1$. In such a situation, the first $\lambda_1 - \lambda_2$ paths of a λ_1-tuple of nonintersecting paths $(p'_1, \ldots, p'_{\lambda_1}) \in NP^{a-1\; a\; \cdots\; a+\lambda_1-2}_{\hat{\lambda}_1^*-1\; \hat{\lambda}_2^*-1\; \cdots\; \hat{\lambda}^*_{\lambda_1}-1}$ are completely forced. That is, p'_1 goes from $(a-1, 0)$ to $(0, 0)$ and hence has only vertical steps. The nonintersecting property forces the first $i-1$ segments of path p'_i to be horizontal so that if $i \leqslant \lambda_1 - \lambda_2$, then path p'_i must end at $(i-1, i-1)$ and hence we must have all vertical segments on p'_i after the initial $i-1$ horizontal segments. If $i > \lambda_1 - \lambda_2$, then after the forced $i-1$ horizontal steps, we have a path of length $a-1$ with $\hat{\lambda}_i^* - 1 - i + 1 = \hat{\lambda}_i^* - 1$ horizontal segments. As before, $\lambda_i^* - 1$ represents the $\lambda_1 - i + 1$-th column height in $F_{\lambda-\lambda_1}$. Suppose the j_1-th, $\ldots$, $j_{\lambda_i^*-1}$-th segments of p'_i after the point $(a-1+i-1, i-1)$ are horizontal, then we fill the λ_1-i+1-th column of $F_{\lambda-\lambda_1}$ from top to bottom with $j_1, \ldots, j_{\lambda_i-1}$. If we do this for each p_i with $i>\lambda_1-\lambda_2$, we get a filling of $F_{\lambda-\lambda_1}$ which we denote by $T'(p_1, \ldots, p_{\lambda_1})$. By exactly the same reasoning as used to prove Theorem 1.2, we can show that the correspondence $(p'_1, \ldots, p'_{\lambda_1}) \to T'(p'_1, \ldots, p'_{\lambda_1})$ is a one–one correspondence between λ_1-tuples in $NP^{a-1\; a\; \cdots\; a+\lambda_1-2}_{\hat\lambda_1^*-1\; \hat\lambda_2^*-1\; \cdots\; \hat\lambda_{\lambda_1}^*-1}$ and elements in $\mathscr{T}^{a-1}_{\lambda-\lambda_1}$. In fact, if we start with $(p'_1, \ldots, p'_{\lambda_1})$ and remove the first $\lambda_1-\lambda_2$ columns (which will eliminate entirely the first $\lambda_1-\lambda_2$ paths) we will get a λ_2-tuple of nonintersecting paths $(q_1, \ldots, q_{\lambda_2})$ and it is easy to see that $T'(p'_1, \ldots, p'_{\lambda_1})$ is just $T(q_1, \ldots, q_{\lambda_2})$ where $(q_1, \ldots, q_{\lambda_2}) \to T(q_1, \ldots, q_{\lambda_2})$ under the old correspondence of Theorem 1.2. This fact is illustrated in Figure 2.2.

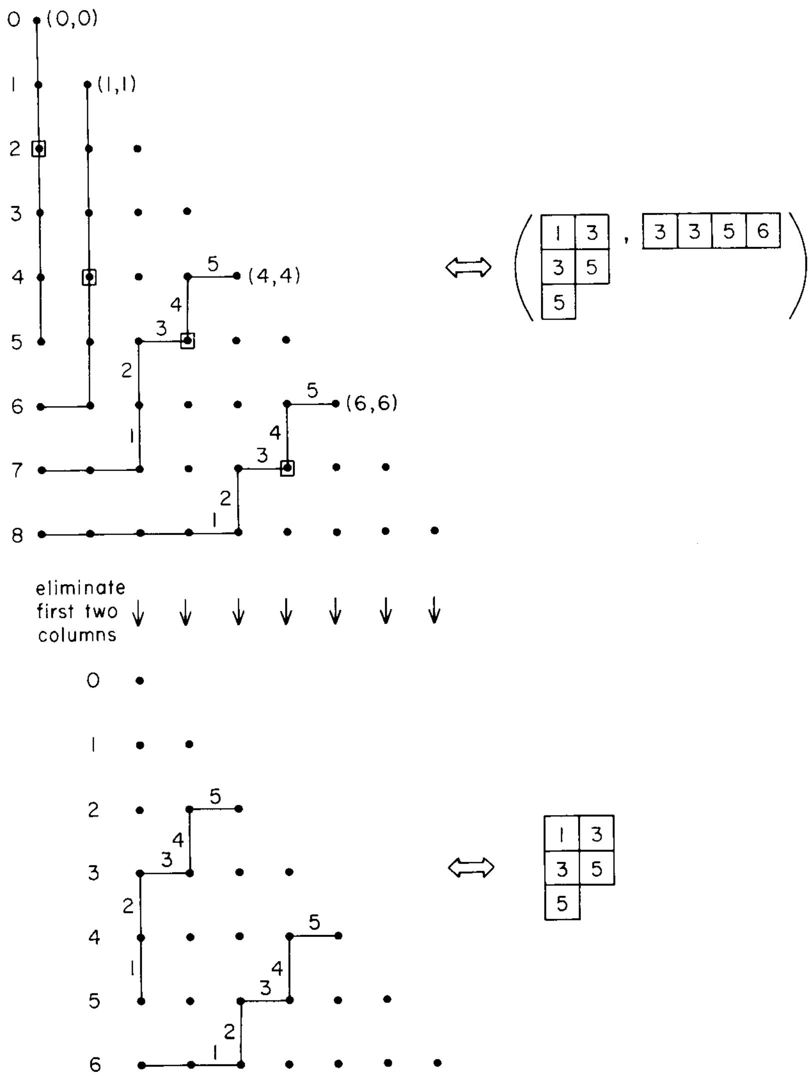


FIGURE 2.2. Here $\lambda=(4,2,2,1)$, $\lambda^*=(1,1,3,4)$, $\hat\lambda^*=(1,2,5,7)$, $\lambda-\lambda_1=(2,2,1)$, and $a-1=5$.

Thus for each $T' \in \mathscr{T}^{a-1}_{\lambda-\lambda_1}$, we can associate a λ_1-tuple $(p'_1, \ldots, p'_{\lambda_1}) \in NP^{a-1\; a\; \cdots\; a+\lambda_1-2}_{\hat\lambda_1^*-1\; \hat\lambda_2^*-1\; \cdots\; \hat\lambda_{\lambda_1}^*-1}$ such that $T'=T'(p'_1, \ldots, p'_{\lambda_1})$. Notice that the number of vertices on $p'_1, \ldots, p'_{\lambda_1}$ are respectively a, $a+1, \ldots, a+\lambda_1-1$ which are exactly the numbers

$a+c_{1,j}$ for $1 \leq j \leq \lambda_1$. We shall consider the collection, $mv(NP^{a-1}_{\hat{\lambda}^*})$, which consists of all λ_1-tuples $(p'_1, \ldots, p'_{\lambda_1})$ of paths as above where one vertex on each path p_i is marked. Then the marked vertices for such λ_1-tuples can code a tabloid $D'(p_1, \ldots, p'_{\lambda_1}) \in C^a_{\lambda_1}$ by letting $d_{1,j} = l_j - 1$ where the l_j-th vertex reading left to right is marked on p_j. We assign a weight, $\omega_{mv}(p'_1, \ldots, p'_{\lambda_1})$ to such a λ_1-tuple in $mv(NP^{a-1}_{\hat{\lambda}^*-1})$ by letting $\omega_{mv}(p'_1, \ldots, p'_k) = \omega(p'_1, \ldots, p'_k) + \sum_{i=1}^{\lambda_1} l_i - 1$. Then by the same type of argument which established expression (2.4), we have

$$r + \omega_{mv}(p'_1, \ldots, p'_{\lambda_1}) = \omega(T'(p'_1, \ldots, p'_{\lambda_1})) + \omega(D'(p'_1, \ldots, p'_{\lambda_1})). \tag{2.5}$$

See Figure 2.2 for an example of this coding where a marked vertex on a path is indicated by putting a box around the vertex.

(Notice that in the case where λ has just one row, all the paths $p'_1, \ldots, p'_{\lambda_1}$ are forced so that we only regard the marked vertices as a code for a tabloid in $\mathscr{C}^a_{\lambda}$.)

We pause at this point to note that we now have explicit interpretations of the hook numbers $[h_{i,j}]$ and the content numbers $[a+c_{i,j}]$ which appear in Theorem 0.1. Namely, the numbers $[h_{i,j}]$ represent the number of horizontal segments on the paths in $ml(NP^a_{\hat{\lambda}^*})$ and the numbers $[a+c_{i,j}]$ represent the number of vertices on the paths in $mv(NP^{a-1}_{\hat{\lambda}^*-1})$. Our next task is to show that these two sets of paths are in bijection.

Given these two lattice path interpretations of $\mathscr{H}_\lambda \,|\, \lambda_1 \times \mathscr{T}^a_\lambda$ and $\mathscr{C}^a_{\lambda_1} \times \mathscr{T}^{a-1}_{\lambda-\lambda_1}$ respectively, we see that by expressions (2.4) and (2.5) we can reduce the problem of constructing the bijection χ to constructing a bijection $\Gamma: ml(NP^a_{\hat{\lambda}^*}) \to mv(NP^{a-1}_{\hat{\lambda}^*-1})$ such that for all $(p_1, \ldots, p_{\lambda_1}) \in ml(NP^a_{\hat{\lambda}^*})$

$$\omega_{ml}(p_1, \ldots, p_{\lambda_1}) = n + \omega_{mv}(\Gamma(p_1, \ldots, p_{\lambda_1})), \tag{2.6}$$

where $n = \sum_{i=1}^k \lambda_i$. We shall construct Γ by the involution principle. First we must interpret $ml(NP^a_{\hat{\lambda}^*})$ and $mv(NP^{a-1}_{\hat{\lambda}^*-1})$ as the fixed point sets of some 'signed' spaces relative to involutions α and β respectively. Given Theorem 1.1, the signed spaces are clear. That is, for any $0 \leq a_1 \leq \cdots \leq a_n$ and $b_1, \ldots, b_n$ where $\varnothing \neq P^{a_1 \cdots a_n}_{b_1 \cdots b_n}$, we let $ml(P^{a_1 \cdots a_n}_{b_1 \cdots b_n})$ denote the collection of all n-tuples of paths $(p_1, \ldots, p_n) \in P^{a_1 \cdots a_n}_{b_1 \cdots b_n}$ where one level on p_i is marked if there are any levels on p_i) and similarly $mv(P^{a_1 \cdots a_n}_{b_1 \cdots b_n})$ will denote the collection of n-tuples of paths $(p_1, \ldots, p_n) \in P^{a_1 \cdots a_n}_{b_1 \cdots b_n}$ where one vertex on each path p_i is marked. Then

$$ml(NP^a_{\hat{\lambda}^*}) \subseteq ml\left(\mathscr{P}\binom{a}{\hat{\lambda}^*}\right) = \sum_{\sigma \in \mathscr{S}_{\lambda_1}} \operatorname{sgn}(\sigma)\, ml\left(P^{a \;\; a+1 \;\; \cdots \;\; a+\lambda_1-1}_{\hat{\lambda}^*_{\sigma_1} \; \hat{\lambda}^*_{\sigma_2} \; \cdots \; \hat{\lambda}^*_{\sigma_{\lambda_1}}}\right), \tag{2.7}$$

and

$$mv(NP^{a-1}_{\hat{\lambda}-1}) \subseteq mv\left(\mathscr{P}\binom{a-1}{\hat{\lambda}^*-1}\right) = \sum_{\sigma \in \mathscr{S}_{\lambda_1}} \operatorname{sgn}(\sigma)\, mv\left(P^{a-1 \;\; a \;\; \cdots \;\; a+\lambda_1-2}_{\hat{\lambda}^*_{\sigma_1}-1 \; \hat{\lambda}^*_{\sigma_2}-1 \; \cdots \; \hat{\lambda}^*_{\sigma_{\lambda_1}}-1}\right). \tag{2.8}$$

We also need to assign weights to the paths in $ml\left(\mathscr{P}\binom{a}{\hat{\lambda}^*}\right)$ and $mv\left(\mathscr{P}\binom{a-1}{\hat{\lambda}^*-1}\right)$ respectively. Given $(p_1, \ldots, p_{\lambda_1}) \in ml\left(\mathscr{P}\binom{a}{\hat{\lambda}^*}\right)$, we let

$$\omega_{ml}(p_1, \ldots, p_{\lambda_1}) = \omega_a(p_1, \ldots, p_{\lambda_1}) + \sum_{i=1}^{\lambda_1} (k_i - 1), \tag{2.9}$$

where $\omega_a(p_1, \ldots, p_{\lambda_1})$ is the old weight assigned to paths in $\mathscr{P}\binom{a}{\hat{\lambda}^*}$ and the k_i-th level is marked on the path that ends at $(\hat{\lambda}^*_i, \hat{\lambda}^*_i)$. Similarly, given $(p'_1, \ldots, p'_{\lambda_1}) \in mv\left(\mathscr{P}\binom{a-1}{\hat{\lambda}^*-1}\right)$, we let

$$\omega_{mv}(p'_1, \ldots, p'_{\lambda_1}) = \omega_{a-1}(p'_1, \ldots, p'_{\lambda_1}) + \sum_{i=1}^{\lambda_1} (l_i - 1), \tag{2.10}$$

where the l_i-th vertex is marked on the path that starts at $(a+i-1, 0)$ and in this case $\omega_{a-1}(p'_1, \ldots, p'_{\lambda_1})$ is the old weight assigned to paths in $\mathscr{P}\binom{a-1}{\hat{\lambda}^*-1}$.

The involutions α on $ml\left(\mathscr{P}\binom{a}{\hat{\lambda}^*}\right)$ and β on $mv\left(\mathscr{P}\binom{a-1}{\hat{\lambda}^*-1}\right)$ are suggested by the involution φ used in theorem 1.1. Given $(p_1, \ldots, p_{\lambda_1}) \in ml\left(\mathscr{P}\binom{a}{\hat{\lambda}^*}\right)$, if $(p_1, \ldots, p_{\lambda_1})$ is nonintersecting then $\alpha(p_1, \ldots, p_{\lambda_1}) = (p_1, \ldots, p_{\lambda_1})$. If $(p_1, \ldots, p_{\lambda_1})$ is intersecting, then just like for φ, we take the lexicographically least lattice point (x, y) which is on two paths and the lexicographically least pair (i, j) such that p_i and p_j pass through (x, y) and switch the tails of p_i and p_j after (x, y) to get a new λ_1-tuple of paths $(q_1, \ldots, q_{\lambda_1})$. Then as with φ, we are insured that the signs assigned to $(p_1, \ldots, p_{\lambda_1})$ and $(q_1, \ldots, q_{\lambda_1})$ are opposite. However, we must say how to deal with the marks. Notice that in expression

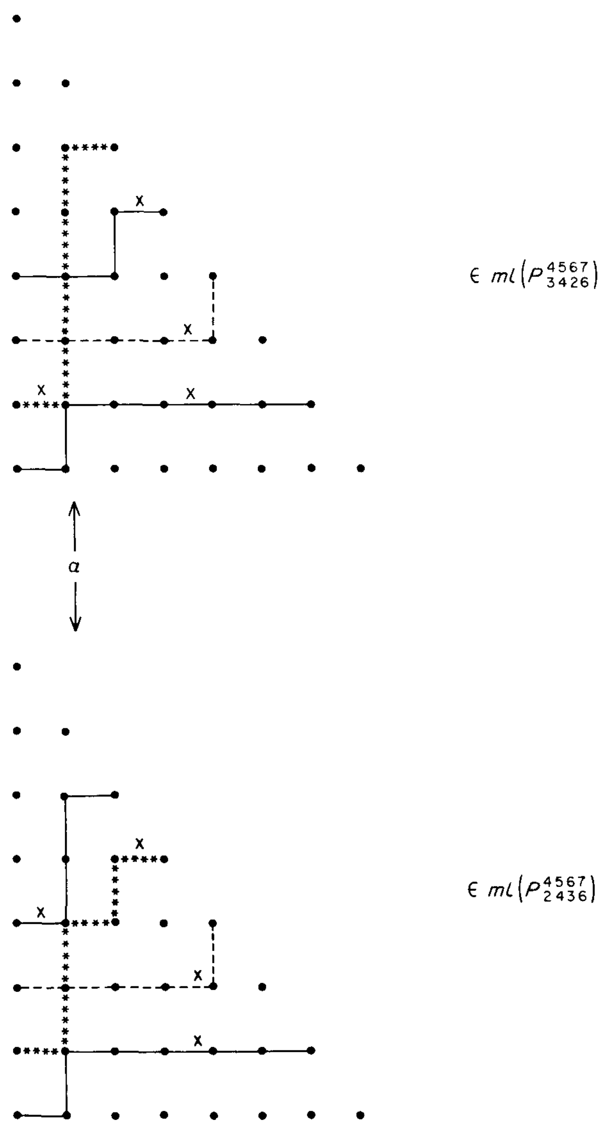


FIGURE 2.3. The α involution.

(2.9) we assigned a weight to a marked level on a path according to its end point. So suppose p_i ends at $(\hat{\lambda}^*_{\sigma_i}, \hat{\lambda}^*_{\sigma_i})$ and p_j ends at $(\hat{\lambda}^*_{\sigma_j}, \hat{\lambda}^*_{\sigma_j})$ so that there are $\hat{\lambda}^*_{\sigma_i}$ horizontal segments or levels on p_i and $\hat{\lambda}^*_{\sigma_j}$ levels on p_j. After the switch, q_i now ends at $(\hat{\lambda}^*_{\sigma_j}, \hat{\lambda}^*_{\sigma_j})$ and q_j ends at $(\hat{\lambda}^*_{\sigma_i}, \hat{\lambda}^*_{\sigma_i})$ so that there are $\hat{\lambda}^*_{\sigma_j}$ levels on q_i and $\hat{\lambda}^*_{\sigma_i}$ on q_j. In this case, if the k_i-th level of the $\hat{\lambda}^*_{\sigma_i}$ levels on p_i is marked (reading left to right), then we mark the k_i-th level of the $\hat{\lambda}^*_{\sigma_i}$ levels on q_j. Similarly, if the k_j-th level of the $\hat{\lambda}^*_{\sigma_j}$ levels of p_j is marked, then we mark the k_j-level of the $\hat{\lambda}^*_{\sigma_j}$ levels on q_i. If $n \notin \{i, j\}$, then $p_n = q_n$ so we ensure that the same level is marked on both paths. The procedure will, in general, move some of the marks but by our definition of weight, it is easy to see that $\omega_{ml}(p_1, \ldots, p_{\lambda_1}) = \omega_{ml}(q_1, \ldots, q_{\lambda_1})$. Moreover, our definition of α ensures $\alpha(q_1, \ldots, q_{\lambda_1}) = (p_1, \ldots, p_{\lambda_1})$ so that α is a weight preserving involution on $ml\left(\mathscr{P}\binom{a}{\hat{\lambda}^*}\right)$ with fixed point set $ml(NP^a_{\hat{\lambda}^*})$. See Figure 2.3 for an example of the α involution.

The definition of β on $mv\left(\mathscr{P}\binom{a-1}{\hat{\lambda}^*-1}\right)$ is similar. Given $(p'_1, \ldots, p'_{\lambda_1}) \in mv\left(\mathscr{P}\binom{a-1}{\hat{\lambda}^*-1}\right)$, we let $\beta(p'_1, \ldots, p'_{\lambda_1}) = (p'_1, \ldots, p'_{\lambda_1})$ if $(p'_1, \ldots, p'_{\lambda_1})$ is nonintersecting. If $(p'_1, \ldots, p'_k)$ is intersecting, then as before we locate the lexicographically least lattice point (x, y) such that at least two paths pass through (x, y) and the lexicographically least pair (i, j) such that p'_i and p'_j pass through (x, y) and then we switch the tails of p'_i and p'_j after (x, y) to get a new λ_1-tuple of paths $(q'_1, \ldots, q'_{\lambda_1})$. Dealing with the marked vertices in this case is easier than with α since the number of vertices on a path depends only on its starting point. Thus, for example, p'_i and q'_i start at $(a-1+i-1, 0)$ and hence have $a+i-1$ vertices on them. Thus, if the l_n-th vertex is marked on p'_n, we mark the l_n-vertex on q'_n for all $n = 1, \ldots, \lambda_1$. This procedure may change the marked vertices associated with the i-th and j-th paths but it will leave all other marked vertices fixed. By our definition of the weight in expression (2.10), it is easy to see that $\omega_{mv}(p'_1, \ldots, p'_{\lambda_1}) = \omega_{mv}(q'_1, \ldots, q'_{\lambda_1})$ so that β is weight preserving. Again it is easy to check that β is in fact an involution. See Figure 2.4 for an example of the β involution.

Finally we must define a sign preserving map between $ml\left(\mathscr{P}\binom{a}{\hat{\lambda}^*}\right)$ and $mv\left(\mathscr{P}\binom{a-1}{\lambda^*-1}\right)$ which we call the *reduction map* and denote by Red. We note that since α and β are weight preserving, to guarantee that expression (2.6) holds we must ensure that for all $(p_1, \ldots, p_{\lambda_1}) \in \mathscr{P}\binom{a}{\hat{\lambda}^*}$

$$\omega_{ml}(p_1, \ldots, p_{\lambda_1}) = \omega_{mv}(\mathrm{Red}(p_1, \ldots, p_{\lambda_1})) + n. \tag{2.11}$$

The idea of the reduction map is very simple. Suppose that we have $d > b > 0$ and we take any path $p \in P^d_b$ with one marked level l. Now if we remove the marked level l and join the two end points of l together we get a path p' with $d-1$ segments of which $b-1$ are horizontal segments. Thus $p' \in P^{d-1}_{b-1}$ and if we mark the vertex on p' where the two end points of l are joined together, it is clear we can recover p. Thus the correspondence $p \to p'$ is a bijection from $ml(P^d_b)$ onto $mv(P^{d-1}_{b-1})$ and this is the reduction map and we write $p' = \mathrm{Red}(p)$. (See Figure 2.5.) Then for $(p_1, \ldots, p_{\lambda_1}) \in ml\left(P^{a\ \ a+1\ \cdots\ a+\lambda_1-1}_{\hat{\lambda}^*_{\sigma_1}\ \hat{\lambda}^*_{\sigma_2}\ \cdots\ \hat{\lambda}^*_{\sigma_{\lambda_1}}}\right)$, let $(\mathrm{Red}(p_1), \ldots, \mathrm{Red}(p_{\lambda_1})) = \mathrm{Red}(p_1, \ldots, p_{\lambda_1}) \in mv\left(P^{a-1\ \ a\ \cdots\ a+\lambda_1-2}_{\hat{\lambda}^*_{\sigma_1}-1\ \hat{\lambda}^*_{\sigma_2}-1\ \cdots\ \hat{\lambda}^*_{\sigma_{\lambda_1}}-1}\right)$. Thus Red is a sign preserving bijection between $ml\left(\mathscr{P}\binom{a}{\hat{\lambda}^*}\right)$ onto $mv\left(\mathscr{P}\binom{a-1}{\hat{\lambda}^*-1}\right)$. [For example if $(p_1, \ldots, p_4) \in P^{4567}_{3426}$ in Figure 2.3 and $(p'_1, \ldots, p'_4) \in P^{3456}_{2315}$ in Figure 2.4, then $\mathrm{Red}(p_1, \ldots, p_4) = (p'_1, \ldots, p'_4)$.]

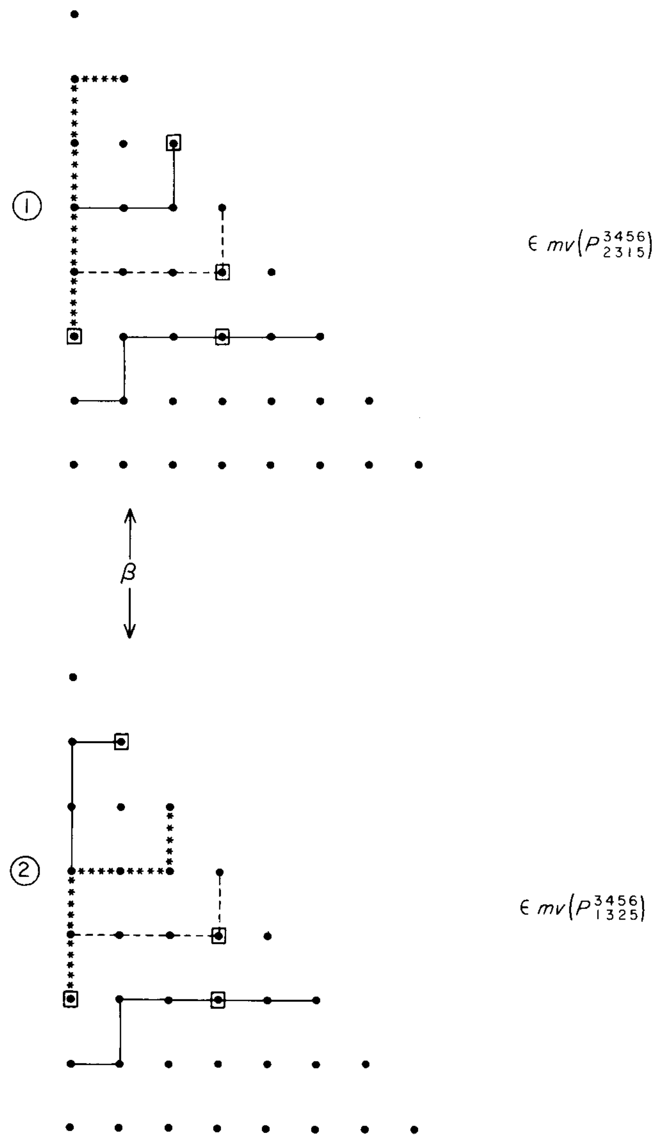


FIGURE 2.4. The β involution.

For the weight preserving properties of the reduction map, consider the i-th path p_i of $(p_1, \ldots, p_{\lambda_1}) \in ml(P^{a\ \ a+1\ \cdots\ a+\lambda_1-1}_{\hat\lambda^*_{\sigma_1}\ \hat\lambda^*_{\sigma_2}\ \cdots\ \hat\lambda^*_{\sigma_{\lambda_1}}})$ and suppose the l_i-th level of p_i is marked and this segment ends at (k, j). Now the weight of l_i-th level of p_i is $a-k+j$. The number of the vertex marked on $\mathrm{Red}(p_i)$ is simply the number of vertices on p_i from $(a+i-1, 0)$ to $(k, j-1)$ which equals $(a+i-1)-(k-1)+j-1 = a+i-k+j-1$. Thus the weight due to the marking of the level on p_i is l_i-1 and the weight due to the marking of the vertex on $\mathrm{Red}(p_i) = a+i-k+j-2$. Let $h_1, \ldots, h_{\hat\lambda^*_{\sigma_i}}$ denote the levels on p_i reading left to right. Now if $j < l_i$ and h_j ends at (x, y), then under the reduction map h_j is shifted up one unit and ends at $(x-1, y)$. (See Figure 2.5.) But the weight of h_j as a segment relative to $\mathcal{P}\binom{a}{\hat\lambda^*}$ is $a-x+y$ and the weight of h_j shifted up vertically one unit relative to $\mathcal{P}\binom{a-1}{\hat\lambda^*-1}$ is $(a-1)-(x-1)+y = a-x+y$ so that h_j contributes the same weight to $\omega(p_i)$ relative

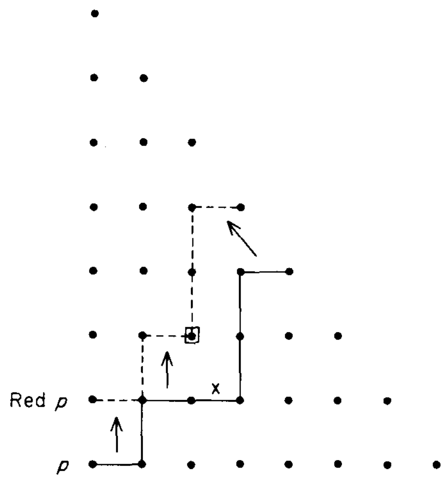


FIGURE 2.5.

to $\mathscr{P}\binom{a}{\hat\lambda^*}$ and $\omega(\text{Red}(p_i))$ relative to $\mathscr{P}\binom{a-1}{\hat\lambda^*-1}$. Now if $j>l_i$, then h_j is shifted diagonally up one unit and ends at $(x-1, y-1)$. Thus its weight relative to $\mathscr{P}\binom{a-1}{\hat\lambda^*-1}$ is $a-1-(x-1)+y-1=a-x+y-1$ and hence is one less than its weight relative to $\mathscr{P}\binom{a}{\hat\lambda^*}$. Thus the difference between $\omega_a(p_i)$ and $\omega_{a-1}(\text{Red}(p_i))$ is given by

$$\omega_a(p_i)-\omega_{a-1}(\text{Red}(p_i))=\omega_a(h_{l_i})+\hat\lambda^*_{\sigma_i}-l_i=a-k+j+\hat\lambda^*_{\sigma_i}-l_i. \tag{2.12}$$

Now taking into account the weights of the marked level on p_i and the marked vertex on $\text{Red}(p_i)$, we have

$$\begin{aligned}\omega_{ml}(p_i)-\omega_{mv}(\text{Red}(p_i))&=a-k+j+\hat\lambda^*_{\sigma_i}-l_i+(l_i-1)-(a+i-k+j-2)\\&=\hat\lambda^*_{\sigma_i}-i+1.\end{aligned} \tag{2.13}$$

Thus summing over all i, we get

$$\begin{aligned}\omega_{ml}(p_1,\ldots,p_{\lambda_1})-\omega_{mv}(\text{Red}(p_1,\ldots,p_{\lambda_1}))&=\sum_{i=1}^{\lambda_1}\hat\lambda^*_{\sigma_i}-\sum_{i=1}^{\lambda_1}(i-1)\\&=\sum_{i=1}^{\lambda_1}(\lambda^*_i+i-1)-\sum_{i=1}^{\lambda_1}(i-1)\\&=\sum_{i=1}^{\lambda_1}\lambda^*_i=n.\end{aligned} \tag{2.14}$$

It now follows from Theorem 1.3 that if $\Gamma=I(\alpha,\beta,\text{Red})$, the iterated (α,β) map in our situation, then Γ is a bijection from $ml\left(NP\binom{a}{\hat\lambda^*}\right)$ onto $mv\left(NP\binom{a-1}{\hat\lambda^*-1}\right)$ such that expression (2.6) holds as desired. Thus the map χ, required by Lemma 2.1, is the map that starts with a pair $(B\,|\,\lambda_1, T)\in\mathscr{H}_\lambda\,|\,\lambda_1\times\mathscr{T}^a_\lambda$, then sends $(B\,|\,\lambda_1, T)$ to a λ_1-tuple $(p_1,\ldots,p_{\lambda_1})\in mlNP^a_{\hat\lambda^*}$, next via $\Gamma(p_1,\ldots,p_{\lambda_1})$ is sent to a λ_1-tuple $(p'_1,\ldots,p'_{\lambda_1})\in mv\,(NP^{a-1}_{\hat\lambda^*-1})$, and finally $(p'_1,\ldots,p'_{\lambda_1})$ is sent to a pair $(D\,|\,\lambda_1, T')\in\mathscr{C}^a_{\lambda_1}\times\mathscr{T}^{a-1}_{\lambda-\lambda_1}$. Schematically, we can picture the χ map as in the following diagram of χ where the values in brackets give the correspondence of the weights:

$$\begin{array}{ccc}
(B\,|\,\lambda_1, T) \in \mathcal{H}_\lambda\,|\,\lambda_1 \times \mathcal{T}^a_\lambda & \Rightarrow & (p_1, \ldots, p_{\lambda_1}) \in mlNP^a_{\lambda^*} \\
[\omega(B\,|\,\lambda_1) + \omega(T)] & & [r + \omega(p_1, \ldots, p_{\lambda_1})] \\
 & r & \Downarrow \\
(D\,|\,\lambda_1, T') \in \mathcal{C}^a_{\lambda_1} \times \mathcal{T}^{a-1}_{\lambda-\lambda_1} & \Leftarrow & (p'_1, \ldots, p'_{\lambda_1}) \in mv(NP^{a-1}_{\lambda^*-1}) \\
[n + \omega(D\,|\,\lambda_1) + \omega(T')] & & [r + n + \omega(p'_1, \ldots, p'_{\lambda_1})]
\end{array}$$

This completes the proof of Lemma 2.1 and by our remarks preceding the lemma, we have the following:

THEOREM 2.1. *Let* $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k > 0)$ *be a partition. There is a bijection* $\theta : \mathcal{H}_\lambda \times \mathcal{T}^a_\lambda \to \mathcal{C}^a_\lambda$ *such that for all* $(B, T) \in \mathcal{H}_\lambda \times \mathcal{T}^a_\lambda$, $\omega(B) + \omega(T) = p + \omega(\theta(B, T))$, *where* $p = \sum_{i=1}^k i\lambda_i$.

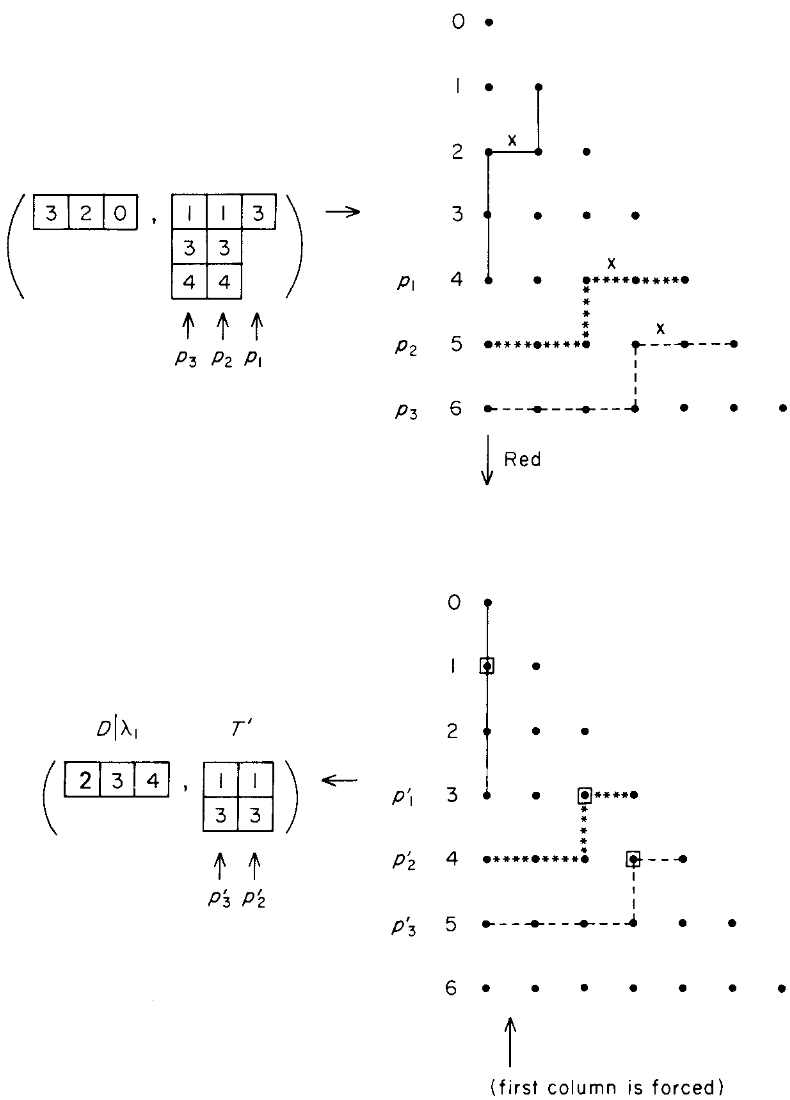


FIGURE 2.6. Step 1.

It is clear that our proof of Lemma 2.1 gives a constructive recursive procedure to calculate θ so we end this section by explicitly carrying out this procedure in the following example.

EXAMPLE. Let $a = 4$ and $\lambda = (3, 2, 2)$ so that $\lambda^* = (1, 3, 3)$ and $\hat{\lambda}^* = (1, 4, 5)$. Consider

$$\left(\underset{B}{\underset{\parallel}{\begin{array}{|c|c|c|}\hline 3 & 2 & 0 \\ \hline 1 & 1 \\ \cline{1-2} 1 & 0 \\ \cline{1-2}\end{array}}} \quad \underset{T}{\underset{\parallel}{\begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 3 & 3 \\ \cline{1-2} 4 & 4 \\ \cline{1-2}\end{array}}} \right) \in \mathscr{H}_\lambda \times \mathscr{T}_\lambda^4$$

Step 1. Our first step is to carry out the χ map on $(B\,|\,\lambda_1, T)$. Recalling our diagram of χ, we proceed as in Figure 2.6. Notice that the reduction map in this case sends a fixed point in $ml\left(\mathscr{P}\binom{4}{\hat{\lambda}^*}\right)$ to a fixed point in $mv\left(\mathscr{P}\binom{3}{\hat{\lambda}^*-1}\right)$ so we require no iterations to find Γ in this case, i.e. $\Gamma(p_1, p_2, p_3) = \mathrm{Red}(p_1, p_2, p_3)$. Now at the end of step 1, we have produced the first row of $D = \theta(B, T)$, $\boxed{4}\boxed{3}\boxed{2}$, and a tableaux $T' \in \mathscr{T}^3_{\lambda-\lambda_1}$.

Step 2. We next take $(B\,|\,\lambda - \lambda_1, T')$ and repeat the procedure as shown in Figure 2.7. In this case, $\mathrm{Red}(p_1, p_2)$ is intersecting which means that Γ, the iterated (α, β) map,

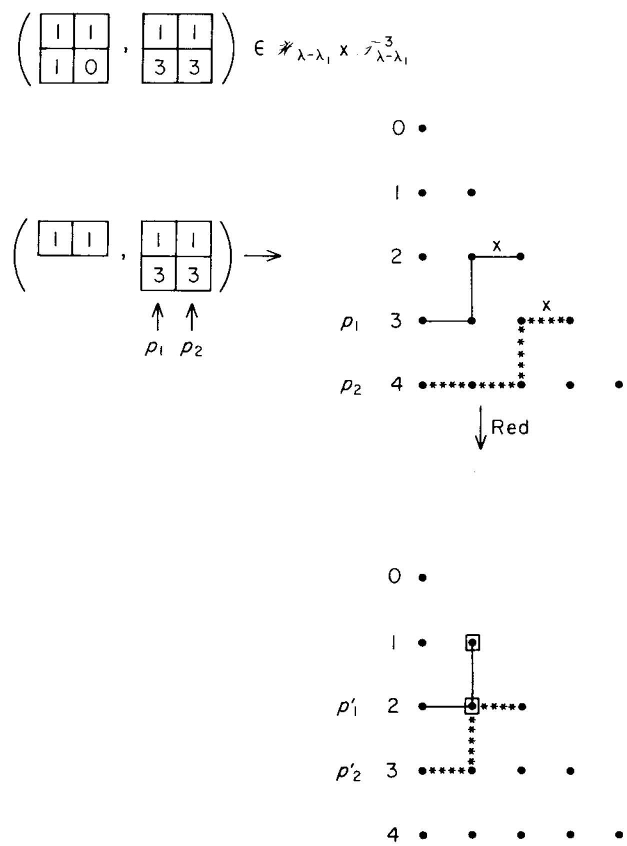


FIGURE 2.7. Step 2.

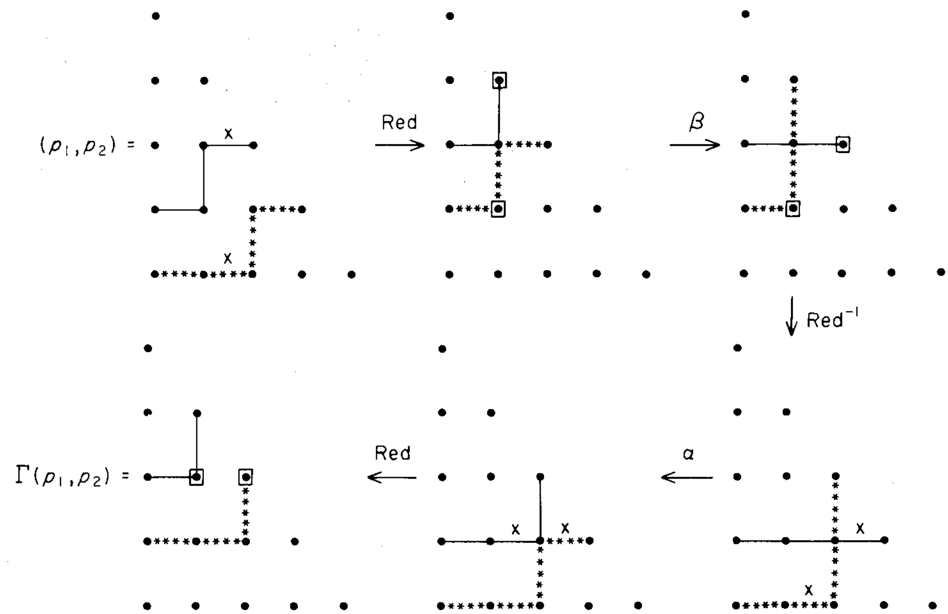
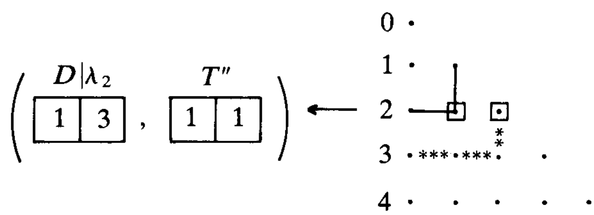


FIGURE 2.8. Iterations used to find $\Gamma(p_1, p_2)$.

requires more than one iteration. For emphasis, we shall explicitly picture the iterations to find $\Gamma(p_1, p_2)$ as shown in Figure 2.8. Now having constructed $\Gamma(p_1, p_2)$, we complete our picture by



Thus at the end of the second step of the procedure we have produced the second row of D, $D\,|\,\lambda_2 = \boxed{3|1}$, and a tableau $T'' \in \mathscr{T}^2_{\lambda-\lambda_1-\lambda_2}$.

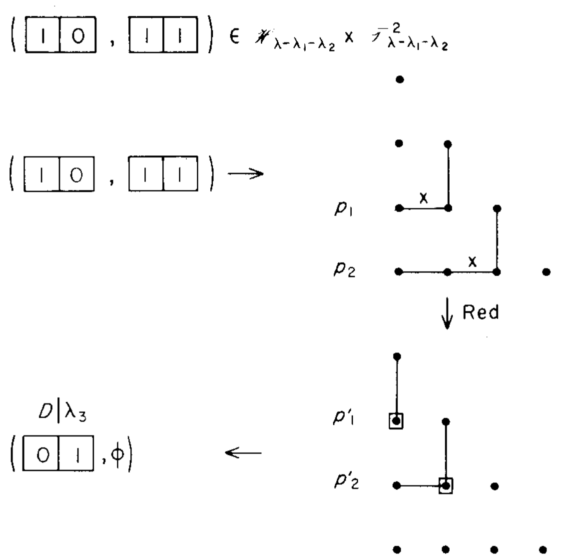


FIGURE 2.9. Step 3.

Step 3. Finally we take $(B \mid \lambda - \lambda_1 - \lambda_2, T'')$ and repeat the procedure as shown in Figure 2.9. Notice that in the last step of the procedure the paths (p_1', p_2') are completely forced and hence the only information they code is $D \mid \lambda_3 \in \mathcal{T}^2_{\lambda_3}$.

At the end of the procedure we have $\theta(B, T) = D = D \mid \lambda_1 \cup D \mid \lambda_2 \cup D \mid \lambda_3$ so that

$$\theta\left(\begin{array}{lll} 3 & 2 & 0 \\ 1 & 1 & \\ 1 & 0 & \end{array}, \begin{array}{lll} 1 & 1 & 2 \\ 2 & 2 & \\ 4 & 4 & \end{array}\right) = \begin{array}{lll} 2 & 3 & 4 \\ 1 & 3 & \\ 0 & 1 & \end{array}$$

One can easily check that, indeed, $\omega(B) + \omega(T) = \sum_{i=1}^{3} i\lambda_i + \omega(D)$ in this case. The interested reader will find it instructive to carry out the procedure on some examples of his own. For example, as an exercise the reader can check that for $\lambda = (4, 3, 3, 2)$ and $a = 5$ that

$$\theta\left(\begin{array}{llll} 6 & 3 & 2 & 0 \\ 1 & 2 & 1 & \\ 0 & 2 & 0 & \\ 1 & 0 & & \end{array}, \begin{array}{llll} 1 & 2 & 2 & 4 \\ 3 & 3 & 4 & \\ 4 & 4 & 5 & \\ 5 & 5 & & \end{array}\right) = \begin{array}{llll} 3 & 4 & 4 & 7 \\ 3 & 3 & 1 & \\ 1 & 0 & 4 & \\ 1 & 2 & & \end{array}$$

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