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A hyperchaotic system from the Rabinovich system

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ABSTRACT

This paper presents a new 4D hyperchaotic system which is constructed by a linear controller to the 3D Rabinovich chaotic system. Some complex dynamical behaviors such as boundedness, chaos and hyperchaos of the 4D autonomous system are investigated and analyzed. A theoretical and numerical study indicates that chaos and hyperchaos are produced with the help of a Liénard-like oscillatory motion around a hypersaddle stationary point at the origin. The corresponding bounded hyperchaotic and chaotic attractors are first numerically verified through investigating phase trajectories, Lyapunov exponents, bifurcation path and Poincaré projections. Finally, two complete mathematical characterizations for 4D Hopf bifurcation are rigorously derived and studied.

1. Introduction

Hyperchaos is characterized as a chaotic system with more than one positive exponent [1], this implies that its dynamics are expended in several different directions simultaneously. Thus, hyperchaotic systems have more complex dynamical behaviors than ordinary chaotic systems. At the same time, due to its theoretical and practical applications in technological fields, such as secure communications, lasers, nonlinear circuits, neural networks, generation, control, synchronization, hyperchaos has recently become a central topic in nonlinear sciences research (see e.g. [1–7] as well as their references).

On the one hand, the ultimate boundedness of a chaotic system is very important for the study of the qualitative behavior of a chaotic or hyperchaotic system. If one can show that a chaotic or hyperchaotic system under consideration has a globally attractive set, one knows that the system cannot have equilibrium points, periodic or quasi-periodic solutions, or other chaotic or hyperchaotic attractors existing outside the attractive set. This greatly simplifies the analysis of dynamics of the system. However, the estimate of the ultimate boundedness of a chaotic system is still a very difficult task [8–10]. Due to their complexity, the study of the ultimate boundedness of hyperchaotic systems is a more difficult task. So far, their ultimate boundedness has not been systematically studied [7,11]. On the other hand, the hyperchaos theory is still in its infancy. Very little has been achieved on hyperchaotic systems. The dynamics of the hyperchaotic systems have not been completely understood by mathematicians until now. For example, some dynamical behaviors such as boundedness, Hopf bifurcation and chaotic property of the 4D hyperchaotic system are still at the exploratory stage. Therefore, it is necessary to make a new study for the hyperchaotic system. This situation motivates us to further study the properties of chaos and hyperchaos and some subtle characteristics of 4D Hopf bifurcation of the new hyperchaotic system which is generated from the chaotic system, so as to benefit more systematic studies of 4D quadratic systems, and to reveal the true geometrical structures of the lower-dimensional chaotic and hyperchaotic attractors.

The Rabinovich differential system, which was firstly introduced in [12], is defined by

$$\begin{cases} \dot{x} = hy - ax + yz \\ \dot{y} = hx - by - xz \\ \dot{z} = -dz + xy, \end{cases} \tag{1.1}$$

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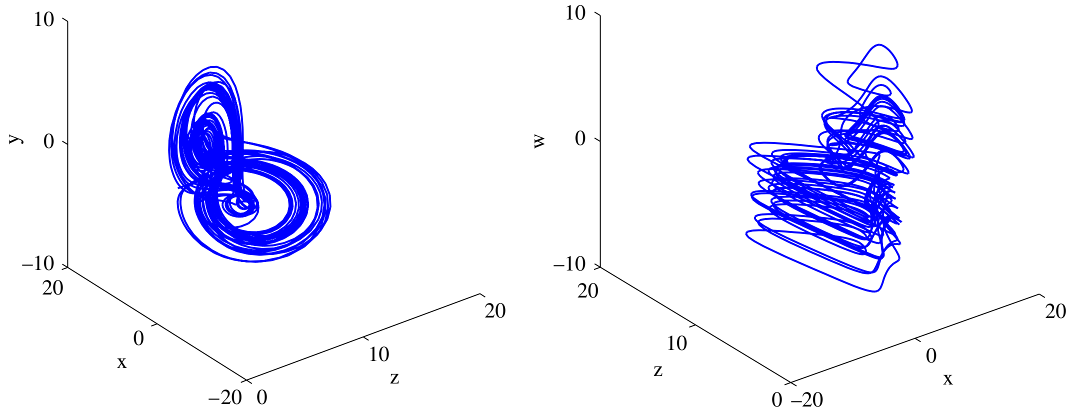

(a) z–x–y space.

(b) x–z–w space.

Fig. 1. Hyperchaotic attractor of the Rabinovich system (2.1) with $a = 4, b = d = 1, h = 6.75$, and $k = 2$.

where $(x, y, z)^T \in \mathbb{R}^3$. It has a chaotic attractor for some values of parameters, for example, $a = 4, b = d = 1, 4.84 \leq h \leq h_0$, for some $h_0 \geq 4.92$, and this attractor looks similar with the chaotic attractor of the Lorenz system. As comparing with the famous Lorenz system [13,14], the Rabinovich system resembles many of its properties and it was stated in the book [15] that both the Rabinovich system and the Lorenz system can be considered as particular cases of the so-called generalized Lorenz system. There have been extensive investigations on dynamical behaviors of the Rabinovich system [12,16,17].

In this paper, a new hyperchaotic is generated from the 3D Rabinovich system via adding a linear controller to it and its basic dynamics and properties are investigated, such as the stability of the system and the geometry of the attractor. The corresponding bounded hyperchaotic and chaotic attractor is first numerically verified through investigating phase trajectories, Lyapunov exponents, bifurcation path and Poincaré projections. Two complete mathematical characterizations for 4D Hopf bifurcation are also rigorously derived and studied. The fact that chaos and hyperchaos are created via a Liénard-like oscillatory motion around a hypersaddle stationary point at the origin is shown by numerical experiments. The 4D system preserves some properties of the 3D system, such as the z-axis symmetry and the attractor's double-lobe structure.

The rest of this paper is organized as follows. In Section 2, the new hyperchaotic system is introduced and its boundedness is also proved. The stability of the system and the geometry of the attractor are discussed, and the dynamical behaviors of this hyperchaotic system such as Lyapunov exponents, fractal dimension and chaotic behaviors are also analyzed in Section 3. In Section 4, by using the normal form theory and symbolic computations, two complete mathematical characterizations for the 4D Hopf bifurcations are derived and investigated. Finally, conclusions are drawn in Section 5.

2. A new hyperchaotic system

2.1. Formulation of the system

The proposed dynamical system is given by the following Rabinovich equations linearly extended to 4D:

$$\begin{cases} \dot{x} = hy - ax + yz \\ \dot{y} = hx - by - xz + w \\ \dot{z} = -dz + xy \\ \dot{w} = -ky, \end{cases} \tag{2.1}$$

where k is positive constant parameter, determining the chaotic and hyperchaotic behaviors and bifurcations of the system. Thus, the controller w has made the chaotic system (1.1) a 4D hyperchaotic system (2.1), which has four Lyapunov exponents. When $(a, b, d, h, k) = (4, 1, 1, 6.75, 2)$, the four Lyapunov exponents are

$$\lambda_{LE_1} = 0.3066, \qquad \lambda_{LE_2} = 0.0582, \qquad \lambda_{LE_3} = -0.000, \qquad \lambda_{LE_4} = -6.3642,$$

and the Lyapunov dimension is $D_L = 3.0573$. Moreover, numerical simulations have verified that system (2.1) indeed has a hyperchaotic attractor when $(a, b, d, h, k) = (4, 1, 1, 6.75, 2)$, as depicted in Fig. 1. Fig. 2 shows the Poincaré mapping on the $x - z$ plane and power spectrum of the time series $x(t)$ for this hyperchaotic system.

2.2. Boundedness

Theorem 2.1. *Suppose $a > 0, b > 0, d > 0, h > 0$ and $k > 0$. Then all orbits of system (2.1), including hyperchaotic orbits, are trapped in a bound region.*

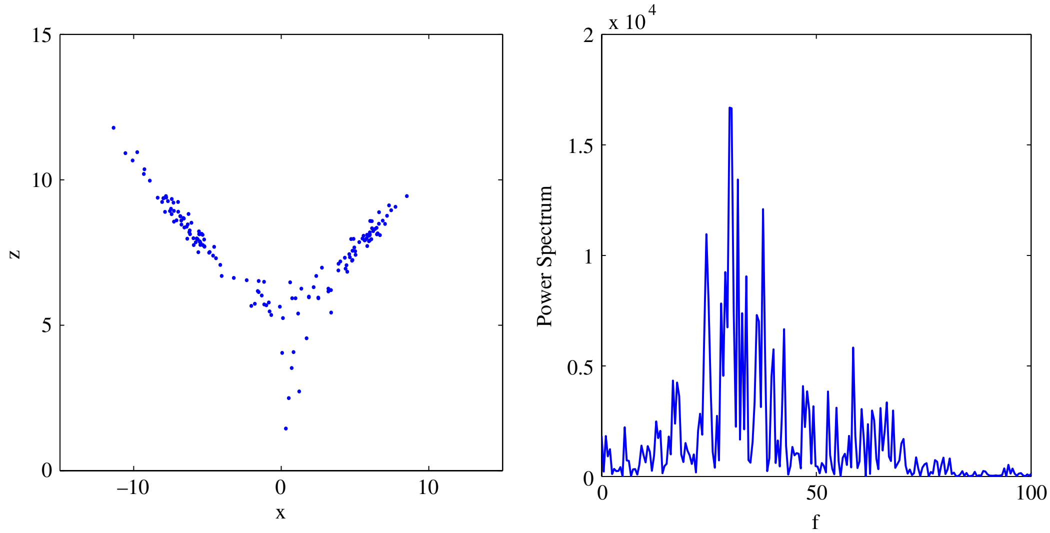

Fig. 2. (a) Poincaré mapping on the x–z plane and (b) power spectrum of time series $x(t)$ for the hyperchaotic Rabinovich system (2.1) with $a = 4$, $b = d = 1$, $h = 6.75$, and $k = 2$.

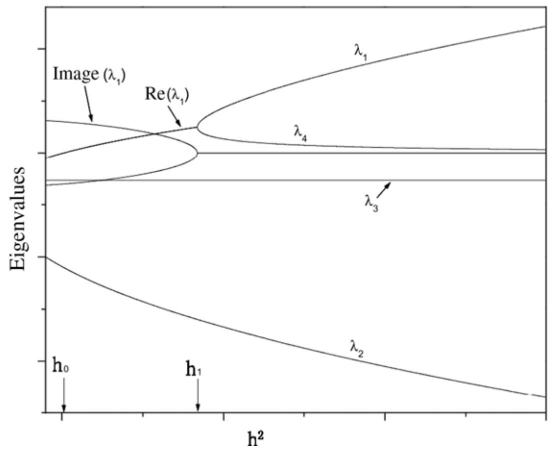

Fig. 3. Graphical illustration of varying eigenvalues.

Proof. Construct the following Lyapunov function:

$$V(x, y, z, w) = x^2 + 2y^2 + (z - 3h)^2 + \frac{2}{k}w^2. \tag{2.2}$$

Along the orbits of system (2.1), one has

$$\begin{aligned}\frac{1}{2}\dot{V}(x, y, z, w) &= -ax^2 - 2by^2 - dz^2 + 3dhz \\ &= -ax^2 - 2by^2 - d\left(z - \frac{3h}{2}\right)^2 + \frac{9dh^2}{4}.\end{aligned}$$

Let $d_0 > 0$ be sufficiently large, so that for all (x, y, z, w) satisfying $V(x, y, z, w) = d_1$ with $d_1 > d_0$, one has

$$ax^2 + 2by^2 + d\left(z - \frac{3h}{2}\right)^2 > \frac{9dh^2}{4}.$$

Consequently, on the surface $\{(x, y, z, w)|V_1(x, y, z, w) = d_1\}$ with $d_1 > d_0$, one has $\dot{V}(x, y, z, w) < 0$, which implies that the set $\Omega = \{(x, y, z)|V(x, y, z, w) \le d_1\}$ is a trapping region of all solutions of system (2.1). Thus, all orbits of system (2.1) are bounded. The proof is thus completed. □

Remark 2.1. From boundedness and two positive Lyapunov exponents, it follows that system (2.1) with $(a, b, d, h, k) = (4, 1, 1, 6.75, 2)$ indeed has a hyperchaos attractor.

3. Dynamical behaviors of the hyperchaotic system

This section further investigates the dynamical behaviors of the hyperchaotic system (2.1), including dissipativity, equilibria and stability, structure of attractor, Lyapunov exponents, and bifurcation diagrams.

First, Figs. 6–10 show some typical dynamical behaviors of the system.

3.1. Dissipativity

For system (2.1), it is noticed that $\nabla V = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} + \frac{\partial \dot{w}}{\partial w} = -(a+b+d)$. Obviously, when system (2.1) can have dissipative structure, with an exponential contraction rate: $\frac{\mathrm{d}V}{\mathrm{d}t} = -(a+b+d)V$. That is, a volume element V_0 is contracted by the flow into a volume element $V_0 \mathrm{e}^{-(a+b+d)t}$ in time t. This means that each volume containing the system orbit shrinks to zero as $t \to \infty$ at an exponential rate, $-(a+b+d)$, which is independent of x, y, z and w. Therefore, all system orbits are ultimately confined to some subset of zero volume, and the asymptotic motion settles on some attractors.

3.2. Equilibria

It is clear that system (2.1) (and thus its solution) is invariant under the transformation $T(x, y, z, w) \to (-x, -y, z, -w)$. This means that any orbit that is not itself invariant under T must have its "twin" orbit in the sense of this transformation.

System (2.1) has the origin as the only stationary point for all positive parameters values. The Jacobian matrix of Eqs. (2.1) is given by:

$$J = \begin{bmatrix} -a & h+z & y & 0 \\ h-z & -b & -x & 1 \\ y & x & -d & 0 \\ 0 & -k & 0 & 0 \end{bmatrix}. \tag{3.1}$$

The characteristic equation $|J - \lambda I| = 0$ at the origin can be written as

$$(\lambda + d)[\lambda^3 + (a+b)\lambda^2 + (ab - h^2 + k)\lambda + ak] = 0, \tag{3.2}$$

which gives $\lambda = -d$ and

$$\Delta(\lambda) = \lambda^3 + (a+b)\lambda^2 + (ab - h^2 + k)\lambda + ak = 0. \tag{3.3}$$

Let $A = a + b$, $B = ab + k - h^2$ and $C = ak$. Then, according to the Routh–Hurwitz criterion, the real parts of all the roots λ in $\Delta(\lambda) = 0$ are negative if and only if $A > 0$, $C > 0$ and $AB - C > 0$. From these inequalities, one obtains $a > 0$, $b > 0$, $k > 0$ and $h^2 < ab + \frac{bk}{a+b}$.

Based on the above discussion, the following property is verified.

Theorem 3.1. *Let $a > 0$, $b > 0$, $d > 0$ and $k > 0$. Then system (2.1) has a unique equilibrium $O(0, 0, 0, 0)$. Furthermore, the necessary and sufficient condition for equilibrium O to be local asymptotical stable is $h^2 < h_0 \equiv ab + \frac{bk}{a+b}$.*

Theorem 3.2. *Let $a > 0$, $b > 0$, $d > 0$, $k > 0$ and $h^2 < \frac{8}{9}ab$. Then the equilibrium O of system (2.1) is globally uniformly and asymptotical stable. Moreover, system (2.1) is neither chaotic nor hyperchaotic.*

Proof. Define the following Lyapunov function

$$V(x, y, z, w) = \frac{1}{2}(x^2 + z^2) + y^2 + \frac{w^2}{k}. \tag{3.4}$$

From $a > 0$, $b > 0$ and $h^2 < \frac{8}{9}ab$, one reduces that matrix

$$\begin{bmatrix} a & -3h/2 \\ -3h/2 & 2b \end{bmatrix}$$

is positive matrix. Its time derivative along the orbit of system (2.1) is

$$\dot{V}(x, y, z, w) = -ax^2 - 2by^2 + 3hxy - dz^2 < 0$$

and by setting

$$\left\{(x, y, z, w) | \dot{V}(x, y, z, w) = 0\right\} = \{(x, y, z, w) | x = 0,\ y = 0,\ z = 0,\ w \in R\}$$

which does not contain a nontrivial trajectory of system (2.1). The Krasnoselskii theorem implies that system (2.1) is globally uniformly and asymptotically stable about the origin. Furthermore, system (2.1) has neither chaotic attractor nor hyperchaotic attractor. The proof is thus completed. □

For $h^2 < h_0$ the solutions of Eq. (3.2) are three roots with negative real part λ_1, λ_2, λ_3 and λ_4, where $\lambda_3 = -d$. At the critical point $h = \sqrt{h_0}$, the roots λ_1 and λ_2 both disappear and give birth to a pair of purely imaginary. $\Re(\lambda_1) = 0$, $\frac{\mathrm{d}\Re(\lambda_1)}{\mathrm{d}h}|_{h=\sqrt{h_0}} = \frac{(a+b)^{3/2}(a^2b+ab^2+bk)^{1/2}}{(a+b)^3+ak} > 0$, $\lambda_2 = -(a+b) < 0$.

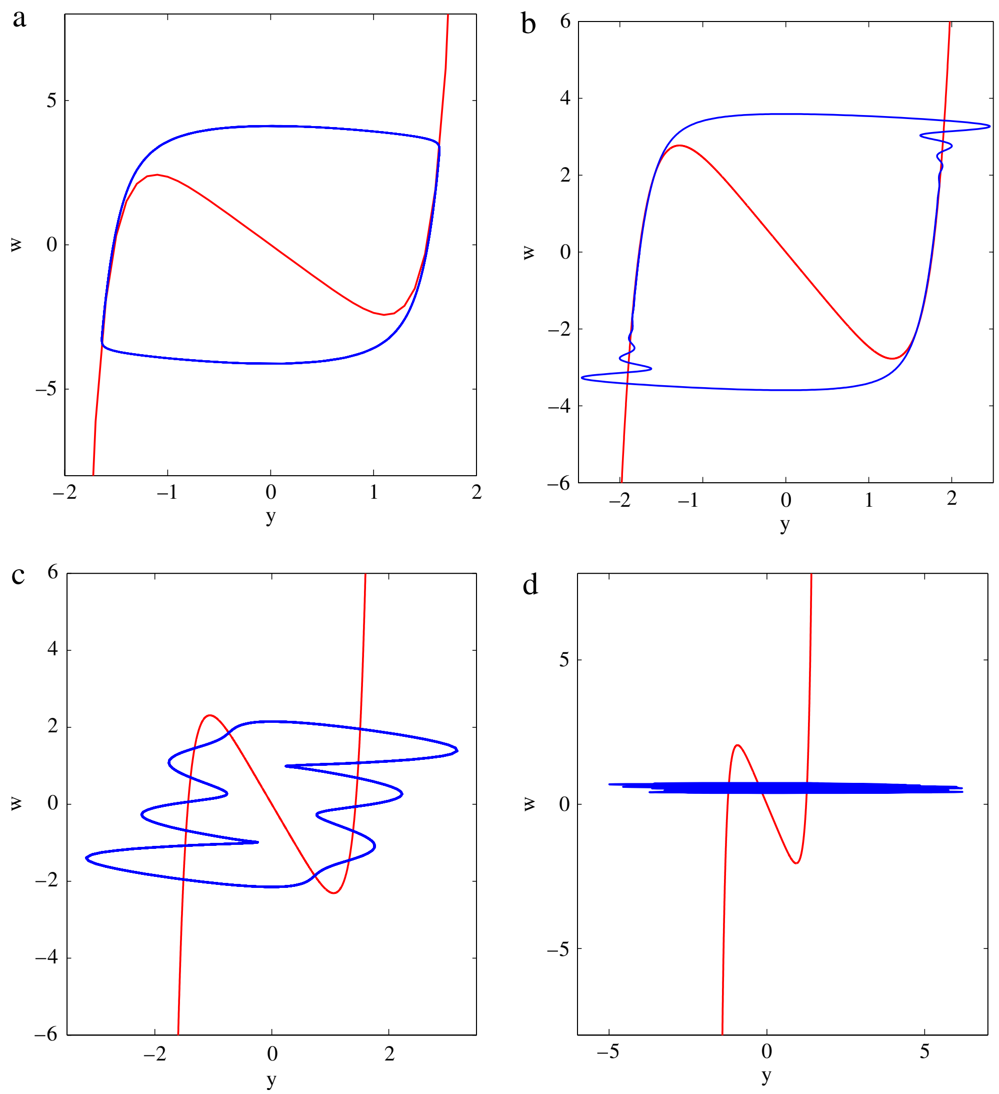

Fig. 4. (a) Solution and nullcline of system (3.5) for $a = 4$, $b = 1$, $d = 1$, $k = 1/2$ and $h = 3$. (b)–(d) Solutions and nullcline of system (2.1) for $a = 4$, $b = 1$, $d = 1$, $k = 1/2$: (b) $h = 2.5$; (c) $h = 3.4$; (d) $h = 10$. Normalization: $y' = y\sqrt{h^2(a-1)/[ad(h^2-ab)]}$, $w' = w\sqrt{27ah^2(a-1)/[4d(h^2-ab)^3]}$.

This result indicates, in accordance with the Hopf theorem [18], the birth of a limit cycle at h_0, which grows in size with h^2, and has initial period given by $T = 2\pi/\Im(\lambda_1)$, where $\Im(\lambda_1) = \frac{[ak(a^2b+ab^2+bk)]^{1/2}}{(a+b)^3+ak}$. The basic role of this limit cycle on the system dynamics will be explained in Section 3.3. The limit cycle persists even when the parameter h goes beyond a certain critical point $h^2 = h_1$, where the imaginary part of λ_1 disappears and its real part remains positive and bifurcates, giving rise to a new real eigenvalue $\lambda_4 > 0$, as shown in Fig. 3. Note that for $h^2 > h_1$ the origin is a hypersaddle stationary point, with $\lambda_1 > \lambda_4 > 0 > \lambda_3 > \lambda_2$.

Instead of calculating the roots exactly, one can get simpler relations by observing that the cubic polynomial in Eq. (3.3) can be written as $\lambda[\lambda^2 + (a+b)\lambda + (ab - h^2)] + (\lambda - a)k$. The bracketed expression is just the polynomial appearing in the characteristic equation of the 3D Rabinovich system, which has the roots $\lambda_1^0 = (1/2)\{-[(a-b)^2 + 4h^2]^{1/2} - (a+b)\}$ and $\lambda_2^0 = (1/2)\{[(a-b)^2 + 4h^2]^{1/2} - (a+b)\}$. One can write the following identity: $\lambda(\lambda - \lambda_1^0)(\lambda - \lambda_2^0) + (\lambda - a)k \equiv (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_4) = 0$. After identifying coefficients of equal powers of λ and arranging the resulting equations, one get the following approximate solutions:

$$\lambda_1 \approx \lambda_1^0 - \frac{(\lambda_1^0 - a)k}{\lambda_1^0(\lambda_1^0 - \lambda_2^0)},$$

$$\lambda_2 \approx \lambda_2^0 - \frac{(\lambda_2^0 - a)k}{\lambda_2^0(\lambda_2^0 - \lambda_1^0)}.$$

The third eigenvalue is $\lambda_3 = \lambda_3^0 = -d$ and the fourth one, introduced by the new variable w, is given by:

$$\lambda_4 = \frac{ak}{\lambda_1\lambda_2} \approx \frac{ak}{ab - h^2}.$$

As an example, for $a = 4, b = 1, d = 1, h = 6.75$ and $k = 2$ one has: $\lambda_1^0 = -9.41$ and $\lambda_2^0 = 4.41$. The exact and approximate values for system (2.1) are $\lambda_1 = -9.33 \approx -9.12$, $\lambda_2 = 4.12 \approx 4.12$ and $\lambda_4 = 0.21 \approx 0.19$. The proximity of the 3D and 4D eigenvalues is the reason for some common features shared by the two systems. It also makes clear the main role of λ_4 in the hyperchaotic behavior. Indeed, for the above parameters the eigenvectors corresponding, respectively, to $\lambda_1, \lambda_2, \lambda_3$ and λ_4 are: $v_1 = [0.78, -0.61, 0, -0.13]^T$, $v_2 = [0.60, 0.72, 0, -0.35]^T$, $v_3 = [0, 0, 1, 0]^T$, $v_4 = [-0.16, -0.10, 0, 0.98]^T$. The related 3D eigenvectors are $v_1^0 = [0.78, -0.63, 0]^T$, $v_2^0 = [0.63, 0.78, 0]^T$, $v_3^0 = [0, 0, 1]^T$. Note that v_4 lies almost entirely along the w axis, and the (x, y, z) subspace preserves the eigenvector structure of the original 3D Rabinovich system.

3.3. Structure of the attractor

To identify the nature of the limit cycle described so far, one first observes that it occurs next to the intersection of the hypersurface $x = hy/a + yz/a$ (given by $\mathrm{d}x/\mathrm{d}t = 0$) and the hypersurface $z = xy/d$ (given by $\mathrm{d}z/\mathrm{d}t = 0$). Using these relations in system (2.1) one obtains the reduced system

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}t} = w - by + \dfrac{dh^2y}{ad - y^2} - \dfrac{h^2y^3}{(ad - y^2)^2} \\ \dfrac{\mathrm{d}w}{\mathrm{d}t} = -ky \end{cases} \tag{3.5}$$

which is a generalized Liénard equation. The nontrivial solutions of system (3.5) converge to a limit cycle, corresponding to a clockwise motion around the origin, over the humps of the nullcline (found by setting $\mathrm{d}y/\mathrm{d}t = 0$), as illustrated in Fig. 4(a). This result indicates that the presence of the extra variable w caused the incorporation of a Liénard-like dynamics into the 4D Rabinovich system.

The nullcline is a useful reference curve for studying the geometric structure of the attractor. In the 4D space, it is the locus of points satisfying $\mathrm{d}x/\mathrm{d}t = \mathrm{d}y/\mathrm{d}t = \mathrm{d}z/\mathrm{d}t = 0$, described by the equations

$$w = by - \frac{dh^2y}{ad - y^2} + \frac{h^2y^3}{(ad - y^2)^2},$$
$$z = xy/b,$$
$$x = dhy/(ad - y^2).$$

Consider Fig. 4(b)–(d). In the case of small k and h^2 the solutions of system (2.1) remain very close to the lateral branches of the nullcline. Numerical calculations using the Jacobian matrix (3.1) at points along these branches give two negative real eigenvalues and a complex conjugate pair with negative real part — this explains the damping oscillations along the orbit shown in Fig. 4(b). One of the real eigenvectors is practically tangential to the nullcline and orthogonal to the eigenspace spanned by the other three eigenvectors. For large enough h^2 the real part of the complex eigenvalues changes from negative to positive. Accordingly, the limit cycle carries a contracting–expanding spiral that whirls transversely around the lateral branches of the nullcline [Fig. 4(c)]. The trajectory is switched from the half-space $y < 0$ to the half-space $y > 0$, and vice versa, as a result of the underlying Liénard-like dynamics combined with the spiral-like flow, thus forming the two lobes of the 4D attractor. Fig. 5 shows a 3D view of the trajectory shown in Fig. 4(c). For h^2 even higher, the attractor goes flattened and confined close to the hypersaddle stationary point at the origin, giving rise to chaos and hyperchaos [Fig. 4(d)].

In the case of large k the trajectory reaches the neighborhood of the origin at smaller h^2 values. For this reason, system (2.1) usually displays chaotic behavior in the low h^2 region where the related 3D system is stable.

3.4. Small k disturbance

It may be interesting to consider the 3D Rabinovich system as arising from the 4D system in the limit $k \to 0$, with initial condition $w(0) = 0$. In this case, the 4D Rabinovich system with $(a, b, d, h, k) = (4, 1, 1, 6.75, 0)$ has four Lyapunov exponents $\lambda_{LE_1} = 0.4627$, $\lambda_{LE_{2,3}} = 0.000$, $\lambda_{LE_4} = -6.4634$. The small k disturbs, which leads that λ_{LE_2} or λ_{LE_3} leaves naught and becomes positive number or negative number. Fig. 6 shows that the very small positive k causes λ_{LE_2} from zero to positive value, and transits 4D chaotic system into 4D hyperchaotic system.

3.5. Lyapunov exponents and bifurcation diagrams

In the two sections below, some properties of the new 4D system are discussed with k varying. And the simulation results are further obtained by using Matlab Tools.

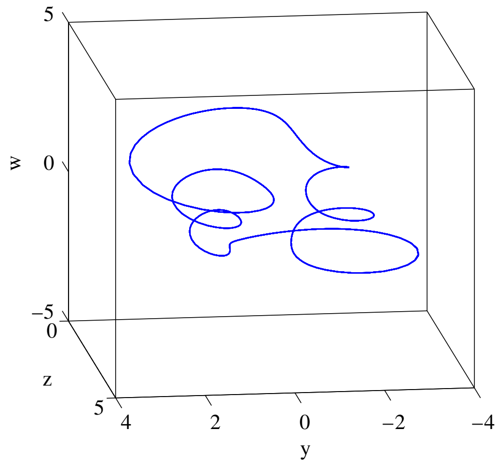

Fig. 5. Solution of system (2.1) with $a = 4$, $b = 1$, $d = 1$, $k = 1/2$ and $h = 3.4$. Normalization: $y' = y\sqrt{h^2(a-1)/[ad(h^2-ab)]}$, $w' = w\sqrt{27ah^2(a-1)/[4d(h^2-ab)^3]}$, $z' = z$.

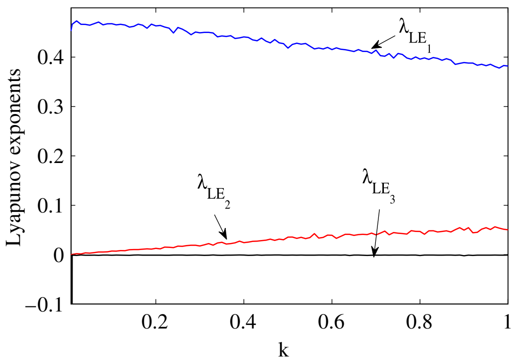

Fig. 6. Lyapunov exponents of system (2.1) with $a = 4$, $b = d = 1$, $h = 6.75$ and $k \in (0, 1]$.

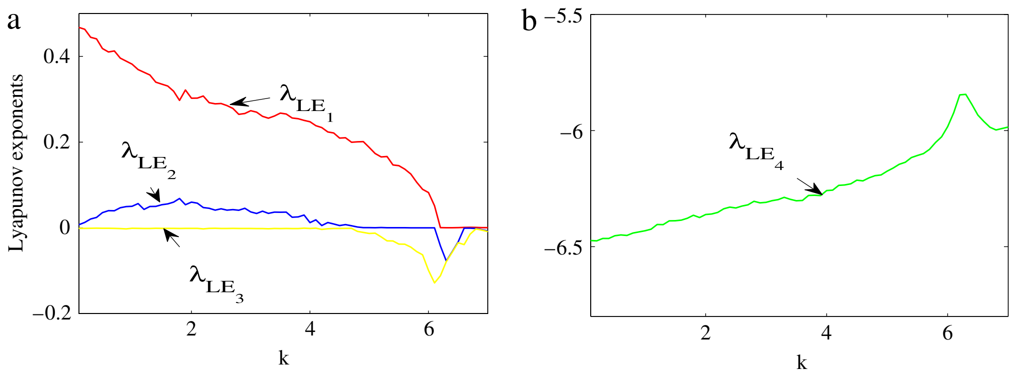

Fig. 7. Lyapunov exponents of system (2.1) with $a = 4$, $b = d = 1$, $h = 6.75$ and $k \in (0, 7]$.

3.5.1. Fix $a = 4$, $b = d = 1$, $h = 6.75$ and vary k

When $k \in (0, 7]$ varies, the corresponding Lyapunov exponent spectrum of system (2.1) are shown in Fig. 7. The bifurcation diagram with respect to $k \in (0, 7]$ is given in Fig. 8. It can be observed that the bifurcation diagram well coincides with the spectrum of Lyapunov exponents. Fig. 7 shows that system (2.1) is hyperchaotic for a very wide range of k, and the system can also evolve into chaotic orbits and periodic orbits. From Figs. 7 and 8, the dynamical behaviors of system (2.1) can be clearly observed. When $k \in (0.1, 4.4)$, the first and the second largest Lyapunov exponents are both positive, which implies that system (2.1) is hyperchaotic. When $k \in (4.4, 6.2)$, it is only one Lyapunov exponent that is bigger than zero, which means that system (2.1) is chaotic. When $k \in (6.2, 6.6)$, the first largest Lyapunov exponents are almost equal zero.

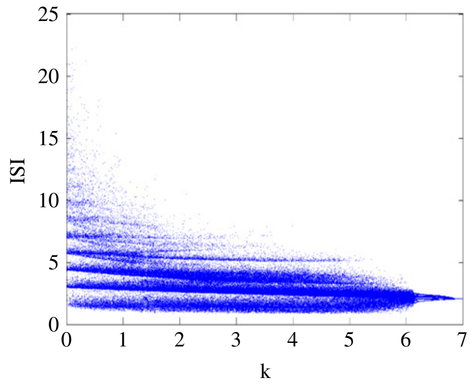

Fig. 8. Bifurcation diagram of system (2.1) with $a = 4$, $b = d = 1$, $h = 6.75$ and $k \in (0, 7]$.

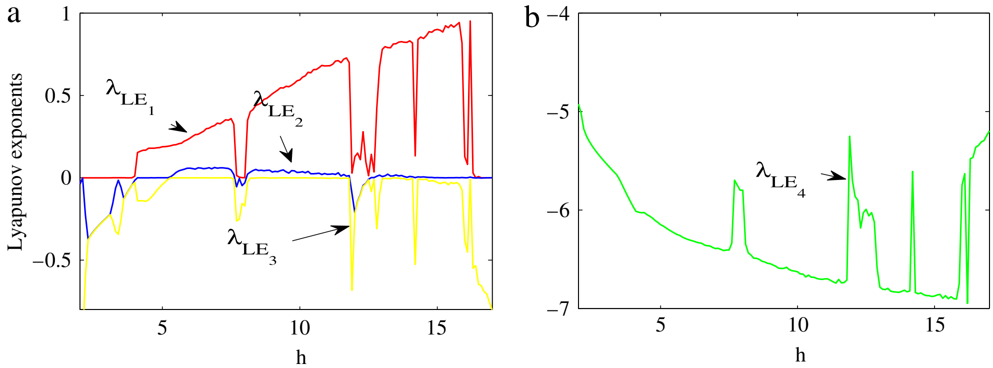

Fig. 9. Lyapunov exponents of system (2.1) with $a = 4$, $b = d = 1$, $k = 2$, and $h \in [2, 17]$.

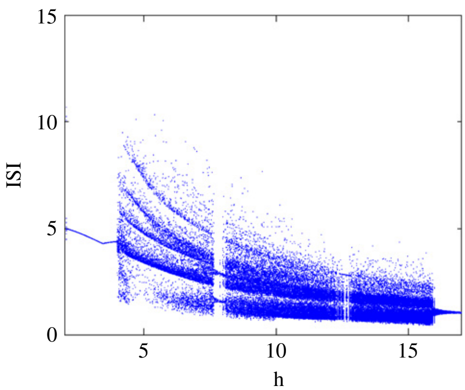

Fig. 10. Bifurcation diagram of system (2.1) with $a = 4$, $b = d = 1$, $k = 2$, and $h \in [2, 17]$.

When $k \in (6.6, 7]$, the first three largest Lyapunov exponents are almost equal zero. Thus, when $k \in (6.2, 7]$, system (2.1) is either periodic or quasi-periodic.

3.5.2. Fix $a = 4$, $b = d = 1$, $k = 2$ and vary h

When $h \in [2, 17]$ varies, the corresponding Lyapunov exponent spectrum of system (2.1) are shown in Fig. 9. The bifurcation diagram with respect to $h \in [2, 17]$ is given in Fig. 10. It can be observed that the bifurcation diagram well coincides with the spectrum of Lyapunov exponents. Fig. 9 shows that system (2.1) is hyperchaotic for a very wide range of h, and the system can also evolve into chaotic orbits and periodic orbits. From Figs. 9 and 10, the dynamical behaviors of system (2.1) can be clearly observed. When $h \in [2, 4), (16.4, 7]$ the first largest Lyapunov exponents are zero, which implies that system (2.1) is periodic. When $h \in (4, 5.4), (7.7, 8.0), (11.9, 13), (13.8, 16.4)$, it is only one Lyapunov exponent that is bigger than zero, which means that system (2.1) is chaotic. When $h \in (5.4, 7.7), (8, 11.9), (13, 13.8)$, the first and the second largest Lyapunov exponents are both bigger than zero, which implies that system (2.1) is hyperchaotic.

4. 4D Hopf bifurcation

This section employs the higher-dimensional Hopf bifurcation theory and applies symbolic computations to perform the analysis of parametric k variations with respect to dynamical bifurcations.

First, consider the existence of Hopf bifurcation.

Theorem 4.1 (*Existence of Hopf Bifurcation*). *Suppose that $a > 0$, $b > 0$, $k > 0$ and $k > h^2 - ab > 0$ hold. Then, as k varies and passes through the critical value $k_0 = (h^2 - ab)(a + b)/b$, system (2.1) undergoes a Hopf bifurcation at the equilibrium $O(0, 0, 0, 0)$.*

Proof. Suppose that Eq. (3.2) has a pure imaginary root $\lambda = \mathrm{i}\omega$, $(\omega \in \mathbb{R}^+, \mathrm{i}^2 = -1)$. Substituting it into Eq. (3.2) yields

$$ak - (a + b)\omega^2 + \mathrm{i}\omega(ab - h^2 + k - \omega^2) = 0.$$

It follows that

$$\omega^2 = ab - h^2 + k = 0, \qquad \omega^2 = ak/(a + b).$$

Solving the above equations gives

$$\omega = \sqrt{a(h^2 - ab)/b}, \qquad k = k_0 = (h^2 - ab)(a + b)/b,$$

under the condition $k > h^2 - ab > 0$. Substituting $k = k_0$ into Eq. (3.2), one obtains

$$\lambda_1 = \mathrm{i}\omega, \qquad \lambda_2 = -\mathrm{i}\omega, \qquad \lambda_3 = -(a + b), \qquad \lambda_4 = -d,$$

where $\omega = \sqrt{a(h^2 - ab)/b}$. Thus, when $k > h^2 - ab > 0$ and $k = k_0$, the first condition for Hopf bifurcation [18] is satisfied.

From Eq. (3.2) and $k > h^2 - ab > 0$, it follows that

$$\Re(\lambda'(k_0))|_{\lambda=\mathrm{i}\omega} = \frac{-b^3}{a(h^2 - ab) + b^2(a + b)^2} \neq 0.$$

Therefore, the second condition for a Hopf bifurcation [18] is also met. Consequently, Hopf bifurcation exists. □

Remark 4.1. When $k \le h^2 - ab$, system (2.1) has no Hopf bifurcation at the equilibrium $O(0, 0, 0, 0)$.

In the following, the stability and expression of the Hopf bifurcation of system (2.1) is investigated, by using the normal form theory [19,20], some rigorous mathematical analysis and symbolic computations.

Theorem 4.2. *Let $a > 0$, $b > 0$, $k > h^2 - ab > 0$ and $L \equiv h^2(2a - 2b - d) + 2b^2(2a - d)$. Then periodic solutions of system (2.1) from Hopf bifurcation at $O(0, 0, 0, 0)$ have the following properties:*

(i) *if $L < 0$ holds, bifurcating periodic solutions exist for sufficient small $0 < k - k_0 < k - (h^2 - ab)(a + b)/b$. Moreover, periodic solutions of system (2.1) from Hopf bifurcation at $O(0, 0, 0, 0)$ is non-degenerate, subcritical and unstable;*
(ii) *if $L > 0$ holds, bifurcating periodic solutions exist for sufficient small $0 < k_0 - k < (h^2 - ab)(a + b)/b - k$. Moreover, periodic solutions of system (2.1) from Hopf bifurcation at $O(0, 0, 0, 0)$ is non-degenerate, supercritical and stable;*
(iii) *the period and characteristic exponent of the bifurcating periodic solution are:*

$$T = \frac{2\pi}{\omega_0}(1 + \tau_2\varepsilon^2 + O(\varepsilon^4)), \qquad \beta = \beta_2\varepsilon^2 + O(\varepsilon^4),$$

where $\omega_0 = \sqrt{a(h^2 - ab)/b}$ and

$$\tau_2 = \frac{a}{4(b^2 + 2ab^2 + ah^2)}\left[\frac{2b^2[2ah^4 + bh^2(a(4b + d) - 2a^2 - bd) + 2ab^3(d - 2a)]}{4ah^2 + bd^2 - 4a^2b} + \frac{(h^2 + b^2)[2b^2(2a - d) - h^2(d + 2b - 2a)][a^2b(b - 1) + b^4 + 2ab^3 + ah^2]}{((a + b)^2b - 1)(4ah^2 + bd^2 - 4a^2b)}\right],$$

$$\beta_2 = -\frac{ab^2(h^2 - ab)[h^2(2a - d - 2b) + 2b^2(2a - d)]}{(b^3 + 2ab^2 + ah^2)[bd^2 + 4a(h^2 - ab)]}, \qquad \varepsilon^2 = \frac{k - k_0}{\mu_2} + O[(k - k_0)^2],$$

$$\mu_2 = -\frac{a(h^2 - ab)[a(h^2 - ab) + b^2(a + b)^2][h^2(2a - d - 2b) + 2b^2(2a - d)]}{2b(b^3 + 2ab^2 + ah^2)[bd^2 + 4a(h^2 - ab)]}$$

(iv) *the expression of the periodic solution of system (2.1) from Hopf bifurcation is*

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} ab\sin\left(\frac{2\pi t}{T}\right) - \sqrt{ab(h^2-ab)}\cos\left(\frac{2\pi t}{T}\right) \\ ah\sin\left(\frac{2\pi t}{T}\right) \\ \frac{a^2bh}{2d} \\ (a+b)h\sqrt{a(h^2-ab)/b}\cos\left(\frac{2\pi t}{T}\right) \end{pmatrix}\varepsilon + \begin{pmatrix} 0 \\ 0 \\ K \\ 0 \end{pmatrix}\varepsilon^2 + O(\varepsilon^3),$$

where $K = -\frac{ab^2h}{2bd^2+8a(h^2-ab)}\left[\frac{ad^2-2a(h^2-ab)}{d}\cos(\frac{4\pi t}{T}) - (2a+d)\sqrt{a(h^2-ab)/b}\sin(\frac{4\pi t}{T})\right]$.

Proof. Let $k > h^2 - ab > 0$, $k = k_0 = (h^2-ab)(a+b)/b$ and $t_1 = \sqrt{a(h^2-ab)/b}$. By straightforward computations with the help of Mathematica 5.0, one can obtain

$$v_1 = \begin{pmatrix} -bt_1 - ab\mathrm{i} \\ -ah\mathrm{i} \\ 0 \\ h(a+b)t_1 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad v_4 = \begin{pmatrix} -h \\ b \\ 0 \\ h^2-ab \end{pmatrix},$$

which satisfy

$$Av_1 = \mathrm{i}t_1v_1, \qquad Av_3 = -dv_3, \qquad Av_4 = -(a+b)v_4.$$

For system (2.1), define

$$P = (\Re v_1, -\Im v_1, v_3, v_4) = \begin{pmatrix} -bt_1 & ab & 0 & -h \\ 0 & ah & 0 & b \\ 0 & 0 & 1 & 0 \\ h(a+b)t_1 & 0 & 0 & h^2-ab \end{pmatrix}, \tag{4.1}$$

and

$$(x, y, z, w)^T = P(x_1, y_1, z_1, w_1)^T.$$

Thus,

$$\begin{cases} \dot{x}_1 = -t_1y_1 + F_1(x_1, y_1, z_1, w_1) \\ \dot{y}_1 = t_1x_1 + F_2(x_1, y_1, z_1, w_1) \\ \dot{z}_1 = -dz_1 + F_3(x_1, y_1, z_1, w_1) \\ \dot{w}_1 = -(a+b)w_1 + F_4(x_1, y_1, z_1, w_1), \end{cases} \tag{4.2}$$

where

$$F_1(x_1, y_1, z_1, w_1) = \frac{(h^2-ab)[a(b^2+h^2)y_1 - t_1b^2x_1]z_1}{ht_1(b^3+2ab^2+ah^2)},$$

$$F_2(x_1, y_1, z_1, w_1) = \frac{[h(b^3+2ab^2+ah^2)w_1 + ab^2(h^2-b)y_1 + abt_1(b^2+h^2)x_1]z_1}{ah(b^3+2ab^2+ah^2)},$$

$$F_3(x_1, y_1, z_1, w_1) = -(hw_1 + bt_1x_1 - aby_1)(bw_1 + ahy_1),$$

$$F_4(x_1, y_1, z_1, w_1) = -\frac{(a+b)[a(h^2+b^2)y_1 - b^2t_1x_1]z_2}{b^3+2ab^2+ah^2}.$$

Furthermore,

$$g_{11} = \frac{1}{4}\left[\frac{\partial^2 F_1}{\partial x_1^2} + \frac{\partial^2 F_1}{\partial y_1^2} + \mathrm{i}\left(\frac{\partial^2 F_2}{\partial x_1^2} + \frac{\partial^2 F_2}{\partial y_1^2}\right)\right] = 0,$$

$$g_{02} = \frac{1}{4}\left[\frac{\partial^2 F_1}{\partial x_1^2} - \frac{\partial^2 F_1}{\partial y_1^2} - 2\frac{\partial^2 F_2}{\partial x_1\partial y_1} + \mathrm{i}\left(\frac{\partial^2 F_2}{\partial x_1^2} - \frac{\partial^2 F_2}{\partial y_1^2} + 2\frac{\partial^2 F_1}{\partial x_1\partial y_1}\right)\right] = 0,$$

$$g_{20} = \frac{1}{4}\left[\frac{\partial^2 F_1}{\partial x_1^2} - \frac{\partial^2 F_1}{\partial y_1^2} + 2\frac{\partial^2 F_2}{\partial x_1\partial y_1} + \mathrm{i}\left(\frac{\partial^2 F_2}{\partial x_1^2} - \frac{\partial^2 F_2}{\partial y_1^2} - 2\frac{\partial^2 F_1}{\partial x_1\partial y_1}\right)\right] = 0,$$

$$G_{21} = \frac{1}{8}\left[\frac{\partial^3 F_1}{\partial x_1^3} + \frac{\partial^3 F_1}{\partial x_1\partial y_1^2} + \frac{\partial^3 F_2}{\partial x_1^2\partial y_1} + \frac{\partial^3 F_2}{\partial y_1^3} + \mathrm{i}\left(\frac{\partial^3 F_2}{\partial x_1^3} + \frac{\partial^3 F_2}{\partial x_1\partial y_1^2} - \frac{\partial^3 F_1}{\partial x_1^2\partial y_1} - \frac{\partial^3 F_1}{\partial y_1^3}\right)\right] = 0.$$

Meanwhile, one has

$$h_{11}^1=\frac{1}{4}\left(\frac{\partial^2 F_3}{\partial x_1^2}+\frac{\partial^2 F_3}{\partial y_1^2}\right)=\frac{1}{2}a^2bh,\qquad h_{11}^2=\frac{1}{4}\left(\frac{\partial^2 F_4}{\partial x_1^2}+\frac{\partial^2 F_4}{\partial y_1^2}\right)=0,$$
$$h_{20}^1=\frac{1}{4}\left(\frac{\partial^2 F_3}{\partial x_1^2}-\frac{\partial^2 F_3}{\partial y_1^2}-2\mathrm{i}\frac{\partial^2 F_3}{\partial x_1\partial y_1}\right)=-\frac{1}{2}abh(a+t_1\mathrm{i}),$$
$$h_{20}^2=\frac{1}{4}\left(\frac{\partial^2 F_4}{\partial x_1^2}-\frac{\partial^2 F_4}{\partial y_1^2}-2\mathrm{i}\frac{\partial^2 F_4}{\partial x_1\partial y_1}\right)=0.$$

By solving the following equations:

$$Dw_{11}=-h_{11}\quad\text{and}\quad(D-2\mathrm{i}t_1I)w_{20}=-h_{20},$$

where

$$D=\begin{pmatrix}-d & 0\\ 0 & -(a+b)\end{pmatrix},\qquad h_{11}=\begin{pmatrix}h_{11}^1\\ h_{11}^2\end{pmatrix},\qquad h_{20}=\begin{pmatrix}h_{20}^1\\ h_{20}^2\end{pmatrix},$$

one obtains

$$w_{11}=\begin{pmatrix}\dfrac{a^2bh}{2d}\\ 0\end{pmatrix},\qquad w_{20}=\begin{pmatrix}-\dfrac{abh(a+t_1\mathrm{i})}{2(d+2t_1\mathrm{i})}\\ 0\end{pmatrix}.$$

Furthermore,

$$\begin{aligned}G_{110}^1&=\frac{1}{2}\left[\left(\frac{\partial^2F_1}{\partial x_1\partial z_1}+\frac{\partial^2F_2}{\partial y_1\partial z_1}\right)+\mathrm{i}\left(\frac{\partial^2F_2}{\partial x_1\partial z_1}-\frac{\partial^2F_1}{\partial y_1\partial z_1}\right)\right]\\&=\frac{1}{2}\left[\frac{-b^3t_1^2+ab^2(h^2-ab)}{ah(b^3+2ab^2+ah^2)}+\mathrm{i}\frac{t_1ab(h^2+b^2)-abt_1(h^2+b^2)}{ah(b^3+2ab^2+ah^2)}\right],\end{aligned}$$
$$G_{110}^2=\frac{1}{2}\left[\left(\frac{\partial^2F_1}{\partial x_1\partial w_1}+\frac{\partial^2F_2}{\partial y_1\partial w_1}\right)+\mathrm{i}\left(\frac{\partial^2F_2}{\partial x_1\partial w_1}-\frac{\partial^2F_1}{\partial y_1\partial w_1}\right)\right]=0,$$
$$\begin{aligned}G_{101}^1&=\frac{1}{2}\left[\left(\frac{\partial^2F_1}{\partial x_1\partial z_1}-\frac{\partial^2F_2}{\partial y_1\partial z_1}\right)+\mathrm{i}\left(\frac{\partial^2F_2}{\partial x_1\partial z_1}+\frac{\partial^2F_1}{\partial y_1\partial z_1}\right)\right]\\&=\frac{1}{2}\left[\frac{-b^3t_1^2-ab^2(h^2-ab)}{ah(b^3+2ab^2+ah^2)}+\mathrm{i}\frac{t_1ab(h^2+b^2)+abt_1(h^2+b^2)}{ah(b^3+2ab^2+ah^2)}\right],\end{aligned}$$
$$G_{101}^2=\frac{1}{2}\left[\left(\frac{\partial^2F_1}{\partial x_1\partial w_1}-\frac{\partial^2F_2}{\partial y_1\partial w_1}\right)+\mathrm{i}\left(\frac{\partial^2F_2}{\partial x_1\partial w_1}+\frac{\partial^2F_1}{\partial y_1\partial w_1}\right)\right]=0,$$
$$\begin{aligned}g_{21}&=G_{21}+\sum_{k=1}^{2}(2G_{110}^k\omega_{11}^k+G_{101}^k\omega_{20}^k)\\&=-ab\left[\frac{(h^2-ab)(h^2(2a-d-2b)+4ab^2-2b^2d)}{(d^2+4t_1^2)(b^3+2ab^2+ah^2)}\right.\\&\quad\left.+\ \mathrm{i}\frac{t_1(2ah^4+bh^2(a(4b+d)-bd-2a^2)+2ab^3(d-2a))}{h(d^2+4t_1^2)(b^3+2ab^2+ah^2)}\right].\end{aligned}$$

Based on the above calculation and analysis, one can compute the following quantities:

$$C_1(0)=\frac{\mathrm{i}}{2\omega_0}\left(g_{20}g_{11}-2|g_{11}|^2-\frac{1}{3}|g_{02}|^2\right)+\frac{1}{2}g_{21}=\frac{1}{2}g_{21},$$
$$\mu_2=-\frac{\Re\,C_1(0)}{\alpha'(0)}=-\frac{a(h^2-ab)[a(h^2-ab)+b^2(a+b)^2][h^2(2a-d-2b)+2b^2(2a-d)]}{2b(b^3+2ab^2+ah^2)[bd^2+4a(h^2-ab)]},$$
$$\begin{aligned}\tau_2&=-\frac{\Im\,C_1(0)+\mu_2\omega'(0)}{\omega_0}\\&=\frac{a}{4(b^2+2ab^2+ah^2)}\left[\frac{2b^2[2ah^4+bh^2(a(4b+d)-2a^2-bd)+2ab^3(d-2a)]}{4ah^2+bd^2-4a^2b}\right.\\&\quad\left.+\ \frac{(h^2+b^2)[2b^2(2a-d)-h^2(d+2b-2a)][a^2b(b-1)+b^4+2ab^3+ah^2]}{((a+b)^2b-1)(4ah^2+bd^2-4a^2b)}\right],\end{aligned}$$

$$\beta_2 = 2\Re C_1(0) = -\frac{ab^2(h^2 - ab)[h^2(2a - d - 2b) + 2b^2(2a - d)]}{(b^3 + 2ab^2 + ah^2)[bd^2 + 4a(h^2 - ab)]},$$

where

$$\omega_0 = \sqrt{a(h^2 - ab)/b}, \qquad \alpha'(0) = \Re(\lambda'(k_0)) = \frac{-b^3}{a(h^2 - ab) + b^2(a + b)^2},$$

$$\omega_0'(0) = \frac{(h^2 + b^2)\sqrt{ab(h^2 - ab)}}{2(h^2 - ab)[a(h^2 - ab) + b(a + b)^2]}.$$

Note $\alpha'(0) < 0$. From $a > 0$, $b > 0$ and $k > h^2 - ac > 0$, one obtains that if $h^2(2a - 2b - d) + 2b^2(2a - d) < 0$ holds, it follows that $\mu_2 > 0$ and $\beta_2 > 0$, which imply that the Hopf bifurcation of system (2.1) at $O(0, 0, 0, 0)$ is non-degenerate and subcritical, and the bifurcating periodic solution exists for $k > k_0$ and is unstable; if $h^2(2a-2b-d)+2b^2(2a-d) > 0$ holds, it follows that $\mu_2 < 0$ and $\beta_2 < 0$, which imply that the Hopf bifurcation of system (2.1) at $O(0, 0, 0, 0)$ is non-degenerate and supercritical, and the bifurcating periodic solution exists for $k < k_0$ and is stable.

Furthermore, the period and characteristic exponent are:

$$T = \frac{2\pi}{\omega_0}(1 + \tau_2\varepsilon^2 + O(\varepsilon^4)), \qquad \beta = \beta_2\varepsilon^2 + O(\varepsilon^4),$$

where $\varepsilon^2 = \frac{k-k_0}{\mu_2} + O[(k - k_0)^2]$. And the expression of the bifurcating periodic solution is (except for an arbitrary phase angle):

$$X = (x, y, z, w)^T = P(x_1, y_1, z_1, w_1)^T = PY,$$

where the matrix P is defined as in (4.1),

$$x_1 = \Re\, u, \qquad y_1 = \Im\, u, \qquad (z_1, w_1)^T = w_{11}\,|u| + \Re(w_{20}u^2) + O(|u|^3),$$

and

$$u = \varepsilon \mathrm{e}^{\frac{2\mathrm{i}t\pi}{T}} + \frac{\mathrm{i}\varepsilon^2}{6\omega_0}[g_{02}\mathrm{e}^{-\frac{4\mathrm{i}t\pi}{T}} - 3g_{20}\mathrm{e}^{\frac{4\mathrm{i}t\pi}{T}} + 6g_{11}] + O(\varepsilon^3) = \varepsilon \mathrm{e}^{\frac{2\mathrm{i}t\pi}{T}} + O(\varepsilon^3).$$

By tedious computations, one can obtain

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} \varepsilon\left[ab\sin\left(\frac{2\pi t}{T}\right) - \sqrt{ab(h^2 - ab)}\cos\left(\frac{2\pi t}{T}\right)\right] \\ \varepsilon ah\sin\left(\frac{2\pi t}{T}\right) \\ \varepsilon\frac{a^2bh}{2d} + \varepsilon^2 K \\ \varepsilon(a + b)h\sqrt{a(h^2 - ab)/b}\cos\left(\frac{2\pi t}{T}\right) \end{pmatrix} + O(\varepsilon^3),$$

where K is defined as in Theorem 4.2.

Based on the above discussion, the results of Theorem 4.2 are indeed established. □

In order to verify the above theoretical analysis, let $a = 4$, $b = d = 1$, and $h = 6.75$. and $k = 1300$. According to Theorem 4.1, one has $k_0 = 166.25$. Then, from Theorem 4.2, one can get $L = 241.8125 > 0$, which imply that the Hopf bifurcation of system (2.1) at $O(0, 0, 00)$ is non-degenerate and supercritical, a bifurcating periodic solution exists for $k < k_0 = 166.25$, and the bifurcating periodic solution is stable. In fact, Hopf bifurcation occurs when $k < k_0 = 166.25$, as shown in Fig. 11 (a)–(b).

5. Conclusions

In this paper, a 4D hyperchaotic system has been constructed by linearly adding a new variable to the 3D Rabinovich system. Some complex dynamical behaviors such as boundedness, chaos and hyperchaotic of the 4D autonomous system are investigated and analyzed. The corresponding bounded hyperchaotic and chaotic attractor is first numerically verified through investigating phase trajectories, Lyapunov exponents, bifurcation path and Poincaré projections. The geometric structure of the attractor is also investigated. A theoretical and numerical study indicates that chaos and hyperchaos are produced with the help of a Liénard-like oscillatory motion around a hypersaddle stationary point at the origin. Meanwhile, by using the normal form theory, two complete mathematical characterizations for 4D Hopf bifurcation are rigorously derived and studied. In particular, the ultimate bound for the new hyperchaotic system and the geometric structure of the attractor are investigated in detail. It will benefit future theoretical analysis and practical applications of new hyperchaotic systems. It is hoped that the investigation of the paper will shed some light to more systematic studies of 4D quadratic systems.

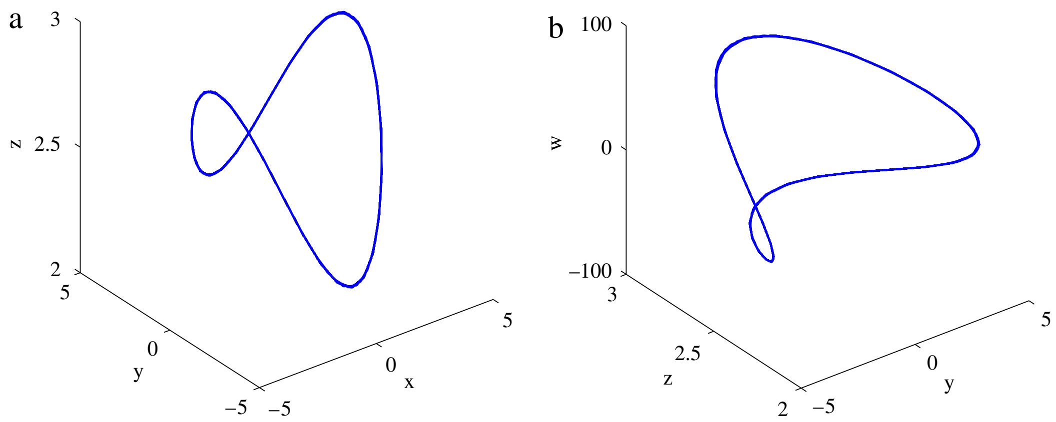

Fig. 11. Phase portraits of system (2.1) with $a = 4$, $b = d = 1$, $h = 6.75$ and $k = 166$.

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