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$[r, s, t]$-Colorings of graphs

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Abstract

Given non-negative integers r, s, and t, an $[r, s, t]$-*coloring* of a graph $G = (V(G), E(G))$ is a mapping c from $V(G) \cup E(G)$ to the color set $\{0, 1, \ldots, k-1\}$ such that $|c(v_i) - c(v_j)| \geqslant r$ for every two adjacent vertices v_i, v_j, $|c(e_i) - c(e_j)| \geqslant s$ for every two adjacent edges e_i, e_j, and $|c(v_i) - c(e_j)| \geqslant t$ for all pairs of incident vertices and edges, respectively. The $[r, s, t]$-*chromatic number* $\chi_{r,s,t}(G)$ of G is defined to be the minimum k such that G admits an $[r, s, t]$-coloring.

This is an obvious generalization of all classical graph colorings since c is a vertex coloring if $r = 1$, $s = t = 0$, an edge coloring if $s = 1$, $r = t = 0$, and a total coloring if $r = s = t = 1$, respectively.

We present first results on $\chi_{r,s,t}(G)$ such as general bounds and also exact values, for example if $\min\{r, s, t\} = 0$ or if G is a complete graph.

MSC: 05C15

Keywords: Chromatic number; Chromatic index; Total chromatic number

1. Introduction

We consider colorings of finite and simple graphs. In a *vertex coloring* of a graph G colors are assigned to the vertices such that adjacent vertices are colored differently. Analogously, adjacent edges must obtain different colors in an *edge coloring* of G. In a *total coloring* of G the vertices and edges—together called the *elements* of G—are colored in such a way that *neighbored elements*—two adjacent vertices or two adjacent edges or a vertex and an incident edge—must be colored differently. The minimum number of colors such that G admits a vertex coloring, an edge coloring, or a total coloring, respectively, is the *chromatic number* $\chi(G)$, the *chromatic index* $\chi'(G)$, or the *total chromatic number* $\chi''(G)$, respectively.

In this paper, we investigate the following generalization of all these classical colorings.

Given non-negative integers r, s, and t, an $[r, s, t]$-*coloring* of a graph G with vertex set $V(G)$ and edge set $E(G)$ is a mapping c from $V(G) \cup E(G)$ to the color set $\{0, 1, \ldots, k-1\}$ such that $|c(v_i) - c(v_j)| \geqslant r$ for every two adjacent vertices v_i, v_j, $|c(e_i) - c(e_j)| \geqslant s$ for every two adjacent edges e_i, e_j, and $|c(v_i) - c(e_j)| \geqslant t$ for all pairs of incident vertices and edges, respectively. The $[r, s, t]$-*chromatic number* $\chi_{r,s,t}(G)$ of G is defined to be the minimum k such that G admits an $[r, s, t]$-coloring.

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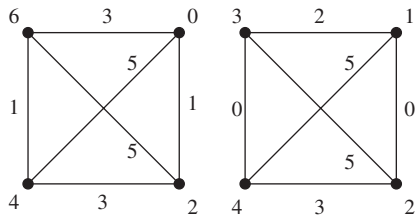

Fig. 1. A [2, 2, 1]- and a [1, 2, 1]-coloring of K_4.

Obviously, a [1, 0, 0]-coloring is an ordinary vertex coloring, a [0, 1, 0]-coloring is an edge coloring, and a [1, 1, 1]-coloring is a total coloring.

Fig. 1 shows as examples a [2, 2, 1]-coloring with seven colors and a [1, 2, 1]-coloring with six colors of the complete graph K_4.

There are different applications for such colorings. For example, assume that in a soccer tournament there are four teams in an elimination round such that each team plays one match against each other team (as in the soccer world championship tournament in Korea and Japan in 2002). During this round each team should get the possibility of a training day. Since there is only one training field, different training days must be assigned to the teams. Of course, a training day of a team should be different from a playing day. Another natural condition should be that no team should play at two successive days.

All the required conditions are fulfilled in a [1, 2, 1]-coloring of a complete graph K_4 if one assigns the training days of the teams to the vertices of K_4 and the matches between them to the edges. The right picture of Fig. 1 shows that one can arrange a schedule fulfilling all the desired conditions in six days.

As for the classical colorings, the following hereditary property holds for $[r, s, t]$-colorings.

Lemma 1. *If $H \subseteq G$ then*

$$\chi_{r,s,t}(H) \leqslant \chi_{r,s,t}(G).$$

Proof. The restriction of an $[r, s, t]$-coloring of G to the elements of $H \subseteq G$ is an $[r, s, t]$-coloring of H. □

Lemma 2. *If $r' \leqslant r$, $s' \leqslant s$, $t' \leqslant t$ then*

$$\chi_{r',s',t'}(G) \leqslant \chi_{r,s,t}(G).$$

Proof. An $[r, s, t]$-coloring of G is by definition also an $[r', s', t']$-coloring of G if $r' \leqslant r$, $s' \leqslant s$, $t' \leqslant t$. □

These obvious monotonicity properties of Lemmas 1 and 2 are often useful to determine bounds and exact values of the $[r, s, t]$-chromatic number of classes of graphs.

In this paper, we establish some general bounds on $\chi_{r,s,t}(G)$. Moreover, we determine $\chi_{r,s,t}(G)$ if $\min\{r, s, t\} = 0$ (with the exception of partial cases of $\chi_{r,0,t}(G)$ and $\chi_{0,s,t}(G)$ where bounds are given). Finally, we present bounds and exact values for the $[r, s, t]$-chromatic number of complete graphs.

2. General bounds

The following multiplication property implies that we can restrict ourselves to $[r, s, t]$-colorings such that the greatest common divisor of r, s, and t is 1.

Theorem 1. *If $a \geqslant 0$ is an integer then*

$$\chi_{ar,as,at}(G) = a(\chi_{r,s,t}(G) - 1) + 1.$$

Proof. If $a = 0$ or 1 then the assertion is obvious. Therefore, let $a \geqslant 2$ and c be an $[r, s, t]$-coloring of a graph G with $\chi_{r,s,t}(G)$ colors. If we multiply all assigned colors by a then we obtain a coloring c' such that $c'(x) = a \cdot c(x)$ for all elements $x \in V(G) \cup E(G)$ and $|c'(v_i) - c'(v_j)| = a|c(v_i) - c(v_j)| \geqslant ar$ if vertices v_i and v_j are adjacent, $|c'(e_i) - c'(e_j)| = a|c(e_i) - c(e_j)| \geqslant as$ if edges e_i and e_j are adjacent, and $|c'(v_i) - c'(e_j)| = a|c(v_i) - c(e_j)| \geqslant at$ if v_i and e_j are incident, respectively. Therefore, c' is an $[ar, as, at]$-coloring of G with colors in $\{0, 1, \ldots, a(\chi_{r,s,t}(G) - 1)\}$ which implies that $a(\chi_{r,s,t}(G) - 1) + 1$ is an upper bound of $\chi_{ar,as,at}(G)$.

On the other hand, assume that the graph G has an $[ar, as, at]$-coloring c with $a(\chi_{r,s,t}(G) - 1)$ colors ($a \geqslant 2$). We partition the elements of G in $\chi_{r,s,t}(G) - 1$ color classes by joining all the elements with colors $za, za+1, \ldots, za+a-1$ for $z = 0, \ldots, \chi_{r,s,t}(G) - 2$. If two elements x_i and x_j belong to the same color class then $|c(x_i) - c(x_j)| \leqslant a - 1$. This implies that if two elements x_i and x_j of the same color class are neighbored then the corresponding parameter r, s, or t must be 0 since c is an $[ar, as, at]$-coloring.

We define a coloring c' by $c'(x) = \lfloor c(x)/a \rfloor$, i.e., all elements belonging to the same color class are colored with one color. If, for example, x_i and x_j are adjacent vertices then $|c(x_i)/a - c(x_j)/a| \geqslant r$ by assumption which implies $|\lfloor c(x_i)/a \rfloor - \lfloor c(x_j)/a \rfloor| = |c'(x_i) - c'(x_j)| \geqslant r$. Therefore, c' is an $[r, s, t]$- coloring of G with $\chi_{r,s,t}(G) - 1$ colors which contradicts the definition of the $[r, s, t]$-chromatic number of G. $\square$

This result immediately gives the $[r, s, t]$-chromatic number of graphs G in case $r = s = t$.

Corollary 1. $\chi_{r,r,r}(G) = r(\chi''(G) - 1) + 1$.

In the following we give some obvious general lower and upper bounds for $\chi_{r,s,t}(G)$. Nevertheless, these bounds are attained for some specific classes of graphs.

Theorem 2.

$$\max\{r(\chi(G) - 1) + 1,\ s(\chi'(G) - 1) + 1,\ t + 1\} \leqslant \chi_{r,s,t}(G) \leqslant r(\chi(G) - 1) + s(\chi'(G) - 1) + t + 1.$$

Proof. By Lemma 2 and Theorem 1, $\chi_{r,s,t}(G) \geqslant \chi_{r,0,0}(G) = r(\chi(G) - 1) + 1$ as well as $\chi_{r,s,t}(G) \geqslant \chi_{0,s,0}(G) = s(\chi'(G) - 1) + 1$. Obviously, $\chi_{r,s,t}(G) \geqslant t + 1$ since we need colors with distance t to color vertices and edges.

If we color the vertices of G with colors $0, r, \ldots, r(\chi(G) - 1)$ and the edges with colors $r(\chi(G) - 1) + t, r(\chi(G) - 1) + t + s, \ldots, r(\chi(G) - 1) + t + s(\chi'(G) - 1)$ we obtain an $[r, s, t]$-coloring of G. $\square$

The lower bound of Theorem 2 is attained, for example, for complete graphs with $s = t = 1$ and $r \geqslant 2$: $\chi_{r,1,1}(K_p) = r\Delta(K_p) + 1$ (see Theorem 6), where $\Delta(G)$ is the maximum degree of G. The upper bound is attained, for example, for complete graphs of even order with $r = s = 1$ and $t > 2\Delta(K_{2n}) - 1$: $\chi_{1,1,t}(K_{2n}) = 2\Delta(K_{2n}) + t$ (see Corollary 2). It would be an interesting task to characterize all graphs for which the upper bound is achieved. Corollary 2 provides a subclass of these graphs.

Lemma 3. *If $t > r(\chi(G) - 1) + s(\chi'(G) - 1)$ then*

$$\chi_{r,s,t}(G) \geqslant r(\chi(G) - 1) + s(\delta(G) - 1) + t + 1,$$

where $\delta(G)$ is the minimum degree of G.

Proof. Let c be an $[r, s, t]$-coloring of G with $\chi_{r,s,t}(G)$ colors. By Theorem 2 and the assumption on t we obtain $2t + 1 > r(\chi(G) - 1) + s(\chi'(G) - 1) + t + 1 \geqslant \chi_{r,s,t}(G)$. If there would be a vertex v and incident edges e_1, e_2 such that $c(e_1) < c(v) < c(e_2)$ or an edge e with end-vertices v_1, v_2 such that $c(v_1) < c(e) < c(v_2)$ then the number of colors of c must be at least $2t + 1$ which contradicts $\chi_{r,s,t}(G) < 2t + 1$. Therefore, if x is an arbitrary element of G then $c(x) < c(y)$ for all elements y that are incident to x or $c(x) > c(y)$ for all y. Using induction this implies that either $c(v) < c(e)$ for all vertices v and all edges e incident to v or always $c(v) > c(e)$ since we can assume that G is connected (otherwise consider each component separately). Without loss of generality, let $c(v) < c(e)$.

Consider a vertex u with greatest color $c(u)$. Since $c(u) \geqslant \chi_{r,0,0}(G) - 1 = r(\chi(G) - 1)$ by Theorems 2 and 4(a) and u is incident to at least $\delta(G)$ edges e with $c(e) > c(u)$ we obtain $\chi_{r,s,t}(G) \geqslant c(u) + t + s(\delta(G) - 1) + 1 \geqslant r(\chi(G) - 1) + s(\delta(G) - 1) + t + 1$. $\square$

Vizing [9] proved that $\Delta(G) \leqslant \chi'(G) \leqslant \Delta(G)+1$ for every graph G. A graph G is called a *class*-1 *graph* if $\chi'(G)=\Delta(G)$ and a *class*-2 *graph* if $\chi'(G) = \Delta(G) + 1$, respectively. A graph G is called *regular* if $\delta(G) = \Delta(G)$.

Corollary 2. *If* $t > r(\chi(G) - 1) + s(\chi'(G) - 1)$ *and G is a regular class*-1 *graph then*

$$\chi_{r,s,t}(G) = r(\chi(G) - 1) + s(\chi'(G) - 1) + t + 1.$$

Proof. Since G is class 1 and regular, we have $\delta(G)=\Delta(G)=\chi'(G)$ and therefore the result is an immediate implication of Theorem 2 and Lemma 3. □

It is also an interesting task to determine all graphs whose $[r, s, t]$-chromatic numbers coincide with the lower bound of Theorem 2.

It was proved by Molloy and Reed [7] that the total chromatic number of every graph G is bounded from above by the maximum degree of G plus a constant $c \geqslant 2$. Therefore,

$$\Delta(G) + 1 \leqslant \chi''(G) \leqslant \Delta(G) + c. \tag{1}$$

Using this result the following bounds for the $[r, s, t]$-chromatic number can be obtained.

Theorem 3. *If* $m = \min\{r, s, t\}$ *and* $M = \max\{r, s, t\}$ *then*

$$m\Delta(G) + 1 \leqslant \chi_{r,s,t}(G) \leqslant M(\Delta(G) + c - 1) + 1$$

with a constant $c \geqslant 2$.

Proof. By Lemma 2, Theorem 1 and (1), we obtain $\chi_{r,s,t}(G) \geqslant \chi_{m,m,m}(G) = m(\chi_{1,1,1}(G) - 1) + 1 = m(\chi''(G) - 1) + 1 \geqslant m\Delta(G) + 1$ and $\chi_{r,s,t}(G) \leqslant \chi_{M,M,M}(G) = M(\chi''(G) - 1) + 1 \leqslant M(\Delta(G) + c - 1) + 1$. □

The total coloring conjecture states that $\chi''(G) \leqslant \Delta(G) + 2$ for every graph G [1,9], i.e., $c = 2$ in (1). So far, the total coloring conjecture is proved for some specific classes of graphs, e.g., for complete graphs, for bipartite graphs, for complete multipartite graphs [10], for graphs G with $\Delta(G) \geqslant \frac{3}{4}|V(G)|$ [4] or $\Delta(G) \leqslant 5$ [6], and for planar graphs G with $\Delta(G) \neq 6$ [2,5,8].

The validity of the total coloring conjecture in general would imply that $\chi''(G)$ attains one of two values for every graph G. A graph G is called a *type*-1 *graph* if $\chi''(G) = \Delta(G) + 1$ and a *type*-2 *graph* if $\chi''(G) = \Delta(G) + 2$, respectively.

The lower bound of Theorem 3 is attained for some classes of graphs, for example if $r = s = t$ and G is a type-1-graph. The upper bound with $c = 2$ is attained if $r = s = t$ and $G = K_{2n}$, for example.

On the other hand, if at least one of the parameters r, s, t is 0 then the bounds can be improved. In the next section we consider this case, namely that the minimum of r, s, t is 0.

3. $\min\{r, s, t\} = 0$

In this section, we investigate such $[r, s, t]$-colorings that at least one of the parameters r, s, t is 0.

Theorem 4. *If G is non-trivial then*

(a) $\chi_{r,0,0}(G) = r(\chi(G) - 1) + 1$,
(b) $\chi_{0,s,0}(G) = s(\chi'(G) - 1) + 1$,
(c) $\chi_{0,0,t}(G) = t + 1$,
(d) $\chi_{r,s,0}(G) = \max\{\chi_{r,0,0}(G), \chi_{0,s,0}(G)\}$,
(e) $r(\chi(G) - 1) + 1 \leqslant \chi_{r,0,t}(G) \leqslant r(\chi(G) - 1) + t + 1$,
(f) $s(\chi'(G) - 1) + 1 \leqslant \chi_{0,s,t}(G) \leqslant s(\chi'(G) - 1) + t + 1$.

Proof. (a) and (b) are consequences of Theorem 1, (c) of Theorem 2. (d) The equation follows by Lemma 2 and the fact that vertices and edges can be colored independently. (e) and (f) are implications of Theorem 2. □

The above theorem only provides lower and upper bounds for $\chi_{r,0,t}(G)$ and $\chi_{0,s,t}(G)$. In the following we will determine exact values for most cases.

Lemma 4. *If $\chi(G)=2$ then*

$$\chi_{r,0,t}(G)=\begin{cases} r+1 & \text{if } r\geqslant 2t,\\ 2t+1 & \text{if } t\leqslant r<2t,\\ r+t+1 & \text{if } r<t.\end{cases}$$

Proof. By Theorem 4(e) we get

$$r(\chi(G)-1)+1=r+1\leqslant\chi_{r,0,t}(G)\leqslant r(\chi(G)-1)+t+1=r+t+1.$$

Since G is bipartite we can color all vertices with colors 0 and r and all edges with color t if $r\geqslant 2t$. In case $t\leqslant r<2t$ color the vertices with colors 0 and $2t$ and the edges with color t. If $r<t$ then color the vertices with colors 0 and r and the edges with color $r+t$.

If $r<2t$ then consider an edge $e=uv$ and an $[r,0,t]$-coloring c of G. Let $c(u)\leqslant c(v)$ without loss of generality. If $c(e)\leqslant c(u)$ or $c(e)\geqslant c(v)$ then at least $r+t+1$ colors are needed. If $c(u)\leqslant c(e)\leqslant c(v)$ then at least $2t+1$ colors are needed. Therefore, $2t+1$ is a lower bound for $\chi_{r,0,t}(G)$ if $t\leqslant r<2t$ and $r+t+1$ is a lower bound if $r<t$. □

Lemma 4 shows that the lower bound as well as the upper bound of part (e) of Theorem 4 can be attained.

Lemma 5. *If $\chi(G)\geqslant 3$ and $r\geqslant t$ then*

$$\chi_{r,0,t}(G)=r(\chi(G)-1)+1.$$

Proof. Let c be an $[r,0,0]$-coloring of G with colors $0,r,\ldots,r(\chi(G)-1)$. To extend c to an $[r,0,t]$-coloring of G we successively color the edges $e=uv$ with a color of the set $\{0,r,2r\}\backslash\{c(u),c(v)\}$ which is possible since $\chi(G)\geqslant 3$ and $r\geqslant t$. □

According to Lemmas 4 and 5, $\chi_{r,0,t}(G)$ is determined with the exception that $r<t$ and $\chi(G)\geqslant 3$. The next result provides lower and upper bounds which improve the bounds of Theorem 2.

Lemma 6. *If $\chi(G)\geqslant 3$ and $r<t$ then*

$$\max\{r(\chi(G)-1)+1,t+1\}\leqslant\chi_{r,0,t}(G)\leqslant r(\chi(G)-3)+t+1+\min\{t,2r\}.$$

Proof. The lower bound is an immediate consequence of Theorem 2.

By Theorem 4(e), $\chi_{r,0,t}(G)\leqslant r(\chi(G)-1)+t+1$.

Consider an $[r,0,0]$-coloring of G with colors $0,r,\ldots,r(\chi(G)-1)$ which does exist since $\chi_{r,0,0}(G)=r(\chi(G)-1)+1$ by Theorem 4(a). Recolor all vertices of color $r(\chi(G)-2)$ with color $r(\chi(G)-2)+t-r$ and all vertices of color $r(\chi(G)-1)$ with color $r(\chi(G)-1)+2(t-r)$. Color all the edges of G which are not incident to vertices of color $r(\chi(G)-2)+t-r$ with this color. Color all edges of G which are incident to vertices of color $r(\chi(G)-2)+t-r$ with color $r(\chi(G)-1)+2(t-r)$. If there are edges joining vertices of colors $r(\chi(G)-2)+t-r$ and $r(\chi(G)-1)+2(t-r)$ then recolor these edges with color 0 to obtain a proper $[r,0,t]$-coloring of G with $r(\chi(G)-3)+2t+1$ colors. □

The same argument as in the proof of Lemma 5 yields that $\chi_{r,0,t}(G)=\chi_{r,0,0}(G)$ if $\chi_{r,0,0}(G)=r(\chi(G)-1)+1\geqslant 4t-1$.

Now we consider $\chi_{0,s,t}(G)$. If $\Delta(G)=1$ then obviously $\chi_{0,s,t}(G)=t+1$.

Lemma 7. *If $\Delta(G)\geqslant 2$ and G is class 1 then*

$$\chi_{0,s,t}(G)=\begin{cases} s(\Delta(G)-1)+1 & \text{if } s\geqslant 2t,\\ s(\Delta(G)-1)+2t-s+1 & \text{if } t\leqslant s<2t,\\ s(\Delta(G)-1)+t+1 & \text{if } s<t.\end{cases}$$

Proof. By Theorem 4(f), $\chi_{0,s,t}(G)\leqslant s(\chi'(G)-1)+t+1=s(\Delta(G)-1)+t+1$.

If $s < 2t$ then consider a $[0, s, 0]$-coloring c of G with colors $0, s, \ldots, s(\Delta(G) - 1)$. Since $r = 0$, all vertices can obtain the same color. Assign color $s(\Delta(G) - 2) + t$ to the vertices and recolor the edges e of G with $c(e) = s(\Delta(G) - 1)$ with color $s(\Delta(G) - 1) + 2t - s$ to obtain a $[0, s, t]$-coloring of G with $s(\Delta(G) - 1) + 2t - s + 1$ colors.

If $s \geqslant 2t$ then color the edges with colors $0, s, \ldots, s(\Delta(G) - 1)$ and all vertices with so far unused color t.

If $s \geqslant 2t$ then $\chi_{0,s,t}(G) \geqslant s(\Delta(G) - 1) + 1$ by Theorem 4(f).

Let $s < 2t$. By Lemma 1, $\chi_{0,s,t}(G) \geqslant \chi_{0,s,t}(K_{1,\Delta(G)})$. Let c be a $[0, s, t]$-coloring of $K_{1,\Delta(G)}$. Assume without loss of generality that $c(e_1) \leqslant c(e_2) \leqslant \ldots \leqslant c(e_{\Delta(G)})$. If v is the center of the star $K_{1,\Delta(G)}$ and $c(v) < c(e_1)$ or $c(v) > c(e_{\Delta(G)})$ then at least $s(\Delta(G) - 1) + t + 1$ colors are needed. If $c(e_i) \leqslant c(v) \leqslant c(e_{i+1})$ for $1 \leqslant i \leqslant \Delta(G) - 1$ then at least $s(i - 1) + 2t + s(\Delta(G) - (i + 1)) + 1 = s(\Delta(G) - 2) + 2t + 1$ colors are needed. Therefore, $\chi_{0,s,t}(G) \geqslant s(\Delta(G) - 1) + 1 + \min\{t, 2t - s\}$. □

Lemma 8. *If $\Delta(G) \geqslant 2$ and $s \geqslant t$ and G is class 2 then*

$$\chi_{0,s,t}(G) = s(\chi'(G) - 1) + 1.$$

Proof. By Theorem 4(f), $\chi_{0,s,t}(G) \geqslant s(\chi'(G) - 1) + 1 = s\Delta(G) + 1$. Consider a $[0, s, 0]$-coloring of G with colors of the set $S = \{0, s, \ldots, s\Delta(G)\}$. Since every vertex v has at most $\Delta(G)$ incident edges, at least one of the colors of S is not contained in the color set of these edges. Assign this color to v to obtain a $[0, s, t]$-coloring of G with $s(\chi'(G) - 1) + 1$ colors. □

Combining the results of Lemmas 7 and 8 we obtain that $\chi_{0,s,t}(G)$ is determined with the exception of the case that $s < t$ and G is a class-2 graph. For those graphs we can prove the following bounds.

Lemma 9. *If G is class 2 and $s < t$ then*

$$s(\Delta(G) - 1) + t + 1 \leqslant \chi_{0,s,t}(G) \leqslant \min\{s\Delta(G) + t + 1, t\Delta(G) + 1\}.$$

Proof. By Lemmas 1 and 7, $\chi_{0,s,t}(G) \geqslant \chi_{0,s,t}(K_{1,\Delta(G)}) = s(\Delta(G) - 1) + t + 1$. Theorem 4(f) provides the upper bound $\chi_{0,s,t}(G) \leqslant s\Delta(G) + t + 1$ and Lemma 2 together with Lemma 8 the upper bound $\chi_{0,s,t}(G) \leqslant \chi_{0,t,t}(G) = t\Delta(G) + 1$. □

4. $\chi_{r,1,1}(G)$, $\chi_{1,s,1}(G)$, $\chi_{1,1,t}(G)$

In this section, we consider $[r, s, t]$-colorings such that two of the parameters r, s, t are 1.

The idea to construct an $[r, 1, 1]$-coloring of G is to extend an $[r, 0, 0]$-coloring of G. This is always possible if $r \geqslant r_0$ where r_0 depends on G.

Lemma 10. *If $r \geqslant \dfrac{\chi'(G)}{\chi(G) - 1} + 1$ then*

$$\chi_{r,1,1}(G) = \chi_{r,0,0}(G).$$

Proof. Let c be an $[r, 0, 0]$-coloring of G with colors $0, r, \ldots, r(\chi(G) - 1)$. There are $(r - 1)(\chi(G) - 1)$ colors less than $r(\chi(G) - 1)$ which are not used by c. Since $\chi'(G) \leqslant (r - 1)(\chi(G) - 1)$ by the assumption on r these colors can be used to extend c to an $[r, 1, 1]$-coloring of G. □

If $L = \{L(v) : v \in V(G)\}$ is a set of lists of colors then an *L-list* (*vertex*) *coloring* of a graph G is a coloring of the vertices of G such that each vertex obtains a color from its own list and adjacent vertices are colored differently. A graph G is called *k-choosable* if such a coloring exists for each choice of lists $L(v)$ of cardinality at least k. The minimum k such that G is k-choosable is the *choice number* $\mathrm{ch}(G)$ of G.

The *coloring number* $\mathrm{col}(G)$ of a graph G is defined by

$$\mathrm{col}(G) = \max_{H \subseteq G} \delta(H) + 1$$

and fulfills $\chi(G) \leqslant \mathrm{ch}(G) \leqslant \mathrm{col}(G) \leqslant \Delta(G) + 1$.

The *total graph* $T(G)$ of a graph G is defined having vertex set $V(T(G)) = V(G) \cup E(G)$ and two vertices x_i and x_j are joined by an edge if and only if x_i and x_j are neighbored in G.

In some cases the lower bound for r in the following lemma is better than that of Lemma 10 (for example for odd cycles).

Lemma 11. *Let $G' \subseteq T(G)$ be the subgraph of $T(G)$ which is induced by the edge set $E(G)$ of G. If $r \geqslant 2$ and $r \geqslant (\mathrm{col}(G') + 1)/(\chi(G) - 1)$ then*

$$\chi_{r,1,1}(G) = \chi_{r,0,0}(G).$$

Proof. Let c be an $[r, 0, 0]$-coloring of G with $\chi_{r,0,0}(G) = r(\chi(G) - 1) + 1$ colors. To obtain a vertex coloring f of $T(G)$ we set $f(x) = c(x)$ if $x \in V(G)$ and assign the list $L(xy) = \{0, 1, \ldots, r(\chi(G) - 1)\} \backslash \{f(x), f(y)\}$ to every edge xy which is of length $r(\chi(G) - 1) - 1 \geqslant \mathrm{col}(G') \geqslant \mathrm{ch}(G')$. Obviously, f induces an $[r, 1, 1]$-coloring of G with $\chi_{r,0,0}(G)$ colors. □

For example, Lemma 10 implies $\chi_{r,1,1}(C_{2n}) = \chi_{r,0,0}(C_{2n}) = r + 1$ for even cycles C_{2n} if $r \geqslant 3$ and Lemma 11 implies $\chi_{r,1,1}(C_{2n+1}) = \chi_{r,0,0}(C_{2n+1}) = 2r + 1$ for odd cycles C_{2n+1} if $r \geqslant 2$.

The following theorem covers completely $[1, s, 1]$-colorings of graphs G if $s \geqslant 2$.

Theorem 5. *If $\Delta(G) \geqslant 2$ and $s \geqslant 2$ then*

$$\chi_{1,s,1}(G) = \begin{cases} 2\Delta(G) & \text{if } s = 2 \text{ and } G \text{ is class 1 and } \chi(H) \geqslant \Delta(G), \\ s(\chi'(G) - 1) + 1 & \text{otherwise}, \end{cases}$$

where H is the core of G which is the subgraph of G that is induced by the vertices of maximum degree.

Proof. By Lemma 2 and Theorem 4(b), $\chi_{1,s,1}(G) \geqslant \chi_{0,s,0}(G) = s(\chi'(G) - 1) + 1$.

If $s \geqslant 3$ then color the edges with colors $0, s, 2s, \ldots, s(\chi'(G) - 1)$ and the vertices with the $(s - 1)(\chi'(G) - 1)$ unused colors in between which is possible by Brooks' theorem [3] with the exception $G = C_{2n+1}$ since $(s-1)(\chi'(G) - 1) \geqslant 2(\Delta(G) - 1)$ is at least $\Delta(G) + 1$ if $\Delta(G) \geqslant 3$ and is equal to $\Delta(G)$ if $\Delta(G) = 2$. If $s \geqslant 3$ and $G = C_{2n+1}$ then color the edges with colors $0, s, 2s$, a vertex which is incident to edges colored by s and $2s$ with color 0 and the other vertices successively by colors 1 and $s + 1$.

If $s = 2$ and G is class 2 then $\chi'(G) - 1 = \Delta(G)$ which implies that we can color the vertices again with the colors in between the edge colors if $G \neq K_{2n+1}$ and $G \neq C_{2n+1}$. If $G = C_{2n+1}$ color the edges by colors 0, 2, 4 and the vertices by 0, 1, 3 as above. If $G = K_{2n+1}$ then $\chi_{1,2,1}(G) = 2\Delta(G) + 1$ (see Theorem 6(c)).

If $s = 2$ and G is class 1 and $\chi(H) < \Delta(G)$ where H is the core of G then we color the edges of G with colors $0, 2, \ldots, 2\Delta(G) - 2 = 2(\chi'(G) - 1)$ and the vertices of H with colors $1, 3, \ldots, 2\Delta(G) - 3$. The remaining vertices of degree at most $\Delta(G) - 1$ have at most $2\Delta(G) - 2$ neighbored elements and therefore can be colored in an arbitrary order.

If $s = 2$ and G is class 1 and $\chi(H) = \Delta(G)$ then we color the edges of G with colors $0, 2, \ldots, 2\Delta(G) - 2$ and the vertices of G with colors $1, 3, \ldots, 2\Delta(G) - 1$ to obtain a $[1, 2, 1]$-coloring of G with $2\Delta(G)$ colors. If we assume that there is a $[1, 2, 1]$-coloring of G with $2\Delta(G) - 1$ colors then the edges incident to a vertex of H must be colored with colors $0, 2, \ldots, 2\Delta(G) - 2$. Therefore, there are only $\Delta(G) - 1$ colors to color the vertices of H which is impossible since $\chi(H) = \Delta(G)$.

If $s = 2$ and G is class 1 and $\chi(H) = \Delta(G) + 1$ then $G = K_{2n}$ and $\chi_{1,2,1}(K_{2n}) = 2\Delta(G)$ (see Theorem 6(f)). □

In the following we give some partial results for $\chi_{1,1,t}(G)$.

Theorem 2 yields a general upper bound: $\chi_{1,1,t}(G) \leqslant \chi(G) + \chi'(G) + t - 1$. On the other hand, Lemma 7 implies $\chi_{1,1,t}(G) \geqslant \chi_{0,1,t}(G) \geqslant \chi'(G) + t$ if G is class 1. For example, these bounds yield $\Delta(G) + t \leqslant \chi_{1,1,t}(G) \leqslant \Delta(G) + t + 1$ for bipartite graphs G.

Corollary 2 implies that $\chi_{1,1,t}(G) = \chi(G) + \chi'(G) + t - 1$ if $t > \chi(G) + \chi'(G) - 2$ and G is a regular class-1 graph.

For complete graphs K_p we can prove the following lemma.

Lemma 12. *If $t > (r+s)(p-1)$ and $r \geqslant s$ then*

$$\chi_{r,s,t}(K_p) = r(p-1) + s(p-2) + t + 1.$$

Proof. If p is even then the result follows from Corollary 2.

If p is odd then $\chi_{r,s,t}(K_p) \geqslant r\Delta + s(\Delta-1) + t + 1 = r(p-1) + s(p-2) + t + 1$ if $t > (r+s)(p-1)$ by Lemma 3 where $\Delta = \Delta(K_p)$. We color the edges of K_p with colors $t + r\Delta - s, t + r\Delta, \ldots, t + r\Delta + s(\Delta-1)$ and then the vertices with colors $0, r, \ldots, r\Delta$ in such a way that the vertex of color $r\Delta$ is not incident to the edges of color $t + r\Delta - s$. Since this is an $[r, s, t]$-coloring of K_p if $r \geqslant s$ we obtain $\chi_{r,s,t}(K_p) \leqslant t + r\Delta + s(\Delta-1) + 1$ which concludes the proof. □

This lemma gives immediately the following result for $[1, 1, t]$-colorings of complete graphs K_p.

Corollary 3. *If $t > 2(p-1)$ then*

$$\chi_{1,1,t}(K_p) = 2(p-1) + t.$$

5. Complete graphs K_p

According to Lemma 1, $\chi_{r,s,t}(K_{\omega(G)}) \leqslant \chi_{r,s,t}(G) \leqslant \chi_{r,s,t}(K_{|V(G)|})$ for arbitrary graphs G where $\omega(G)$ is the clique number and $|V(G)|$ the order of G. Therefore, it is an interesting task to determine the $[r, s, t]$-chromatic number of complete graphs K_p.

Due to the results of Section 3, $\chi_{r,s,t}(K_p)$ is already determined in the case $\min\{r, s, t\} = 0$ with the exceptions of $\chi_{r,0,t}(K_p)$ iff $r < (4t-2)/(p-1)$ and of $\chi_{0,s,t}(K_p)$ iff $p = 2n+1$ and $s < t$. Now we consider $[r, s, t]$-colorings of complete graphs K_p such that $\min\{r, s, t\} \geqslant 1$.

Theorem 6. *If $\min\{r, s, t\} \geqslant 1$, $p \geqslant 3$ and $\Delta = \Delta(K_p)$ then*

(a) $\chi_{r,r,r}(K_p) = \begin{cases} r\Delta + 1 & \text{if } p \text{ odd}, \\ r(\Delta+1) + 1 & \text{if } p \text{ even}, \end{cases}$
(b) $\chi_{r,s,t}(K_{2n+1}) = r\Delta + 1$ *if* $1 \leqslant s \leqslant r$, $1 \leqslant t \leqslant r$,
(c) $\chi_{r,s,t}(K_{2n+1}) = s\Delta + 1$ *if* $1 \leqslant r \leqslant s$, $1 \leqslant t \leqslant s$,
(d) $\chi_{r,s,t}(K_{2n}) = r\Delta + 1$ *if* $r \geqslant 2$, $1 \leqslant s \leqslant r$, $1 \leqslant t \leqslant \lfloor r/2 \rfloor$,
(e) $\chi_{1,s,1}(K_{2n}) = s(\Delta-1) + 1$ *if* $s \geqslant 3$,
(f) $\chi_{1,2,1}(K_{2n}) = 2\Delta$,
(g) $\chi_{r,s,t}(K_p) = r\Delta + s(\Delta-1) + t + 1$ *if* $t > (r+s)\Delta$ *and* $r \geqslant s$.

Proof.

(a) This is an immediate consequence of Corollary 1 since $\chi''(K_p) = \Delta + 1$ if p is odd and $\Delta + 2$ if p is even.
(b) We have $\chi_{r,s,t}(K_p) \geqslant \chi_{r,0,0}(K_p) = r\Delta + 1$ by Theorem 4(a) and Lemma 2 and $\chi_{r,s,t}(K_{2n+1}) \leqslant \chi_{r,r,r}(K_{2n+1}) = r\Delta + 1$ by (a) and Lemma 2 if $1 \leqslant s \leqslant r$ and $1 \leqslant t \leqslant r$.
(c) The proof is analogous to that of part (b) since $\chi_{0,s,0}(K_{2n+1}) = s\Delta + 1$.
(d) It holds $\chi_{r,s,t}(K_{2n}) \geqslant \chi_{r,0,0}(K_{2n}) = r\Delta + 1$ by Lemma 2 and Theorem 4(a). Color the vertices of K_{2n} with colors $0, r, 2r, \ldots, \Delta r$ ($r \geqslant 2$) and the edges with the Δ colors $\lfloor r/2 \rfloor, \lfloor r/2 \rfloor + r, \lfloor r/2 \rfloor + 2r, \ldots, \lfloor r/2 \rfloor + (\Delta-1)r$ to obtain a $[r, r, \lfloor r/2 \rfloor]$-coloring of K_{2n} since $\chi'(K_{2n}) = \Delta$. Therefore, by Lemma 2, $\chi_{r,s,t}(K_{2n}) \leqslant \chi_{r,r,\lfloor r/2 \rfloor}(K_{2n}) \leqslant r\Delta + 1$ if $1 \leqslant s \leqslant r$, $1 \leqslant t \leqslant \lfloor r/2 \rfloor$ and $r \geqslant 2$.
(e) By Lemma 2 and Theorem 4(b), $\chi_{1,s,1}(K_{2n}) \geqslant \chi_{0,s,0}(K_{2n}) = s(\Delta-1) + 1$. Color the edges of K_{2n} with colors $0, s, 2s, \ldots, (\Delta-1)s$ and the vertices with $2n = \Delta + 1$ arbitrary of the so far unused $(s-1)(\Delta-1)$ colors to obtain a $[1, s, 1]$-coloring of K_{2n} which is possible since $s \geqslant 3$.
(f) If we assume that $\chi_{1,2,1}(K_{2n}) \leqslant 2\Delta - 1$ then in a $[1, 2, 1]$-coloring of K_{2n} with $2\Delta - 1$ colors we need to color the edges with colors $0, 2, \ldots, 2(\Delta-1)$. Therefore, the vertices must be colored by the remaining $\Delta - 1$ colors $1, 3, \ldots, 2(\Delta-1) - 1$ which is impossible. This implies $\chi_{1,2,1}(K_{2n}) \geqslant 2\Delta$. To obtain a $[1, 2, 1]$-coloring of K_{2n} with 2Δ colors we color the edges with the Δ colors $0, 2, \ldots, 2(\Delta-1)$. Recolor one of the edges with color

$2(\Delta-1)$ by $2\Delta-1$. Color the incident vertices of this edge with colors 1 and $2(\Delta-1)$ and all other $\Delta-1$ vertices successively by colors $3, 5, \ldots, 2\Delta-1$ in an arbitrary order.

(g) See Lemma 12. □

The smallest cases with $\min\{r, s, t\} \geqslant 1$ which are not covered by Theorem 6 are $\chi_{2,3,1}(K_{2n})$ and $\chi_{1,1,2}(K_p)$. Nevertheless, Theorem 6(d) and (e) and Lemma 2 yield bounds for $\chi_{2,3,1}(K_{2n})$: $6n-5=\chi_{1,3,1}(K_{2n}) \leqslant \chi_{2,3,1}(K_{2n}) \leqslant \chi_{3,3,1}(K_{2n})= 6n-2$. The lower bound can be improved by 1 since we are forced in a [2, 3, 1]-coloring of K_{2n} with $6n-5$ colors to color the edges with the $\Delta(K_{2n}) = 2n-1$ colors $0, 3, \ldots, 3(2n-2)$ and therefore there are left only $2n-2$ colors of pairwise distance at least 2 to color the vertices of K_{2n} which is impossible.

If we color the edges of K_{2n} with colors $1, 4, 7, \ldots, 6n-5$ and the vertices with colors $0, 2, 5, 8, 11, \ldots, 6n-4$ then we result in a [2, 3, 1]-coloring of K_{2n} with $6n-3$ colors. Therefore, $6n-4 \leqslant \chi_{2,3,1}(K_{2n}) \leqslant 6n-3$. If $n=2$ we obtain $\chi_{2,3,1}(K_4) = 9$ by some case analysis.

The bounds of Sections 2 and 3 yield $p+1 \leqslant \chi_{1,1,2}(K_p) \leqslant 2p-1$ if p is odd and $\leqslant 2p$ if p is even. A suitable coloring yields the upper bound $2p-1$ also for even order p. If $p=3$ or 4 then this upper bound is achieved: $\chi_{1,1,2}(K_3) = 5$, $\chi_{1,1,2}(K_4) = 7$.

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