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On Infinite-Dimensional Differential Equations*

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Differential equations $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), t)$ are exhibited in a general infinite-dimensional Banach space, failing each of the following in turn. (i) The set S_t of solution values $\mathbf{x}(t)$ from a given point $\mathbf{x}(0)$ is compact. (ii) S_t is connected. (iii) Any point on the boundary ∂S_t of S_t can be reached by a solution $\mathbf{x}$ with $\mathbf{x}(s) \in \partial S_s$, $0 \leqslant s \leqslant t$.

1. INTRODUCTION

Consider the differential equation

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), t), \qquad t \in \mathbb{R}_+, \tag{1.1}$$

where $\mathbf{f}$ is (jointly) continuous. Let S_t be the set of points $\mathbf{x}(t)$ as $\mathbf{x}$ runs through all solutions of (1.1) for fixed $\mathbf{x}(0)$ in the Banach space X which has its norm topology throughout. Since $\mathbf{f}$ maps some neighborhood of $\mathbf{x}(0)$ into a bounded set and

$$(d/dt)\,\|\mathbf{x}(t)\| \leqslant \|\dot{\mathbf{x}}(t)\|,$$

we may scale t to an interval I on which S_t (if nonempty) lies in a fixed ball of radius r, say.

When $X = \mathbb{R}^n$ with Euclidean topology, the following results are well known (cf. [2]) for all $t \in I$:

$$S_t \text{ is nonempty.} \tag{1.2}$$

$$S_t \text{ is compact.} \tag{1.3}$$

$$S_t \text{ is connected.} \tag{1.4}$$

$$\begin{gathered}\textit{Given } \mathbf{b} \in \partial S_t, \textit{ there is a solution } \mathbf{x} \textit{ satisfying} \\ \mathbf{x}(t) = \mathbf{b}, \qquad \mathbf{x}(s) \in \partial S_s \qquad 0 \leqslant s \leqslant t.\end{gathered} \tag{1.5}$$

Let c_0 denote the Banach space of real valued sequences $\mathbf{y} = (y_1, y_2, \ldots)$ satisfying

$$\lim_{n\to\infty} y_n = 0, \qquad \|\mathbf{y}\|_\infty = \sup\{|y_n| : n > 0\},$$

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while l_2 is the corresponding Hilbert space for which

$$\| \mathbf{y} \|_2^2 = \sum_{n=1}^{\infty} | y_n |^2 < \infty.$$

Dieudonné [1] gave an example failing (1.2) in c_0 and since then various authors have considered (1.2) in more general spaces. Recently Godunov [5] has shown that (1.2) fails in any Banach space X, if (as is assumed from now on) $\dim X = \infty$. Godunov's example is based on Day's result [2], that X has a closed subspace Y with a basis, and on an earlier example [4] failing (1.2) in l_2. See [5] for further references to work along these lines: in particular Godunov's work has much in common with Yorke's earlier example [7] in l_2.

The main object here is to negate (1.3), (1.4), and (1.5) in turn for any X. Dieudonné's example [1] has no solutions at all in c_0, regardless of $\mathbf{x}(0)$, as pointed out by Yorke [7]. Section 3 contains examples with exactly one and exactly two solutions in c_0, and these are then used to counter (1.5) and (1.4), respectively. The examples for general X in Section 4 on (1.4) and Section 5 on (1.5) roughly combine the S_t behavior of those in Section 3 with Godunov's example [5] cited above.

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Notation. The following functions will be needed:

$$\mathbf{g}(\mathbf{0}) = \mathbf{0}, \qquad \mathbf{g}(\mathbf{x}) = 2\mathbf{x} \| \mathbf{x} \|^{-1/2} \qquad \text{if} \quad \mathbf{x} \neq \mathbf{0} \tag{1.6}$$

defines $\mathbf{g}\colon X \to X$, and when $x \in \mathbb{R}$ this function is denoted by g. Further, two continuous piecewise linear functions h and $k_{\alpha\beta} : \mathbb{R} \to \mathbb{R}$ are defined by

$$h(-\infty) = h(0) = 0, \qquad 1 = h(1) = h(+\infty), \tag{1.7}$$

$$k_{\alpha\beta}(\pm\infty) = k_{\alpha\beta}(\alpha) = k_{\alpha\beta}(\beta) = 0, \qquad k_{\alpha\beta}[(\alpha + \beta)/2] = 2. \tag{1.8}$$

Throughout, k_{01} will simply be written as k; observe that $\int_0^1 k = 1$.

2. Compactness

Although this example is elementary, it will be needed later. Let

$$\mathbf{f}(\mathbf{x}, t) = \mathbf{g}(\mathbf{x})\, k(t), \qquad \mathbf{x}(0) = \mathbf{0}. \tag{2.1}$$

Obviously

$$\begin{aligned} \mathbf{x}(t) &= \mathbf{0} \qquad \text{for} \quad 0 \leqslant t \leqslant \alpha, \\ &= \left(\int_\alpha^t k \right)^2 \mathbf{e} \qquad \text{for} \quad t \geqslant \alpha \end{aligned} \tag{2.2}$$

satisfies (1.1) for any $\alpha \geqslant 0$ and unit $\mathbf{e} \in X$.

It is easy to show that these are the only solutions, so S_1 is the unit ball of X, hence is not compact. It is in fact possible to modify Dieudonné's second example in [2] (on continuation) to give S_1 noncompact yet with empty interior, but this will not be pursued here.

3. Examples in c_0

Let $\mathbf{e}_1 = (1, 0, 0, \ldots)$, $P_1\mathbf{x} = \mathbf{x} - x_1\mathbf{e}_1$ and consider the three equations

$$ {}_1f_n(\mathbf{x}, t) = |g(x_n)| + n^{-1} \qquad n \geqslant 1, \tag{3.1}$$

$$ {}_2f_n(\mathbf{x}, t) = |g(x_n)| + \|\mathbf{x}\|\, n^{-1}, \qquad n \geqslant 1, \tag{3.2}$$

$$ {}_3f_1(\mathbf{x}, t) = \| P_1\mathbf{x} \|, \qquad {}_3f_n(\mathbf{x}, t) = |g(x_n)| + |x_1{}^2 - 1|\, n^{-1}, \qquad n \geqslant 2. \tag{3.3}$$

Evidently if (1.1) is defined by (3.1) then $|x_n(t) - x_n(0)|$ exceeds the maximal value without the n^{-1} term, and from 2 (in $\mathbb{R}$ instead of X), this is t^2 so does not tend to zero as $n \to \infty$. Thus $\mathbf{x} \notin c_0$, i.e., no solutions exist, a fact observed by Dieudonné [1] for $\mathbf{x}(0) = \mathbf{0}$ and Yorke [7] for general $\mathbf{x}(0)$.

Likewise (3.2) gives exactly one solution, viz., $\mathbf{x} \equiv \mathbf{0}$. For if not then $\|\mathbf{x}(t)\| > 0$ on an interval $[\sigma, \tau]$ of positive length so $|x_n(\tau) - x_n(\sigma)| > (\tau - \sigma)^2$ which is again positive and independent of n. Similarly (3.3) gives precisely two solutions $\mathbf{x} \equiv \pm\mathbf{e}_1$.

Statement (1.2) is countered by (3.1) and (1.3) by (2.1). To negate (1.5), take (2.1) again on [0, 1] and

$$ \mathbf{f}(\mathbf{x}, t) = {}_2\mathbf{f}(\mathbf{x}, t)\, k(t - 1) \tag{3.4}$$

on [1, 2]. Evidently $\mathbf{f}$ is continuous, $S_2 = \{\mathbf{0}\}$ so $\mathbf{0} \in \partial S_2$ but $\mathbf{0} \notin \partial S_1$.

Statement (1.4) fails by taking $\mathbf{x}(0) = \mathbf{0}$ and

$$ f_1(\mathbf{x}, t) = g(x_1)\, k(t), \qquad f_n(\mathbf{x}, t) = 0, \qquad n \geqslant 2 \tag{3.5}$$

over [0, 1] to give S_1 as the line segment joining $\pm\mathbf{e}_1$. Then continue over [1, 2] with

$$ \mathbf{f}(\mathbf{x}, t) = {}_3\mathbf{f}(\mathbf{x}, t)\, k(t - 1) \tag{3.6}$$

to give $S_2 = \{-\mathbf{e}_1, \mathbf{e}_1\}$. In the next section this S_t behavior will be reproduced, although it will be convenient to replace (3.6) by

$$ \mathbf{f}(\mathbf{x}, t) = P_1({}_1\mathbf{f}[P_1\mathbf{x}, t])\, |x_1{}^2 - 1|\, k(t - 1), \tag{3.7}$$

which behaves in the same way on the subspace $P_1^{-1}(\mathbf{0})$.

4. Kneser's Property

First we collect some results and notation to be used. A construction of Day [2] gives a closed subspace $Y \subseteq X$ with a basis $\mathbf{e}_1, \mathbf{e}_2, \ldots$. For each $\mathbf{y} \in Y$, which will play the role of c_0 in 3, let

$$P_n\mathbf{y} = \sum_{m=n+1}^{\infty} y_m \mathbf{e}_m, \tag{4.1}$$

where y_m are the coordinates of $\mathbf{y}$, so P_0 is the identity on Y.

In what follows, it will sometimes be convenient to define $\mathbf{f}(\mathbf{y}, t) \in Y$ and extend $\mathbf{f}$ to $(\mathbf{x}, t)$ using Dugundji's construction [3]. This preserves continuity and Y-valuedness of $\mathbf{f}$, so solutions to (1.1) once in Y remain there, as observed by Godunov [5].

As in (3.5) let $\mathbf{x}(0) = \mathbf{0}$ and

$$\mathbf{f}(\mathbf{y}, t) = \mathbf{g}(y_1\mathbf{e}_1)\, k(t) \tag{4.2}$$

on $[0, 1]$. The analog of (3.7) will be a slight modification of Godunov's equation. Let $\theta: \mathbb{R} \to \mathbb{R}_+$ be continuous and

$$\mathbf{q}(\mathbf{y}, t, \theta) = \mathbf{g}(P(t)\mathbf{y}) + \sum_{n=1}^{\infty} \varphi_n(\mathbf{y}, t, \theta)\, \mathbf{e}_n, \qquad \mathbf{y} \in Y, \qquad 0 \leqslant t \leqslant 1, \tag{4.3}$$

where the projector $P(t)$ on Y and the $\mathbb{R}$-valued functions φ_n are defined as follows. Take three sequences

$$a_n = 1/(2n+1), \qquad c_n = \tfrac{1}{2}(a_n + b_n), \qquad b_n = 1/2n$$

and define P countably piecewise linearly by

$$P(0) = 0, \qquad P(b_{n+1}) = P(c_n) = P_n, \qquad P(1) = P_0.$$

Referring to (1.7, 8), let

$$\varphi_n(\mathbf{y}, t, \theta) = n^{-1} k_{a_n c_n}(t)\, h(w[t - b_{n+1}] - \| P_n\mathbf{y} \|), \qquad w(s) = \left[\int_0^s \theta\right]^2.$$

Godunov in fact takes $\mathbf{x}(0) = \mathbf{0}$, $\theta(s) = s$ in (4.3) and $\mathbf{f}(\mathbf{y}, t) = \frac{1}{2}\mathbf{q}(\mathbf{y}, t, \theta)$ in (1.1). He concludes that $\mathbf{x}(b_n) = \mathbf{0}$ from some n onwards is impossible, the argument going through provided $\theta(s) > 0$ if $s > 0$. The alternative case, that $\mathbf{x}(b_n) \neq \mathbf{0}$ for arbitrarily large n, is contradicted provided w is not less than the maximum solution of

$$\dot{\xi} = \tfrac{1}{2} g(\xi), \qquad \xi(0) = 0; \tag{4.4}$$

i.e., $w(s) \geqslant s^2/4$. (The reasoning involves showing that (1.1) reduces to $\dot{\mathbf{x}} = \frac{1}{2}\mathbf{g}(\mathbf{x})$, a device typical of "nonexistence" examples.)

In order to continue (4.2) over [1, 2], set

$$\mathbf{f}(\mathbf{y}, t) = P_1\mathbf{q}(P_1\mathbf{y}, t - 1, | y_1^2 - 1 | k) | y_1^2 - 1 | k(t - 1),$$

cf. (3.7). Nonexistence of solutions for $y_1 \not\equiv \pm 1$ follows essentially as in [5], (4.4) showing that $\theta = | y_1^2 - 1 | k$ is a suitable choice.

5. Hukuhara's Property

This example is based on a similar idea to that in c_0 (3.4), to generate S_1 as the unit ball and then kill enough nonzero solutions to give $\mathbf{0} \in \partial S_2$. Since the nonexistence example (4.3) does not have the extreme non-Lipschitz behavior of (3.2) in c_0, some nonzero solutions will remain in S_2 this time, although it will be shown how to remove these too in case $Y = X$.

Over [0, 1], define $\mathbf{f}$ by (2.1), so S_1 is the unit ball and $\mathbf{0} \notin \partial S_1$ is reached only via the zero solution. Now using (4.2) suppose that we chose

$$\mathbf{f}(\mathbf{y}, t) = P_1\mathbf{q}(P_1\mathbf{y}, t - 1, | y_1 | k) | y_1 | k(t - 1). \tag{5.1}$$

Evidently $y_1(1) = 0$ would give constant $\mathbf{y}$. Further $y_1(1) \neq 0$ but $P_1\mathbf{y}(1) = \mathbf{0}$ would give no solution, following similar reasoning to that in 4, since y_1 would remain a nonzero constant.

In fact we reverse (5.1), replacing $\mathbf{f}(\mathbf{y}, t)$ by $\mathbf{f}(\mathbf{y}, 3 - t)$, i.e.,

$$\mathbf{f}(\mathbf{y}, t) = -P_1\mathbf{q}(P_1\mathbf{y}, 2 - t, | y_1 | k) | y_1 | k(t - 1) \tag{5.2}$$

over [1, 2]. The key conclusion is that S_2 contains no points with $y_1(2) \neq 0$ but $P_1\mathbf{y}(2) = \mathbf{0}$, or else we could again reverse the flow to obtain solutions emanating from such points, and this was ruled out above. Further $\mathbf{0} \in S_2$ and is reached only via $\mathbf{x} \equiv \mathbf{0}$. Thus $\mathbf{0} = \lim_{n\to\infty} n^{-1}\mathbf{e}_1 \in \partial S_2$ and this negates (1.5).

In case $Y = X$ (for example $X = l_2$) we can kill the remaining nonzero solutions as follows. Note that y_1 is bounded (by 2 in Day's construction [2]) if $\| \mathbf{y} \| \leqslant 1$, so since y_1 is constant over [1, 2] we have $| y_1(2)| \leqslant \alpha$ say. Choose

$$\mathbf{f}(\mathbf{y}, t) = -\alpha^{1/2}\mathbf{g}(y_1\mathbf{e}_1)\, k(t - 2)$$

to give $y_1(3) = 0$.

So far then, $P_1S_3 = S_3$ and $\mathbf{0}$ is reached only via $\mathbf{y} \equiv \mathbf{0}$. With $\eta = \| P_1\mathbf{y} \|$, observe that

$$\eta(1) \leqslant \| \mathbf{y}(1)\| + | y_1(1)| \leqslant 1 + \alpha,$$

and over [1, 2] that

$$\dot{\eta}(t) \leqslant \| \dot{\mathbf{y}}(t)\| \leqslant \alpha(\eta(t)^{1/2} + 1)\, k(t - 1) \leqslant \alpha(\eta(t) + 1)\, k(t - 1).$$

Thus

$$\eta(3) = \eta(2) \leqslant \beta = (1 + \alpha)\, e^{\alpha} - 1$$

so if we choose

$$\mathbf{f}(\mathbf{y}, t) = [-\beta^{1/2}\mathbf{g}(P_1\mathbf{y}) + \eta\mathbf{e}_1]\, k(t - 3)$$

over [3, 4] we obtain $\eta(4) = 0$, i.e., $P_1\mathbf{y}(4) = \mathbf{0}$. Note again that $\mathbf{0}$ is reached only via $\mathbf{y} \equiv \mathbf{0}$.

It remains to repeat (5.1) without reversal, explicitly

$$\mathbf{f}(\mathbf{y}, t) = P_1\mathbf{q}(P_1\mathbf{y}, t - 4, | y_1 | k) | y_1 | k(t - 4)$$

over [4, 5], leaving just the zero solution.

References

1. J. Dieudonné, Deux exemples singuliers d'équations différentielles, *Acta Sci. Math. (Szeged) B* 12 (1950), 38–40.
2. M. Day, On the basis problem in normed spaces, *Proc. Amer. Math. Soc.* 13 (1962), 655–658.
3. J. Dugundji, An extension of Tietze's theorem, *Pacific J. Math.* 1 (1951), 353–367.
4. A. Godunov, A counterexample to Peano's theorem in infinite dimensional Hilbert space, *Moscow Univ. Math. Bull.* 27 (1972), 24–26.
5. A. Godunov, On Peano's theorem in Banach spaces (in Russian), *Funktsional. Anal. i Priložen.* 9 (1975), 59–60.
6. P. Hartman, "Ordinary Differential Equations," Wiley, New York, 1964.
7. J. Yorke, A continuous differential equation in Hilbert space without existence, *Funkcial. Ekvac.* 13 (1970), 19–21.