Numerical range of composition operators on a Hilbert space of Dirichlet series

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Abstract

We study the numerical range of composition operators on a Hilbert space of Dirichlet series with square-summable coefficients. We first describe the numerical range of “nice” composition operators (as invertible, normal and isometric ones). We also focus on the zero-inclusion question for more general symbols.

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1. Introduction

We work with H, the Hilbert space of Dirichlet series with square-summable coefficients, equipped with the norm ∥f∥ = (∑∞ n=1 |an|2)1/2 if f(s) = ∑∞ n=1 anzn−s belongs to H. By the Cauchy–Schwarz inequality, the functions in H are all holomorphic on the half-plane C1/2 = {s ∈ C, Re s > 1/2}. Thus H appears as a Hilbert space of analytic functions on the half-plane C1/2, with reproducing kernel Kα (a ∈ C1/2), i.e. f(a) = ⟨f, Ka⟩ and Ka(s) = ζ(s + iπ), where ζ denotes the Riemann zeta-function (observe that ζ(s + iπ) > 1 for a, s ∈ C1/2) (cf. [13]). The space H can be viewed as a Dirichlet series analog of the Hardy space H2 of functions.
For the Hardy space, it follows from the Littlewood subordination principle [16] that any analytic self-map \( \phi : D \to D \) induces a bounded "composition operator" on \( H^2 \) by the formula: \( C_\phi(f) = f \circ \phi \). Such operators have been intensively studied during the two last decades [16]. In particular, their numerical range has been studied in [4,5,14]. A rather complete description of this numerical range when \( \phi \) is an automorphism of \( D \) (equivalently, when \( C_\phi \) is invertible) has been obtained.

We will proceed to a similar study for the composition operators on \( H \) (see [9] for an enumeration of the results). For the space \( H \), mainly due to the fact that not any analytic function in a half-plane can be represented as a Dirichlet series, the situation is different. Yet, Gordon and Hedenmalm [10] have obtained the following Dirichlet series analog of the classical Littlewood subordination principle, where we denote by \( C_\theta \) the half-plane \( C_\theta = \{ s \in \mathbb{C}, R s > \theta \} \).

**Theorem 1.** An analytic self-map \( \phi : \mathbb{C}_{1/2} \to \mathbb{C}_{1/2} \) induces a bounded composition operator \( C_\phi : f \mapsto f \circ \phi \) on \( H \) if and only if

1. \( \phi \) is "representable" i.e. \( \phi(s) = c_0 s + \varphi(s) \), where \( c_0 \) is a non-negative integer, and where the analytic function \( \varphi \) can be written as a convergent Dirichlet series \( \sum_{n=0}^{\infty} c_n n^{-s} \) for \( R s > \theta \) (in short \( \varphi \in \mathcal{D} \)).
2. \( \phi \) is "extendable" with "controlled range", namely \( \phi \) has an analytic extension to \( \mathbb{C}_0 \), still denoted by \( \phi \), and such that
   (a) \( \phi(\mathbb{C}_0) \subseteq \mathbb{C}_0 \) if \( c_0 \geq 1 \).
   (b) \( \phi(\mathbb{C}_0) \subseteq \mathbb{C}_{1/2} \) if \( c_0 = 0 \).

Let us mention that the cases \( c_0 \geq 1, c_0 = 0 \), are significantly different. It will be also convenient to extract from Theorem 4.2 of the Gordon–Hedenmalm paper [10] the following lemma.

**Lemma 2.** Let \( \phi(s) = c_0 s + \varphi(s) : \mathbb{C}_0 \to \mathbb{C}_0, \varphi(s) = \sum_{n=1}^{\infty} c_n n^{-s} \). Then

1. If \( \varphi(s) = c_1 \), we have \( R c_1 \geq 0 \).
2. If \( \varphi \) is not constant, we have \( R c_1 > 0 \).

**Proof.** Theorem 4.2 of [10] states, more generally, that if \( \phi : \mathbb{C}_\theta \to \mathbb{C}_v \), then \( \phi : \mathbb{C}_\theta \to \mathbb{C}_{v-c_0 \theta} \) if \( \phi \) is constant, and \( \phi : \mathbb{C}_\theta \to \mathbb{C}_{v-c_0 \theta} \) if \( \phi \) is not constant. Applying this with \( \theta = v = 0 \) gives:

1. If \( \varphi(s) = c_1 \), then \( c_1 \in \overline{\mathbb{C}_0} \), i.e. \( R c_1 \geq 0 \).
2. If \( \varphi \) is not constant, then \( R \varphi(s) > 0 \) if \( R s > 0 \). Let \( c_q (q \geq 2) \) be the first non-zero coefficient of \( \varphi \); adjust \( t \in \mathbb{R} \) so that \( c_q q^{-it} = -|c_q| \), and take \( s = \sigma + it \), with \( \sigma > 0 \) so large that \( \sum_{n>q} |c_n| n^{-\sigma} \leq \frac{1}{2} |c_q| q^{-\sigma} \). Then we have
Let us recall the following results of Bayart [2].

**Theorem 3.** For a bounded composition operator $C_\phi : \mathcal{H} \to \mathcal{H}$, the following are equivalent:

1. $C_\phi$ is invertible,
2. $C_\phi$ is Fredholm,
3. $\phi(s) = s + ik$, where $k$ is a real number.

**Theorem 4.** Let $C_\phi : \mathcal{H} \to \mathcal{H}$ be a bounded composition operator. Then:

1. $C_\phi$ is normal if and only if $\phi(s) = s + c_1$, where $\Re c_1 \geq 0$.
2. $\phi(s) = c_0 s + \phi(s)$, and if the Dirichlet series of $\phi$ converges uniformly for $Re s \geq 0$, $C_\phi$ is isometric if and only if $\phi(s) = c_0 s + ik$, where $c_0 \geq 1$ and $k \in \mathbb{R}$.

These theorems show that in some sense the situation is poorer in the case of $\mathcal{H}$ than in the case of $H^2$. On the other hand, as we shall see, new phenomena appear, like weighted shifts; and some interesting (open) questions emerge.

2. **Main results**

Let us first recall some general facts; if $H$ is a complex (separable) Hilbert space and $T : H \to H$ is a bounded operator, the numerical range of $T$ is the set: $W(T) = \{ \langle Tf, f \rangle; \| f \| = 1 \}$.

The numerical range is known to have the following general properties:

(a) it contains every eigenvalue of $T$ (obvious),
(b) it lies in the disk $|w| \leq \| T \|$ (obvious),
(c) its closure contains the spectrum of $T$ (easy),
(d) it is convex (Toeplitz–Hausdorff theorem), therefore Lebesgue measurable,
(e) it is even a Borel set [1],
(f) for compact $T$, it is closed if and only if it contains 0 [6].
This set is often difficult to describe, but it encodes much information on $T$, and (to quote Bourdon and Shapiro) “plays a role in spectral location similar to that of the Gershgorin sets in matrix theory”. We now start the study of the numerical range for composition operators on the space $H$. We shall always assume that $\phi$ is not the identity map of $C$ range for composition operators on the space $H$ of the Gershgorin sets in matrix theory”. We now start the study of the numerical

Proposition 5. Suppose that $\phi$ is a constant $c_1 \in \mathbb{C}_{1/2}$(i.e. $c_0 = 0$). Then, $W$ is the closed elliptic disk with foci at 0 and 1 and major axis of length

$$\|K_{c_1}\| = \left(\zeta(2)\|c_1\|\right)^{1/2}.$$

Proof. We have $C_\phi(f) = f(c_1) = (f, K_{c_1})\mathbf{1}$, i.e. $C_\phi$ is the rank-one operator $\mathbf{1} \otimes K_{c_1}$; since the two-dimensional subspace $V$ generated by $\mathbf{1}$ and $K_{c_1}$ is reducing for $C_\phi$, $W$ is the same as $W(A)$, where $A$ is the restriction of $C_\phi$ to $V$; and it is well-known [12] that $W(A)$ is the closed elliptic disk with foci at $\alpha$ and $\beta$, and major axis of length $\frac{\sqrt{1 - |\langle f, g \rangle|^2}}{\sqrt{1 - |\langle f, f \rangle|^2}} = c = \|K_{c_1}\| > 1$. We see that $|\langle f, g \rangle| = \sqrt{c^2 - 1}$ and that $\sqrt{1 - |\langle f, f \rangle|^2} = c = \|K_{c_1}\|$, and this gives the result. □

Proposition 6. Suppose that $\phi(s) = s + c_1$, with $c_1 \neq 0$. Then:

(a) If $c_1 > 0$, $W = [0, 1]$. If $\Re c_1 > 0$ and $c_1 \notin \mathbb{R}$, $W$ is a closed polygon containing the origin in its interior.

(b) If $\Re c_1 = 0$, $W = D \cup \{n^{-c_1}, n \geq 1\}$, where $D$ is the open unit disk.

Proof. (a) Clearly, $C_\phi$ is (on the canonical basis of $H$) the diagonal operator with eigenvalues $n^{-c_1}$, therefore $W = \left\{ \sum_{n=1}^{\infty} \lambda_n n^{-c_1}; \lambda_n \geq 0, \sum_{n=1}^{\infty} \lambda_n \leq 1 \right\}$.

If $c_1 > 0$, we have $W = [0, 1]$ by convexity. If $\Re c_1 > 0$ and $\Im c_1 \neq 0$, there are points $n^{-c_1}$ in each of the four quadrants, so that by convexity, 0 belongs to the interior of $W$; moreover, $W$ is the convex hull of the sequence $(n^{-c_1})$ that spirals monotonically towards zero, and it is closed (for example since $\phi$ is compact and since 0 $\in W$), therefore [14] it is a closed polygon.

(b) Write $c_1 = ik$, $k$ real and $\neq 0$, and set $E = \left\{ n^{-ik}, n \geq 1 \right\}$.

Since $|n+1|^{-ik} - n^{-ik}| \leq \frac{|k|}{n}$, $(n+1)^{-ik} - n^{-ik}$ tends to zero, and $E$ is dense in the unit circle, which implies:
\[ D \subset \text{co}E \subset W \] (1)

(where co denotes the convex hull).

Indeed, take \( re^{i\theta} \in D \), set \( \epsilon = 1 - r \) and take \( e^{i\omega}, e^{i\beta} \in E \) such that \( \alpha < \theta < \beta \) and \( \beta - \alpha \leq \epsilon \). Define \( \rho, 0 < \rho < 1 \), by \( \rho e^{i\theta} = \lambda e^{i\omega} + (1 - \lambda)e^{i\beta} \), i.e. \( \rho e^{i\theta} \) is the intersection point of the segments \([0, e^{i\omega}]\) and \([e^{i\omega}, e^{i\beta}]\). Then: \( 1 - \rho = \lambda \left( 1 - e^{i(\alpha - \theta)} \right) + (1 - \lambda)(1 - e^{i(\beta - \theta)}) \), so that \( 1 - \rho \leq \lambda |\alpha - \theta| + (1 - \lambda)|\beta - \theta| \leq (\lambda + 1 - \lambda)\epsilon = 1 - r \). This shows that \( \rho \geq r \). But, as in (a), we know that \( 0 \in \text{co}W \); by the convexity of \( W \), we see that \( \rho e^{i\theta} \in W \), since \( \rho e^{i\theta} \in [e^{i\omega}, e^{i\beta}] \), and that \( re^{i\theta} \in W \), since \( re^{i\theta} \in [0, e^{i\omega}] \); this proves (1). Since \( W \subset D \), it remains to show that if \( w = \sum_1^{\infty} \lambda_n n^{-i\theta} \in \partial D \), with \( \lambda_n \geq 0 \) and \( \Sigma \lambda_n = 1 \), then \( w \in E \). But if \( p^{-ik} \neq q^{-ik} \) and \( \lambda p, \lambda q \neq 0 \), we have \( |\lambda_p p^{-ik} + \lambda_q q^{-ik}| < \lambda_p + \lambda_q \), and \( |w| < 1 \). Therefore, if \( \lambda_n \neq 0 \), \( w = n^{-i\theta} \in E \). \( \square \)

Proposition 7. Suppose that \( \phi(s) = c_0 s + c_1 \), with \( c_0 \geq 2 \). Then:

(a) If \( c_1 = ik, k \in \mathbb{R} \), one has \( W = D \cup \{1\} \). And this remains true if \( \phi \) is any symbol such that \( \phi \) is a non-surjective isometry of \( \mathcal{H} \) into itself.

(b) If \( \Re c_1 > 0 \), one has \( W = \text{co}(D(0, r) \cup \{1\}) \), where \( r < 1 \) is given by the relation

\[
(\star \star) \quad r = \sup \left\{ \sum_{h=0}^{\infty} a_h a_{h+1} 2^{-\epsilon_h n}; a_h \geq 0, \sum_0^{\infty} a_h^2 = 1 \right\}, \quad \gamma_1 = \Re c_1.
\]

(c) In particular, we always have \( 0 \in \text{co}W \).

**Proof.** (a) Denote by \( S \) the unilateral shift on a separable Hilbert space, with an orthonormal basis \( e_1, e_2, \ldots \), i.e.: \( S(e_n) = e_{n+1} \) for all \( n \geq 1 \). Then, \( W(S) = \{ \sum_1^{\infty} x_n e_{n+1}; \sum_1^{\infty} |x_n|^2 = 1 \} \), and it is well-known that \( W(S) = D \), the open unit disk. More precisely, if \( S_n \) is the \( n \)-dimensional shift defined by \( S(e_1) = e_2, \ldots, S(e_{n-1}) = e_n, S(e_n) = 0 \), then \( W(S_n) = D \left( 0, \cos \frac{n}{n+1} \right) \), and \( W(S_n) \subset W(S) \) for each \( n \) \[12\]. We will show that

\[ W \supset W(S). \]

Indeed, define inductively a sequence \( (\lambda_n)_{n\geq1} \) of integers by: \( \lambda_1 = 2; \lambda_{n+1} = \lambda_n^2 \); take \( f(s) = \sum_1^{\infty} b_n s^{-\lambda_n} \), with \( \sum_1^{\infty} |b_n|^2 = 1 \).

Then, \( \|f\| = 1 \), and \( C_\phi f(s) = \sum_2^{\infty} b_n s^{-\lambda_n^{-1}} \lambda_n^{-1} = \sum_2^{\infty} b_n s^{-\lambda_n^{-1}} \lambda_n^{-1} \), so that \( (C_\phi f, f) = \sum_2^{\infty} b_n b_n^{-\lambda_n^{-1}} \lambda_n^{-1} = \sum_1^{\infty} b_n b_n^{-\lambda_n^{-1}} \lambda_n^{-1} \). Let now \( w \in W(S) \), i.e. \( w = \sum_1^{\infty} x_n e_{n+1}, \Sigma_1^{\infty} |x_n|^2 = 1 \). Define \( (b_n) \) by \( b_n = x_n e^{i\theta_n} \), where \( \theta_1 \in \mathbb{R} \) is arbitrary and \( \theta_n - \theta_{n+1} = k \log \lambda_n \). We get \( w = \sum_1^{\infty} b_n b_n^{-\lambda_n^{-1}} \lambda_n^{-1} \), which proves (2). Therefore (recall that \( I = (C_\phi I, I) \) always belongs to \( W \), \( W \) contains \( D \cup \{1\} \). Moreover, if \( f(s) = \sum_1^{\infty} a_n s^{-\lambda_n} \in \mathcal{H} \) and \( \|f\| = 1 \), then \( |(C_\phi f, f)| \leq |a_1|^2 + \sum_2^{\infty} |a_n|^2 \leq (1 - |a_1|^2)^{-1} \).
\[(\sum_{2}^{\infty} |a_{n}|^{2})^{1/2} (\sum_{2}^{\infty} |a_{n'}|^{2})^{1/2} \leq |a_{1}|^{2} + 1 - |a_{1}|^{2} = 1, \] the first inequality being strict unless \(|a_{n'}| = \alpha|a_{n}|\) for all \(n \geq 2\). But then, \(|a_{n'}| = \alpha^{k}|a_{n}|\) for all \(n \geq 2\).

If \(f(s) = a_{1}, \ (C_{\phi} f, f) = a_{1}a_{1}^{*} = 1\). Otherwise, \(a_{n} \neq 0\) for some \(n \geq 2\), and since \(a_{n_{0}} \to 0\) as \(k \to \infty\), we must have \(\alpha < 1\). But then, the second inequality is strict, i.e.:
\[|C_{\phi} f, f| = |a_{1}|^{2} + \alpha \sum_{n \geq 2} |a_{n}|^{2} < |a_{1}|^{2} + 1 - |a_{1}|^{2} = 1.\]
This proves the reverse inclusion \(W \subset D \cup \{1\}\). Suppose now that \(C_{\phi}\) is a non-surjective isometry of \(\mathcal{H}\).

Following a suggestion of Matache, we use the Wold decomposition of \(C_{\phi}\); recall that, for a non-surjective isometry \(T : H \to H\), one defines \(N_{\infty} = \bigcap_{i} T^{n}(H)\), the space of infinitely divisible vectors; \(N = (T H)\) \(\neq \{0\}\), the wandering subspace; and that one then has the orthogonal decompositions \(H = N_{\infty} \oplus N_{0}\), where \(N_{0} = \bigoplus_{i=0}^{\infty} T^{i}(N)\) [15]. Suppose moreover that \(N_{\infty} = C e, \ \|e\| = 1\), and that \(T e = e\). Then, we claim that
\[W(T) = D \cup \{1\}.\] (3)

In fact, \(W(T) = \text{co}(W(T/N_{\infty}) \cup W(T/N_{0}))\). Since \(T e = e\), we see that \(W(T/N_{\infty}) = \{1\}\). And if \(x = \sum_{j \geq 0} T^{j}(x_{j}) \in N_{0}\), with \(\|x\|^{2} = \sum_{j \geq 0} \|x_{j}\|^{2}\), we see that \((T x, x) = \sum_{j \geq 0} \|x_{j}\|^{2}, T^{j}(x_{j})) = \sum_{j \geq 0} \|T^{j}x_{j}\|^{2} = \sum_{j \geq 0} \|x_{j + 1}\|^{2}\). And, as for the case of the scalar unilateral shift \(S\), we conclude that \(W(T/N_{0}) = D \ (W(T/N_{0}) \text{ is included in } D \text{ and contains the numerical range of } S)\). It follows that \(W(T) = \text{co}([1] \cup D) = D \cup \{1\}\), proving (3). (Recall that \(\text{co}\) denotes the convex hull.) We will see that, in our case, we have \(N_{\infty} = \mathcal{C}^{1}\), with \(C_{\phi}(1) = 1\). In fact, since \(C_{\phi}\) is an isometry, one has \(c_{0} \neq 0\), and since this isometry is non-surjective, one has \(c_{0} \geq 2\). If \(\phi^{n}\) is the \(n\)-th iterate of \(\phi\), one clearly has \(\phi^{n}(s) = c_{0}^{n}s + \sum_{h \geq 1} c_{h}^{(n)} h^{-r}\). Let now \(f \in N_{\infty}\). For each \(n \geq 1\), we can write \(f = g_{n} \circ \phi^{n}\), where \(g_{n}(s) = \sum_{h \geq 1} a_{h}^{(n)} h^{-r} \in \mathcal{H}\). Clearly, then (cf. [10]), one has \(g_{n} \circ \phi^{n}(s) = \sum_{h \geq 1} a_{h}^{(n)} h^{-r}(s) = \sum_{h \geq 1} a_{h}^{(n)} h^{-r} c_{h}^{(n)} = a_{1}^{(n)} + \sum_{j \geq 2} \beta_{j}^{(n)} h^{-r}\), for appropriate coefficients \(\beta_{j}^{(n)}\). Since \(2^{n} \to \infty\) as \(n \to \infty\), this shows that \(f = g_{n} \circ \phi^{n}\) has no point in its Dirichlet spectrum, except possibly the point one. This proves that \(N_{\infty} = \mathcal{C}^{1}\), therefore \(W(C_{\phi}) = D \cup \{1\}\) by (3). Observe that one ignores whether there exist non-surjective isometric composition operators \(C_{\phi}\) for which \(\phi(s) \neq c_{0}s + ik\).

(b) Denote by \(Y\) the set of integers \(\geq 2\) which are not perfect \(c_{0}\)-powers: \(n \in Y\) iff \(n \neq m^{c_{0}}, m \in \mathbb{N}\). For each \(n \in Y\), denote by \(H_{n}\) the closed subspace of \(\mathcal{H}\) generated by the vectors \(n^{-s}, (n^{c_{0}})^{-s}, \ldots\), i.e.:
\[H_{n} = \overline{\text{span}} \{n^{k}s\}^{-s}, k \geq 0\].

Since each integer \(n \geq 2\) can be uniquely written as \(q = n^{k}s\), with \(n \in Y\) and \(k \geq 0\), we have the orthogonal decomposition
\[(\bullet \bullet \bullet) \ \mathcal{H} = \mathcal{C}^{1} \oplus \left(\bigoplus_{n \in Y} H_{n}\right)\].
Observe that each $H_n$ is stable under $C_\phi$, and that $C_\phi$ acts on each $H_n$ as a weighted shift; in fact, if $n \in Y : C_\phi((n^\phi)^{-\tau}) = (n^{\phi + \epsilon_1})^{-\tau}$, which indicates that, for fixed $n$, the restriction $C_\phi / H_n$ is the weighted forward shift $S_n$ acting on a Hilbert space $K$, with orthonormal basis $(e_h)_{h \geq 0}$ by

$$S_n(e_h) = w_h e_{h+1}, \quad \text{with } w_h = n^{-\phi + \epsilon_1}, \ h \geq 0.$$ 

If $x = \sum_0^\infty x_h e_h \in K$, we have $\langle S_n x, x \rangle = \sum_0^\infty w_h x_h x_{h+1}$ and we see that $W(S_n)$ is circularly symmetric, as in the proof of (a); $\langle S_n(e_0), e_0 \rangle = 0$, therefore $0 \in W(S_n); S_n$ is compact (it is Hilbert–Schmidt), therefore (see (f) of this section) $W(S_n)$ is a closed disk $D(0, r_n)$, where the sequence $(r_n)_{n \in Y}$ clearly decreases, since

$$r_n = \sup \left\{ \sum_0^\infty x_h x_{h+1} n^{-\phi + \epsilon_1}; \sum_0^\infty x_h^2 = 1, x_h \geq 0 \right\}.$$

Let now $f \in H$, with $\|f\| = 1$. In view of (**★★**), write $f = f_1 + \sum_{n \in Y} f_n$, with $f_1 \in C, f_n \in H_n, |f_1|^2 + \sum_{n \in Y} \|f_n\|^2 = 1$.

Let us put $\lambda_1 = |f_1|^2, \lambda_n = \|f_n\|^2, g_n = \frac{f_n}{\|f_n\|}$ if $n \in Y$. We have:

$$\langle C_\phi f, f \rangle = |f_1|^2 + \sum_{n \in Y} \langle S_n f_n, f_n \rangle = \lambda_1 + \sum_{n \in Y} \lambda_n \langle S_n g_n, g_n \rangle \in \overline{co} \left( 1 \cup \bigcup_{n \in Y} W(S_n) \right)$$

$$= \overline{co} \left( 1 \cup \bigcup_{n \in Y} D(0, r_n) \right) = \overline{co} (1 \cup D(0, r_2))$$

since $r_n \leq r_2$ if $n \in Y$.

This clearly ends the proof of Proposition 7. □

For more on the numerical range of weighted shifts (see [17]). Propositions 5–7 have shown that, for symbols $\phi(s) = c_0 s + \varphi(s)$ where $\varphi$ is constant, we always have $0 \in W$, and even $0 \in W$ if $\phi(s) \neq s + c_1, c_1 > 0$. We shall extend those results for more general symbols; we begin with the following simple proposition, which will be improved later. Recall that we exclude the trivial case $\phi(s) = s$, and that we write $W$ for $W(C_\phi)$.

**Proposition 8.** For any symbol $\phi$, we have $0 \in \overline{W}$.

**First proof.** Since $\phi \neq Id_\mathcal{H}$, there exists $a$ such that $\Re a = \frac{1}{2}$ and $\phi(a) \neq a$. Let $\epsilon > 0$ and set $\alpha = a + \epsilon$, and $f_\epsilon = \frac{K_\alpha}{\|K_\alpha\|}$, where $K_\beta$ is the reproducing kernel of $\mathcal{H}$ at $\beta$.

Then, $w_\epsilon = \langle C_\phi f_\epsilon, f_\epsilon \rangle \in W$, and $w_\epsilon = \frac{1}{\zeta(2, \alpha_\epsilon)} \langle K_{\alpha_\epsilon}, K_{\phi(a_\epsilon)} \rangle = \frac{\zeta(\phi(a_\epsilon) + \pi \epsilon)}{\zeta(2, \alpha_\epsilon)} =: N_\epsilon$.
Since \( \phi(a) \neq a \), we have \( \phi(a) + \overline{a} = \phi(a) + 1 - a \neq 1 \), and \( N_\epsilon \) tends to the finite value \( \zeta(\phi(a) + \overline{a}) \) as \( \epsilon \to 0 \), whereas the denominator \( D_\epsilon \) tends to \( \zeta(1) = \infty \). Therefore, \( w_\epsilon \to 0 \), and \( 0 \in W \).

**Second proof.** If \( C_\phi \) is not invertible, \( 0 \) belongs to the spectrum of \( C_\phi \) and therefore to \( W \), by property (c) of the numerical range recalled at the beginning of this section. If \( C_\phi \) is invertible, it follows from Bayart’s Theorem 3 in the Introduction that \( \phi(s) = s + ik \), where \( k \in \mathbb{R} \); but then, by (b) of Proposition 6, \( 0 \) belongs not only to the closure of \( W \), but to its interior. \( \square \)

We will now prove a much more precise result, implying Proposition 8 as a special case:

**Theorem 9.** Let \( \phi \) be the symbol of a composition operator on \( \mathcal{H} \). Then:

(a) Either \( \phi(s) = s + c_1, c_1 > 0 \). Or \( 0 \) belongs to the interior of \( W \).

(b) If \( \phi(s) \neq s + c_1, c_1 > 0 \), \( W \) is closed as soon as \( C_\phi \) is compact.

**Proof.** Observe first that (b) is an immediate consequence of (a), modulo property (f) recalled at the beginning of this section. If \( \phi(s) = s + c_1, c_1 > 0 \), we know that \( C_\phi \) is compact and that \( W = \{0\} \). We can now assume that \( \phi \) is not of this form; write \( \phi(s) = c_0s + \psi(s) \), where \( c_0 \in \mathbb{N} \) and \( \psi \in \mathcal{D} \). We shall treat separately the cases \( c_0 = 0, c_0 = 1, c_0 > 2 \).

**Case 1.** \( c_0 = 0 \). Then, \( \phi = \psi \in \mathcal{D} \), and according to a well-known result [7, p. 131] of the theory of analytic, almost-periodic functions, a Dirichlet series \( \psi(s) = \sum_{n=1}^{\infty} \alpha_n n^{-s} \), absolutely convergent in a half-plane \( \Re s > \theta \), will be injective on no vertical strip \( \alpha < \Re s < \beta \) of this half-plane; we can therefore (we also assume that \( \phi \) is not constant, this case having been treated in Proposition 5) find \( a \) and \( b \in \mathbb{C}_{1/2} \) with \( a \neq b \) and \( \phi(a) = \phi(b) \). This shows that \( C_\phi^* (K_a - K_b) = K_{\phi(a)} - K_{\phi(b)} = 0 \), so that, since moreover \( C_\phi \) is injective, \( \phi \) being non-constant, we have:

\[
0 \text{ is an eigenvalue of } C_\phi^*; 0 \text{ is not an eigenvalue of } C_\phi.
\]

Let us now recall that an eigenvalue \( \lambda \) of an operator \( T \) on a Hilbert space is normal if \( \ker(T - \lambda I) = \ker(T^* - \overline{\lambda} I) \neq 0 \). It is known [5] that every eigenvalue of \( T \) lying on the boundary of \( W(T) \) is normal. And (4) shows that \( 0 \) is not a normal eigenvalue of \( C_\phi^* \); therefore, \( 0 \in \text{Int}(W(C_\phi^*)) = \text{Int}(\overline{W}) \), where here the bar indicates conjugation; and \( \overline{0} = 0 \in \text{Int}W = W \).

On the other hand, \( \phi \) may very well be injective if \( c_0 > 0 \). For example:

\[
\text{If } c_0 \geq \sum_{n=1}^{\infty} |c_n| \log n > 0, \text{ then } \phi \text{ is injective on } C_0.
\]
In fact, we then have $\Re \phi(s) > 0$ on the convex open set $C_0$. Similarly, $\phi$ is injective on $C_{1/2}$ if $c_0 \geq \sum_{i=1}^{\infty} |c_n| n^{-1/2} \log n$.

**Case 2.** $c_0 \geq 2$. It follows from [10] that, for $n \geq 2$, one has $n^{-\phi(s)} = (n^{c_0})^{-s} n^{-c_1}$.

Case 3. \( c_0 = 1 \). Write $\phi(s) = s + c_1 + c_2 q^{-s} + \cdots$, with $c_2 \neq 0$. (The case $\phi(s) = s + c_1, R_1 = 0$, $c_1 \neq 0$, has already been treated in Proposition 6.) For $n \geq 2$, we have:

\[
R \phi(s) = n^{-s} n^{-c_1} \prod_{r > q} e^{-c_r r^{-s}} \log n \]

\[
= n^{-s} n^{-c_1} \prod_{r > q} \left(1 + \sum_{e, b \geq 1} (-c_e \log n)^b \frac{(r, b^{-1})}{j_r!} \right) \]

\[
= n^{-s} n^{-c_1} \left(1 - c_q \log n \ q^{-s} + \cdots \right). \]

Now take, for an arbitrary integer $p \geq 1 : f(s) = a(q^p)^{-s} + b\left(q^{p+1}\right)^{-s}$, where $|a|^2 + |b|^2 = 1$. We see that

\[
(C_{\phi}f)(s) = a(q^p)^{-s} q^{-pc_1} \left(1 - c_q \log q^p q^{-s} + \cdots \right) + b \left(q^{p+1}\right)^{-s} q^{-(p+1)c_1} \left(1 - c_q \log q^{p+1} q^{-s} + \cdots \right), \]

so that

\[
(C_{\phi}f, f) = |a|^2 q^{-pc_1} + |b|^2 q^{-(p+1)c_1} - c_q a \bar{b} \log q^p q^{-pc_1} \]

\[
= q^{-pc_1} \left(|a|^2 + |b|^2 q^{-c_1} - pc_q a \bar{b} \log q \right). \]

Here, it is convenient to recall the

**Lemma 10** [11]. Let $A : \mathbb{C}^2 \to \mathbb{C}^2$ be represented on the canonical basis of $\mathbb{C}^2$ by the matrix $A = \begin{bmatrix} \lambda_1 & 0 \\ a & \lambda_2 \end{bmatrix}$, with $\lambda_1 \neq \lambda_2$. Then, the numerical range of $A$ is the closed elliptic disk $D$ with foci $\lambda_1$ and $\lambda_2$, and minor axis of length $|a|$. 


It follows in particular that

\[
|\lambda_1| + |\lambda_2| < |a|, \quad \text{then } 0 \in \mathcal{E}.
\] (6)

In fact, if \( \alpha \geq |a| \) is the length of the major axis of \( \mathcal{E} \), \( z \in \mathcal{E} \) if and only if \(|z - \lambda_1| + |z - \lambda_2| < \alpha\).

Let now \( A_p : \mathbb{C}^2 \to \mathbb{C}^2 \) be represented by the matrix

\[
A_p = \begin{bmatrix} 1 & 0 \\ -pcq \log q & q^{-c_1} \end{bmatrix}.
\]

The previous relation \( \langle C_{\phi} f, f \rangle = q^{-pc_1} (|a|^2 + |b|^2 q^{-c_1} - pcq a\bar{b} \log q), |a|^2 + |b|^2 = 1 \), indicates that, for each integer \( p \geq 2 \), we have

\[
W \supset q^{-pc_1} W(A_p).
\]

From Lemma 2, we know that \( \Re c_1 > 0 \), since \( \psi \) is not constant. We can therefore take \( p \) large enough to ensure that \( 1 + |q^{-c_1}| < p|c_q| \log q \). Then, (6) shows that \( 0 \in \mathcal{W}(A_p) \), and (7) shows that \( 0 \in \mathcal{W} \), which ends the proof of Theorem 9. \( \square \)

Remark 11. In view of (b) in Theorem 9, it is interesting to know when the operator \( C_{\phi} \) is compact. In [2,3,8] the compactness of such composition operators is studied.

References


