Some determinantal inequalities for Hadamard product of matrices

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Abstract

The main result of this paper is the following: if both $A = (a_{ij})$ and $B = (b_{ij})$ are $M$-matrices or positive definite real symmetric matrices of order $n$, the Hadamard product of $A$ and $B$ is denoted by $A \circ B$, and $A_k$ and $B_k$ ($k = 1, 2, \ldots, n$) are the $k \times k$ leading principal submatrices of $A$ and $B$, respectively, then

$$\det(A \circ B) \geq \det(AB)^n \prod_{k=2}^{n} \left( \frac{a_{kk} \det A_{k-1}}{\det A_k} + \frac{b_{kk} \det B_{k-1}}{\det B_k} - 1 \right).$$

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1. Introduction

Let $N = \{1, 2, \ldots, n\}$, and $C^{m \times n}$ ($R^{m \times n}$) denote the set of all $m \times n$ complex (real) matrices. Let $Z_{n \times n} = \{A = (a_{ij}) \in R^{n \times n} : a_{ij} \leq 0, \ i \neq j, \ i, j \in N\}$. For any $A = (a_{ij}) \in C^{n \times n}$, its $k \times k$ leading principal submatrix is denoted by $A_k$ ($k \in N$), and its comparison matrix is defined by $u(A) = (u_{ij})$, where $u_{ii} = |a_{ii}|$, $u_{ij} = -|a_{ij}|$ ($i \neq j, \ i, j \in N$). Let $S_n^+$ denote the set of $n \times n$ positive definite real symmetric
matrices. For both $A = (a_{ij})$ and $B = (b_{ij})$ belong to $C^{m \times n}$, the Hadamard product of $A$ and $B$ is defined by $A \circ B = (a_{ij}b_{ij}) \in C^{m \times n}$.

A matrix $A \in R^{m \times n}$ is called a $P$-matrix if all $k \times k$ principal minors of $A$ are positive for any $k \in N$; and $A$ is called an $M$-matrix if $A \in Z^{m \times n}$ and $\det A_k > 0$ ($\forall k \in N$), and denote it by $A \in M_n$. A matrix $A \in C^{m \times n}$ is called an $H$-matrix if $u(A)$ is an $M$-matrix, and denote it by $A \in H_n$.

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For both $A = (a_{ij})$ and $B = (b_{ij})$ belong to $R^{n \times n}$, we write $A \geq B$ if $a_{ij} \geq b_{ij}$ for all $i, j \in N$.

It is well known that the class of $H$-matrices is closed under positive diagonal multiplication, and every principal submatrix of an $H$-matrix is also an $H$-matrix. The class of $M$-matrices has the same properties.

On the estimations of bounds for determinant of Hadamard product of matrices, we have the following results.

1. Oppenheim’s inequality [1, p. 480]: if $A = (a_{ij}) \in S_n^+$ and $B = (b_{ij}) \in S_n^+$, then

$$\det(A \circ B) \geq \left( \prod_{i=1}^{n} a_{ii} \right) \det B.$$  (1)

2. Lynn [2] and Ando [3] have given the following result: if $A = (a_{ij}) \in M_n$ and $B = (b_{ij}) \in M_n$, then

$$\det(A \circ B) + \det A \cdot \det B \geq (\det A) \prod_{i=1}^{n} b_{ii} + (\det B) \prod_{i=1}^{n} a_{ii},$$

i.e.

$$\det(A \circ B) \geq \det(AB) \left( \prod_{i=1}^{n} b_{ii} \det B - \prod_{i=1}^{n} a_{ii} \det A + 1 \right).$$  (2)

3. Liu and Zhu [4] have improved Oppenheim’s inequality as follows: if $A = (a_{ij}) \in M_n$, $B = (b_{ij}) \in M_n \cup S_n^+$, then

$$\det(A \circ B) \geq a_{11}b_{11} \prod_{k=2}^{n} \left[ \frac{b_{kk} \det A_k}{\det A_{k-1}} + \frac{\det B_k}{\det B_{k-1}} \left( \sum_{i=1}^{k-1} \frac{a_{ik}a_{ki}}{a_{ii}} \right) \right].$$  (3)

4. Li and Li [5] have obtained the following result: if both $A = (a_{ij})$ and $B = (b_{ij})$ belong to $H_n \cap R^{n \times n}$, then

$$\det(A \circ B) \geq |a_{11}|b_{11} \prod_{k=2}^{n} \left[ \frac{|b_{kk}| \det u(A_k)}{\det u(A_{k-1})} + \frac{\det u(B_k)}{\det u(B_{k-1})} \left( \sum_{i=1}^{k-1} \frac{|a_{ik}a_{ki}|}{|a_{ii}|} \right) \right].$$  (4)

In this paper, we shall obtain some inequalities for the determinants of Hadamard products which are better than all the above inequalities.
2. Main results

In this section, for both 
\[ A = \begin{pmatrix} A_{n-1} & A_{12} \\ A_{21} & a_{nn} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} B_{n-1} & B_{12} \\ B_{21} & b_{nn} \end{pmatrix} \]
belong to \( C^{n \times n} \), we define
\[ A(x) = \begin{pmatrix} A_{n-1} & A_{12} \\ A_{21} & x \end{pmatrix} \quad \text{and} \quad B(x) = \begin{pmatrix} B_{n-1} & B_{12} \\ B_{21} & x \end{pmatrix}, \]
then \( (A \circ B)(xy) = A(x) \circ B(y) \).

Lemma 2.1
(a) Let \( A = (a_{ij}) \in R^{n \times n} \), if \( \det A_{n-1} > 0 \), and \( x \) is a real number, then \( \det A(x) > 0 \) if and only if \( x > a_{nn} - \det A / \det A_{n-1} \).
(b) If \( A = (a_{ij}) \in H_n \), then \( A(x) \in H_n \) if and only if \( |x| > |a_{nn}| - \det u(A) / \det u(A_{n-1}) \).

Proof
(a) From \( \det A_{n-1} > 0 \), and
\[ \det A(x) = \det \left( \begin{pmatrix} A_{n-1} & A_{12} \\ A_{21} & x \end{pmatrix} a_{nn} \right) = \det A + (x - a_{nn}) \det A_{n-1}, \]
we conclude that \( \det A(x) > 0 \) if and only if \( x > a_{nn} - \det A / \det A_{n-1} \).
(b) Since \( A \in H_n \), then both \( u(A) \) and \( u(A_{n-1}) \) are \( M \)-matrices. By case (a), we have
\[ A(x) \in H_n \iff u(A(x)) \in M_n \iff \det u(A(x)) > 0 \iff |x| > |a_{nn}| - \det u(A) / \det u(A_{n-1}). \]

Lemma 2.2 [6]. If \( A \in H_n \), \( B \in H_n \cup S^+_n \), then \( A \circ B \in H_n \).

Lemma 2.3 [1, p. 458]. If \( A \in S^+_n \), \( B \in S^+_n \), then \( A \circ B \in S^+_n \).

Lemma 2.4. If \( A = (a_{ij}) \in R^{n \times n} \cap H_n \), \( a_{ii} > 0 \ (\forall i \in N) \), then
\[ \det A \geq \det u(A) > 0 \]

Proof. It is well known that if \( A \in H_n \), then there exists a positive diagonal matrix \( D \) such that \( AD \) is strictly diagonally dominant, that is
\[ |a_{ii}d_i| > \sum_{j \neq i} |a_{ij}d_j| \quad (i = 1, 2, \ldots, n). \]
Now \( a_i d_i > 0 (\forall i \in N) \), by [4, Lemma 1.4], we have \( \det(AD) \geq \det u(AD) > 0 \). From \( \det(AD) = \det A \cdot \det D \), \( \det u(AD) = \det u(AD) \cdot \det D \), we have \( \det A \geq \det u(A) > 0 \). □

**Lemma 2.5.** If both \( A \) and \( B \) belong to \( M_n \cup S_n^+ \), then \( A \circ B \) is a \( P \)-matrix.

**Proof.** We distinguish the following two cases:

**Case 1.** By Lemma 2.3, if \( A \in S_n^+ \) and \( B \in S_n^+ \), then \( A \circ B \in S_n^+ \). Therefore, \( A \circ B \) is also a \( P \)-matrix.

**Case 2.** If one of \( A \) and \( B \) belongs to \( M_n \), then, by Lemma 2.2, we have \( A \circ B \in H_n \).

Now \( A \circ B \) is a real \( H \)-matrix with positive diagonal entries, and so is every principal submatrix of \( A \circ B \). Apply Lemma 2.4 to all principal submatrices of \( A \circ B \), we conclude that \( A \circ B \) is a \( P \)-matrix. □

**Lemma 2.6**

(a) If both \( A = (a_{ij}) \) and \( B = (b_{ij}) \) belong to \( M_n \cup S_n^+ \), \( x > a_{nn} - \det A / \det A_{n-1} \), \( y > b_{nn} - \det B / \det B_{n-1} \), then

\[
xy > a_{nn} b_{nn} - \det(A \circ B) / \det(A_{n-1} \circ B_{n-1}).
\]

(b) If \( A = (a_{ij}) \in H_n \) and \( B = (b_{ij}) \in H_n \), \( |x| > |a_{nn}| - \det u(A) / \det u(A_{n-1}) \), \( |y| > |b_{nn}| - \det u(B) / \det u(B_{n-1}) \), then

\[
|xy| > |a_{nn} b_{nn}| - \det u(A \circ B) / \det u(A_{n-1} \circ B_{n-1}).
\]

**Proof**

(a) By Lemma 2.1 and our assumption, we have both \( A(x) \) and \( B(y) \) belong to \( M_n \cup S_n^+ \). By Lemma 2.5, both \( A \circ B \) and \( A(x) \circ B(y) \) are \( P \)-matrices. Again by Lemma 2.1, we have

\[
xy > a_{nn} b_{nn} - \det(A \circ B) / \det(A_{n-1} \circ B_{n-1}).
\]

(b) Our assumption and Lemma 2.1 imply that \( A(x) \in H_n \) and \( B(y) \in H_n \). By Lemma 2.2, both \( A \circ B \) and \( A(x) \circ B(y) \) are \( H \)-matrices. According to Lemma 2.1, we have

\[
|xy| > |a_{nn} b_{nn}| - \det u(A \circ B) / \det u(A_{n-1} \circ B_{n-1}). \]

Now we state the main result of this paper as follows.

**Theorem 2.7**

(a) If both \( A = (a_{ij}) \) and \( B = (b_{ij}) \) belong to \( M_n \cup S_n^+ \), then

\[
\det(A \circ B) \geq \det(AB) \prod_{k=2}^{n} \left( \frac{a_{kk} \det A_{k-1}}{\det A_k} + \frac{b_{kk} \det B_{k-1}}{\det B_k} - 1 \right).
\]
(b) If \( A = (a_{ij}) \in H_n \) and \( B = (b_{ij}) \in H_n \), then

\[
\det u(A \circ B) \geq \det(u(A)u(B)) \prod_{k=2}^{n} \left( |a_{kk}| \frac{\det u(A_{k-1})}{\det u(A_k)} + |b_{kk}| \frac{\det u(B_{k-1})}{\det u(B_k)} - 1 \right). \tag{6}
\]

**Proof**

(a) Obviously, both \( A_k \) and \( B_k \) belong to \( M_k \cup S_k^+ \) \((\forall k \in N')\). By Lemma 2.6, \( \forall \varepsilon > 0 \), we have

\[
|a_{kk} - \frac{\det A_k}{\det A_{k-1}} + \varepsilon| |b_{kk} - \frac{\det B_k}{\det B_{k-1}} + \varepsilon| > a_{kk}b_{kk} - \frac{\det(A_k \circ B_k)}{\det(A_{k-1} \circ B_{k-1})}.
\]

Letting \( \varepsilon \to 0 \), we obtain

\[
\left( a_{kk} - \frac{\det A_k}{\det A_{k-1}} \right) \left( b_{kk} - \frac{\det B_k}{\det B_{k-1}} \right) \geq a_{kk}b_{kk} - \frac{\det(A_k \circ B_k)}{\det(A_{k-1} \circ B_{k-1})}.
\]

From this, we can get that

\[
\frac{\det(A_k \circ B_k)}{\det(A_{k-1} \circ B_{k-1})} \geq \frac{\det A_k \cdot \det B_k}{\det A_{k-1} \det B_{k-1}} \left( \frac{a_{kk} \det A_{k-1}}{\det A_k} + \frac{b_{kk} \det B_{k-1}}{\det B_k} - 1 \right),
\]

\[
\prod_{k=2}^{n} \frac{\det(A_k \circ B_k)}{\det(A_{k-1} \circ B_{k-1})} \geq \prod_{k=2}^{n} \frac{\det A_k \cdot \det B_k}{\det A_{k-1} \det B_{k-1}} \left( \frac{a_{kk} \det A_{k-1}}{\det A_k} + \frac{b_{kk} \det B_{k-1}}{\det B_k} - 1 \right).
\]

Finally, we have

\[
\det(A \circ B) \geq \det(AB) \prod_{k=2}^{n} \left( \frac{a_{kk} \det A_{k-1}}{\det A_k} + \frac{b_{kk} \det B_{k-1}}{\det B_k} - 1 \right).
\]

(b) Because \( A_k \in H_k \) and \( B_k \in H_k \) \((2 \leq k \leq n)\), \( \forall \varepsilon > 0 \), by Lemma 2.6, we have

\[
|a_{kk} - \frac{\det(A_k)}{\det(A_{k-1})} + \varepsilon| |b_{kk} - \frac{\det(B_k)}{\det(B_{k-1})} + \varepsilon| > |a_{kk}b_{kk}| - \frac{\det(A_k \circ B_k)}{\det(A_{k-1} \circ B_{k-1})}.
\]

Now (6) can be proved in a similar manner as in the proof of (5). \( \Box \)
Corollary 2.8. If both $A = (a_{ij})$ and $B = (b_{ij})$ belong to $R^{n \times n} \cap H_n$, $\prod_{i=1}^{n} a_{ii} b_{ii} > 0$, then
\[
\det(A \circ B) \geq \det(u(A)u(B)) \prod_{k=2}^{n} \frac{|a_{kk}| \det(A_{k-1})}{\det(A_k)} + \frac{|b_{kk}| \det(B_{k-1})}{\det(B_k)} - 1.
\] (7)

Proof. Consider the $n \times n$ diagonal matrix $D = \text{diag}(d_1, d_2, \ldots, d_n)$ defined by $d_i = \text{sgn}(a_{ii} b_{ii})$ ($\forall i \in N$), where $\text{sgn}(x) = 1$ if $x > 0$, and $-1$ if $x < 0$. Since $\prod_{i=1}^{n} a_{ii} b_{ii} > 0$, and $\prod_{i=1}^{n} |a_{ii} b_{ii}| = \prod_{i=1}^{n} a_{ii} b_{ii} = \left(\prod_{i=1}^{n} a_{ii} b_{ii}\right) \left(\prod_{i=1}^{n} d_i\right)$, we have $\det(D) = \prod_{i=1}^{n} d_i = 1$. Consequently, $\det(A \circ B) = \det([A \circ B]D) = \det(A \circ (BD))$. Now both $A$ and $B$ belong to $R^{n \times n} \cap H_n$, thus, by Lemma 2.2, $A \circ (BD) \in R^{n \times n} \cap H_n$, and $A \circ (BD)$ has positive diagonal entries: $|a_{11} b_{11}|, |a_{22} b_{22}|, \ldots, |a_{nn} b_{nn}|$. By Lemma 2.4, we have $\det([A \circ (BD)]) \geq \det(u(A \circ (BD))) = \det(u(A \circ B))$, and hence $\det(A \circ B) \geq \det(u(A \circ B))$. According to Theorem 2.7, (7) is valid. $\square$

Corollary 2.9. If both $A = (a_{ij})$ and $B = (b_{ij})$ belong to $M_n \cup S_n^+$, then
\[
\det(A \circ B) \geq \det(AB).
\]

Proof. According to the well-known Hadamard–Fischer inequalities [7, p. 179], we have $a_{kk} \det(A_{k-1})/\det(A_k) \geq 1$, and $b_{kk} \det(B_{k-1})/\det(B_k) \geq 1$ ($2 \leq k \leq n$). Therefore, $a_{kk} \det(A_{k-1})/\det(A_k) + b_{kk} \det(B_{k-1})/\det(B_k) - 1 \geq 1$. By Theorem 2.7, we obtain $\det(A \circ B) \geq \det(AB)$. $\square$

3. Relationship to previous results

To see that our results have improved and generalized the corresponding results in [2–5], we need some preliminaries.

Proposition 3.1. If both $A = (a_{ij})$ and $B = (b_{ij})$ belong to $M_n \cup S_n^+ (n \geq 2)$, then
\[
\prod_{k=2}^{n} \left(\frac{a_{kk} \det(A_{k-1})}{\det(A_k)} + \frac{b_{kk} \det(B_{k-1})}{\det(B_k)} - 1\right) \geq \prod_{i=1}^{n} b_{ii} - \prod_{i=1}^{n} a_{ii} - 1.
\] (8)

Proof. We proceed by induction on $n$. It is easy to see that (8) is true even with the equality sign for $n = 2$.

Now assume that $n > 2$ and (8) is true for the case $n - 1$, then it follows from the induction hypothesis that
\[
\prod_{k=2}^{n} \left( \frac{a_{kk} \det A_{k-1}}{\det A_k} + \frac{b_{kk} \det B_{k-1}}{\det B_k} - 1 \right)
\geq \left( \frac{\prod_{i=1}^{n-1} b_{ii} \det B_{n-1}}{\det B_{n-1}} + \frac{\prod_{i=1}^{n-1} a_{ii} \det A_{n-1}}{\det A_{n-1}} - 1 \right) \left( \frac{a_{nn} \det A_{n-1}}{\det A} + \frac{b_{nn} \det B_{n-1}}{\det B} - 1 \right)
\]
\[
= \prod_{i=1}^{n} a_{ii} \det A + \prod_{i=1}^{n} b_{ii} \det B + \prod_{i=1}^{n-1} b_{ii} \left( \frac{a_{nn} \det A_{n-1}}{\det A} - 1 \right)
+ \prod_{i=1}^{n-1} a_{ii} \left( \frac{b_{nn} \det B_{n-1}}{\det B} - 1 \right)
- \left[ \left( \frac{a_{nn} \det A_{n-1}}{\det A} - 1 \right) + \left( \frac{b_{nn} \det B_{n-1}}{\det B} - 1 \right) + 1 \right]
\]
\[
= \prod_{i=1}^{n} a_{ii} \det A + \prod_{i=1}^{n} b_{ii} \det B + \left( \frac{a_{nn} \det A_{n-1}}{\det A} - 1 \right) \left( \prod_{i=1}^{n-1} a_{ii} \det A_{n-1} - 1 \right) - 1
+ \left( \frac{b_{nn} \det B_{n-1}}{\det B} - 1 \right) \left[ \prod_{i=1}^{n-1} a_{ii} \det A_{n-1} - 1 \right] - 1
\geq \prod_{i=1}^{n} a_{ii} \det A + \prod_{i=1}^{n} b_{ii} \det B - 1.
\]

This completes the induction. □

**Proposition 3.2.** If \( A = (a_{ij}) \in M_n \), then
\[
\det A_k \leq \frac{a_{kk} \det A_{k-1}}{a_{kk} \det A_{k-1}} - 1 - \sum_{i=1}^{k-1} \frac{a_{ik}a_{ki}}{a_{ii}a_{kk}} (2 \leq k \leq n).
\] (9)

**Proof.** Set \( \alpha = (a_{k1} \cdots a_{kk-1}) \), and \( \beta = (a_{1k} \cdots a_{k-1k})^T \), then \( \det A_k = \det A_{k-1} \cdot (a_{kk} - \alpha A_{k-1}^{-1} \beta) \). Since \( A_{k-1} \leq \text{diag}(a_{11}, \ldots, a_{k-1k-1}) \), by [4, Lemma 1.3], we have
\[
\alpha A_{k-1}^{-1} \beta \geq \alpha \text{diag} \left( \frac{1}{a_{11}}, \ldots, \frac{1}{a_{k-1k-1}} \right) \beta = a_{kk} \sum_{i=1}^{k-1} \frac{a_{ik}a_{ki}}{a_{ii}a_{kk}}
\]
and thus
\[
\det A_k \leq \det A_{k-1} \left[ a_{kk} - a_{kk} \left( \sum_{i=1}^{k-1} \frac{a_{ik}a_{ki}}{a_{ii}a_{kk}} \right) \right]
= a_{kk} \det A_{k-1} \left( 1 - \sum_{i=1}^{k-1} \frac{a_{ik}a_{ki}}{a_{ii}a_{kk}} \right).
\]

From \( a_{kk} \det A_{k-1} > 0 \), we claim that (9) is valid. □
Remark. The right-hand side of (3)

\[
= a_{11} b_{11} \prod_{k=2}^{n} \frac{\det A_k \det B_k}{\det A_{k-1} \det B_{k-1}} 
\times \left[ \frac{b_{kk} \det B_{k-1}}{\det B_k} + \frac{a_{kk} \det A_{k-1}}{\det A_k} \left( \sum_{i=1}^{k-1} a_{ik} a_{ki} \right) \right] 
= \det(AB) \prod_{k=2}^{n} \left[ \frac{b_{kk} \det B_{k-1}}{\det B_k} + \frac{a_{kk} \det A_{k-1}}{\det A_k} \left( \sum_{i=1}^{k-1} a_{ik} a_{ki} \right) \right].
\]

By (9), we have

\[
\left( \frac{a_{kk} \det A_{k-1}}{\det A_k} - 1 \right) - \frac{a_{kk} \det A_{k-1}}{\det A_k} \left( \sum_{i=1}^{k-1} a_{ik} a_{ki} \right) 
= \frac{a_{kk} \det A_{k-1}}{\det A_k} \left( 1 - \sum_{i=1}^{k-1} \frac{a_{ik} a_{ki}}{a_{ii} a_{kk}} \right) - 1 \geq 0
\]

and hence,

\[
\frac{a_{kk} \det A_{k-1}}{\det A_k} + \frac{b_{kk} \det B_{k-1}}{\det B_k} - 1 
\geq \frac{b_{kk} \det B_{k-1}}{\det B_k} + \frac{a_{kk} \det A_{k-1}}{\det A_k} \left( \sum_{i=1}^{k-1} \frac{a_{ik} a_{ki}}{a_{ii} a_{kk}} \right).
\]

From this, we can easily deduce that the right-hand side of (5) \geq the right-hand side of (3). Similarly, the right-hand side of (7) \geq the right-hand side of (4). Moreover, the Proposition 3.1 shows that the right-hand side of (5) \geq the right-hand side of (2). These complete our work.

References