Elementary Proof for a Van der Waerden's Conjecture and Related Theorems

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To my wife

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Abstract—A well-known conjecture of Van der Waerden says that for the permanent $\text{Per} A$ of an $n \times n$ doubly stochastic matrix $A$ we have (3.0), with equality if and only if all entries of the matrix $A$ equal to $n^{-1}$. In 1977 [1], the author proved that if $A$ is an $n \times n$ doubly stochastic matrix, and $p > 0$, $q > 0$, $p + q = 1$, then (2.0) holds with equality if and only if all entries of the matrix $A$ equal to $n^{-1}$. In this paper, we show that (3.0) and (2.0) are equivalent. On the basis of this equivalence one can say that the equivalent of the Van der Waerden's conjecture was solved already in 1977. A further subject of the paper is to show similar equivalence theorems concerning permanents of doubly stochastic matrices, moreover a refinement of the Van der Waerden's theorem. A separate section deals with the probabilistic interpretation of some previous results.

Keywords—Permanent, Doubly stochastic.

1. INTRODUCTION

Let $n \geq 2$ be a fixed integer, and let

$$\Gamma_k := \{(i_1, \ldots, i_k) \mid 1 < i_1 < \ldots < i_k \leq n\}, \quad (k = 1, \ldots, n),$$

be the set of all $\binom{n}{k}$ combinations of order $k$ of the elements $1, \ldots, n$ without repetition and without permutation.

Let $M$ be the set of all $n \times n$ matrices with real entries where the row and column sums equal to 1. Let $H \subseteq M$ be the set of the matrices with nonnegative entries, i.e., the set of the so-called doubly stochastic matrices. Denote $A_0 \in H$ the matrix with entries $n^{-1}$. Let $A^*$ denote the transpose of the matrix $A$.

If $A = (a_{jk})$ is an $n \times n$ matrix with real or complex entries, then the permanent of $A$, denoted by $\text{Per} A$, is denoted as follows:

$$\text{Per} A = \sum_{(i_1, \ldots, i_n)} a_{i_1i_1} \cdots a_{i_ni_n},$$

where the summation runs over all permutations $(i_1, \ldots, i_n)$ of the elements $1, \ldots, n$. The properties of the permanents, used in this paper, can be found, e.g., in [2].

Let $A = (a_{jk}) \in M$. The matrices

$$A_{i_1, \ldots, i_k}^{j_1, \ldots, j_k} := \begin{pmatrix}
a_{i_1j_1} & \cdots & a_{i_1j_k} \\
\vdots & \ddots & \vdots \\
a_{i_kj_1} & \cdots & a_{i_kj_k}
\end{pmatrix},$$

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are called $k$-rowed matrices of $A$ ($k = 1, \ldots, n$).

The permanent of a $k$-rowed matrix of $A \in \mathcal{M}$ will be called a $k$-rowed permanental minor of $A$. Now, we introduce the following functions of the $k$ rowed permanental minors of $A$. Let

$$T_k(A) := \sum \Per A_{i_1, \ldots, i_k}^{j_1, \ldots, j_k}, \quad (k = 1, \ldots, n).$$

Moreover, let

$$R_k(A) := \sum \Per A_{i_1, \ldots, i_k}^{j_1, \ldots, j_k} \sum_{A_{i_1, \ldots, i_k}^{j_1, \ldots, j_k}} A_{i_1, \ldots, i_k}^{j_1, \ldots, j_k}, \quad (k = 1, \ldots, n),$$

where

$$\sum_{B} = \sum_{j=1}^{n} \sum_{k=1}^{n} b_{jk}$$

denotes the sum of the entries of an $n \times n$ matrix $B = (b_{jk})$. The summation in (1.2) and (1.3) are extended over all combinations $(i_1, \ldots, i_k), (j_1, \ldots, j_k) \in \Gamma_k$.

We introduce the following notations:

$$S_k(A) := \frac{R_k(A)}{T_k(A)}, \quad (k = 1, \ldots, n),$$

$$P_k(A) := \frac{T_{k+1}(A)}{T_k(A)}, \quad (k = 1, \ldots, n - 1).$$

Finally, let

$$t_k(A) := \frac{1}{\binom{n}{k}^2} T_k(A), \quad (k = 1, \ldots, n);$$

i.e., $t_k(A)$ is the arithmetic mean of the $k$-rowed permanental minors of $A$. If $A \in \mathcal{M}$ then for $0 \leq x \leq 1$ obviously

$$A(x) := (1 - x)A_0 + xA \in \mathcal{M}.$$ 

Similarly, if $A \in \mathcal{H}$ then $A(x) \in \mathcal{H}$. The polynomial

$$B(x) := \Per A(x), \quad 0 \leq x \leq 1,$$

is called the Bernstein polynomial adjoined to the matrix $A$. Let $A \in \mathcal{M}$ and denote by $(AA^*)^{1/2}$ and $(A^*A)^{1/2}$ the positive semidefinite square root of $AA^*$ and $A^*A$, respectively.

For $p \geq 0$, $q \geq 0$, $p + q = 1$, $k = 1, \ldots, n$ let us introduce the following notations:

$$T_k^{(p)}(A) := p^2 T_k \left((AA^*)^{1/2}\right) + q^2 T_k \left((A^*A)^{1/2}\right) + 2pq T_k(A),$$

$$R_k^{(p)}(A) := p^2 R_k \left((AA^*)^{1/2}\right) + q^2 R_k \left((A^*A)^{1/2}\right) + 2pq R_k(A).$$

With the help of (1.9) and (1.10), we define the quantities

$$S_k^{(p)}(A) := \frac{R_k^{(p)}(A)}{T_k^{(p)}(A)}, \quad (k = 1, \ldots, n),$$

$$P_k^{(p)}(A) := \frac{T_{k+1}^{(p)}(A)}{T_k^{(p)}(A)}, \quad (k = 1, \ldots, n - 1).$$

Let $A \in \mathcal{M}$. Denote by $B_p(x)$ the weighted mean of the Bernstein polynomials adjoined to the matrices $(AA^*)^{1/2}$, $(A^*A)^{1/2}$ and $A$ with weights $p^2$, $q^2$ and $2pq$, respectively.
2. MAIN THEOREMS

The following propositions of the author play an important role in Sections 3 and 4. The first two of them were proved by the Cauchy-Binet expansion formula (see [2]). We assume that \( p \geq 0, q \geq 0, p + q = 1 \) always hold.

**Proposition 2.1.** [1, Theorem 2.1]. If \( A \in \mathcal{M} \) then

\[
p^2 \text{Per}((AA^*)^{1/2}) + q^2 \text{Per}((A^*A)^{1/2}) + 2pq \text{Per} A \geq \frac{n!}{n^n},
\]

with equality if and only if \( A = A_0 \).

**Proposition 2.2.** [3, Theorem 1.1]. If \( A \in \mathcal{M} \) then

\[
S_k^{(p)}(A) \geq \frac{k^2}{n}, \quad (k = 1, \ldots, n),
\]

with equality in the cases \( k = 1, \ldots, n - 1 \) if and only if \( A = A_0 \). In the case \( k = n \), equality holds for all \( A \in \mathcal{M} \).

The following three propositions are consequences of Proposition 2.2.

**Proposition 2.3.** [3, Theorem 1.2]. If \( A \in \mathcal{M} \) then

\[
P_k^{(p)}(A) \geq \frac{(n-k)^2}{n(n+1)}, \quad (k = 1, \ldots, n-1),
\]

with equality if and only if \( A = A_0 \).

**Proposition 2.4.** [3, Theorem 1.3]. If \( A \in \mathcal{M} \) then

\[
T_k^{(p)}(A) \geq \left(\frac{n}{k}\right)^2 \frac{k!}{n^k}, \quad (k = 1, \ldots, n),
\]

with equality if and only if \( A = A_0 \).

**Proposition 2.5.** [3, Theorem 1.3]. If \( A \in \mathcal{M} \) then

\[
R_k^{(p)}(A) \geq \left(\frac{n}{k}\right)^2 \frac{k^2}{n^{k+1}} \quad (k = 1, \ldots, n),
\]

with equality if and only if \( A = A_0 \).

Proposition 2.3 plays a role in the proof of the following proposition.

**Proposition 2.6.** [3, Theorem 1.4]. If \( A \in \mathcal{M}, A \neq A_0 \), then the weighted Bernstein polynomial \( B_p(x) \) adjoined to \( A \) is strictly monotone increasing on the interval \( 0 \leq x \leq 1 \), and \( B'_p(0) = B'_p(1) = 0 \).

Finally we prove a lemma which has a fundamental role in the proofs of Sections 3 and 4.

**Lemma 2.1.** If \( 0 < x_0 < 1 \) and \( k \) is a positive integer then the matrix equation

\[
(1 - x_0)A + x_0 (AA^*)^{2k+1} = A_0
\]

has only the trivial solution \( A = A_0 \) in \( \mathcal{M} \).

**Proof.** It is known that the spectral representation of \( A_0 \) is

\[
A_0 = V^*AV,
\]

(2.3)
where $V$ is the Vandermonde matrix generated by the $n$ unit roots and $A$ is a diagonal matrix with diagonal elements 1 with multiplicity 1 and zeros. Obviously a solution $A$ of (2.2) is symmetric. Let $A = U^* \Lambda_1 U$ be the spectral representation of $A$, where $\Lambda_1$ is a diagonal matrix with real diagonal elements $1, \alpha_2, \ldots, \alpha_n$. Using (2.3), we obtain
\[(AA^*)^{2k+1} = U^* \Lambda_1^{2k+1} U;\]
i.e., $U = V$ by (2.2). Consequently the equation (2.2) reduces to
\[(1 - x_0) \Lambda_1 + x_0 \Lambda_1^{2k+1} = \Lambda,\]
i.e., to
\[(1 - x_0) \alpha_j + x_0 \alpha_j^{2k+1} = 0, \quad (j = 2, \ldots, n).\]
(2.4)
The solutions of this equation are
\[\alpha_j = 0, \quad (2.5)\]
and
\[\alpha_j = \left(\frac{x_0 - 1}{x_0}\right)^{1/2k} \quad (2.6)\]
Since $(x_0 - 1)/x_0 < 0$ by the assumption, (2.6) is an imaginary number, contradicting that the diagonal elements of $\Lambda_1$ are real. Thus (2.5) is the only solution of (2.4). Consequently, $A = A_0$.

**LEMMA 2.2.** [1, Theorem 3.1]. If $A \in \mathcal{M}$ then
\[\text{Per } A \leq \text{Per } \left( (AA^*)^{1/2} \text{ Per } \left( (A^* A)^{1/2} \right) \right)^{1/2},\]
with equality if and only if $A$ is a symmetric positive semidefinite matrix.

We say that two theorems are equivalent if each of them are consequences of the other.

**3. VAN DER WAERDEN’S THEOREM**

It is known that the so-called Van der Waerden’s conjecture was formulated in 1926 in [4], and the first full solution of this conjecture was published in 1980 in [5] and in 1981 in [6]. Between these two authors a polemical discussion took place on the question who found first the basic idea of the proof. We refer to the proved Van der Waerden’s conjecture as Van der Waerden’s theorem. This theorem states the following.

**THEOREM 3.1.** If $A \in \mathcal{H}$ then
\[\text{Per } A \geq \frac{n!}{n^n}, \quad (3.1)\]
with equality if and only if $A = A_0$.

The proof of this theorem is a consequence of the following theorem, and the basic idea of the new proof of the Van der Waerden’s theorem is quite different from the method used in the papers [5,6].

The aim of this section is to show the following theorem.

**THEOREM 3.2.** Proposition 2.1 and Theorem 3.1 for $A \in \mathcal{H}$ are equivalent.

**PROOF.** Obviously, if Theorem 3.1 holds then Proposition 2.1 is also valid. Namely, if $A \in \mathcal{H}$ then
\[\text{Per } A \geq \frac{n!}{n^n}, \quad \text{Per } \left( (AA^*)^{1/2} \right) \geq \frac{n!}{n^n}, \quad \text{Per } \left( (A^* A)^{1/2} \right) \geq \frac{n!}{n^n},\]
by Theorem 3.1 with equality if and only if
\[A = (AA^*)^{1/2} = (A^* A)^{1/2} = A_0.\]
Consequently
\[ p^2 \text{Per} \left( (AA^*)^{1/2} \right) + q^2 \text{Per} \left( (A^*A)^{1/2} \right) + 2pq \text{Per} A \geq \frac{n!}{n^n}, \]
with equality if and only if
\[ \text{Per} \left( (AA^*)^{1/2} \right) = \text{Per} \left( (A^*A)^{1/2} \right) = \text{Per} A = \frac{n!}{n^n}, \]
i.e., if \( A = A_0 \) by Theorem 3.1. Thus we have to show conversely that Theorem 3.1 is a consequence of Proposition 2.1. First we prove the second part of Theorem 3.1.

**Theorem 3.3.** The only solution of the equation
\[ \text{Per} A = \frac{n!}{n^n} \quad (3.2) \]
is \( A = A_0 \) over \( \mathcal{H} \).

**Proof.** In order to prove this statement it is necessary to start from the following form of the second part of Proposition 2.1. The equation
\[ p^2 \text{Per} \left( (AA^*)^{1/2} \right) + q^2 \text{Per} \left( (A^*A)^{1/2} \right) + 2pq \text{Per} A = \frac{n!}{n^n} \]
has only the trivial solution \( A = A_0 \) over \( \mathcal{H} \). In other words, for a matrix \( A \in \mathcal{H} \) the identity
\[
\begin{align*}
    &p^2 \left[ \text{Per} \left( (AA^*)^{1/2} \right) + \text{Per} \left( (A^*A)^{1/2} \right) - 2\text{Per} A \right] - 2p \left[ \text{Per} \left( (A^*A)^{1/2} \right) - \text{Per} A \right] \\
    &\quad + \left[ \text{Per} \left( (A^*A)^{1/2} \right) - \frac{n!}{n^n} \right] = 0
\end{align*}
\]
holds for all \( 0 \leq p \leq 1 \) if and only if \( A = A_0 \). Consequently, for a matrix \( A \in \mathcal{H} \) the equations
\[
\begin{align*}
    \text{Per} A &= \frac{1}{2} \left[ \text{Per} \left( (AA^*)^{1/2} \right) + \text{Per} \left( (A^*A)^{1/2} \right) \right], \quad (3.3) \\
    \text{Per} A &= \text{Per} \left( (A^*A)^{1/2} \right), \quad (3.4) \\
    \text{Per} \left( (A^*A)^{1/2} \right) &= \frac{n!}{n^n} \quad (3.5)
\end{align*}
\]
are satisfied if and only if \( A = A_0 \).

Using the well-known inequality between arithmetic and geometric means we get
\[ \text{Per} A \geq \left( \text{Per} \left( (AA^*)^{1/2} \right) \text{Per} \left( (A^*A)^{1/2} \right) \right)^{1/2}, \]
by (3.3). Applying Lemma 2.2,
\[ \text{Per} A \leq \left( \text{Per} \left( (AA^*)^{1/2} \right) \text{Per} \left( (A^*A)^{1/2} \right) \right)^{1/2}. \]
Comparing these two inequalities, we have
\[ \text{Per} A = \left( \text{Per} \left( (AA^*)^{1/2} \right) \text{Per} \left( (A^*A)^{1/2} \right) \right)^{1/2}; \]
i.e., \( A \) is a symmetric positive semidefinite matrix by Lemma 2.2. Consequently (3.4) is a triviality, and the equation
\[ \text{Per} A = \frac{n!}{n^n}, \]
obtained by (3.4) and (3.5), has the only solution \( A = A_0 \) over \( \mathcal{H} \) taking into account Proposition 2.1 with \( p = 1 \).

It is known (e.g., [3, Conjecture 4.1]) that the first statement of Theorem 3.1 follows from Theorem 3.3. Now, we give a new proof for completeness.

Suppose that the first statement of Theorem 3.1 is not true; i.e., there exists matrix \( A \in \mathcal{H} \) such that

\[
\text{Per } A < \frac{n!}{n^n}. \tag{3.6}
\]

It is obvious that \( A \neq A_0 \). Let \( k \) be a positive integer. Since

\[
(\boldsymbol{A} \boldsymbol{A}^*)^{2k+1} \neq A_0,
\]

and it is positive semidefinite,

\[
\text{Per } (\boldsymbol{A} \boldsymbol{A}^*)^{2k+1} > \frac{n!}{n^n}, \tag{3.7}
\]

by Proposition 2.1. Let us introduce the matrix function

\[
B(x) := (1 - x)\boldsymbol{A} + x(\boldsymbol{A} \boldsymbol{A}^*)^{2k+1},
\]

defined on \( 0 \leq x \leq 1 \). It is evident that \( B(x) \in \mathcal{H} \) and

\[
\text{Per } B(0) < \frac{n!}{n^n}, \quad \text{Per } B(1) > \frac{n!}{n^n},
\]

by (3.6) and (3.7). Since \( \text{Per } B(x) \) is continuous on the interval \( 0 \leq x \leq 1 \), there exists \( x_0 \), \( 0 < x_0 < 1 \), such that

\[
\text{Per } B(x_0) = \frac{n!}{n^n}.
\]

But this equation holds if and only if

\[
(1 - x_0)\boldsymbol{A} + x_0(\boldsymbol{A} \boldsymbol{A}^*)^{2k+1} = A_0,
\]

by Theorem 3.3. However this matrix equation has the only solution \( A = A_0 \) over \( \mathcal{H} \) by Lemma 2.1, which contradicts the assumption \( A \neq A_0 \). This completes the proof of Theorem 3.1.

4. VAN DER WAERDEN TYPE THEOREMS

The following theorems will be proved in this section.

**THEOREM 4.1.** Let \( S_k(A) \) be defined by (1.4). If \( A \in \mathcal{H} \) then

\[
S_k(A) \geq \frac{k^2}{n}, \quad (k = 1, \ldots, n),
\]

with equality in the cases \( k = 1, \ldots, n - 1 \) if and only if \( A = A_0 \). In the case \( k = n \), equality holds for all \( A \in \mathcal{H} \).

**THEOREM 4.2.** Let \( P_k(A) \) be defined by (1.5). If \( A \in \mathcal{H} \) then

\[
P_k(A) \geq \frac{(n-k)^2}{n(k+1)}, \quad (k = 1, \ldots, n - 1),
\]

with equality if and only if \( A = A_0 \).
THEOREM 4.3. Let $T_k(A)$ be defined by (1.2). If $A \in \mathcal{H}$ then $T_1(A) = n^{-1}$ and

$$T_k(A) \geq \left( \frac{n}{k} \right)^2 \frac{k!}{n^k}, \quad (k = 2, \ldots, n),$$

with equality if and only if $A = A_0$.

For $k = n$ we get Van der Waerden’s theorem from Theorem 4.3. Thus, Theorem 4.3 is a generalization of Van der Waerden’s theorem.

THEOREM 4.4. Let $R_k(A)$ be defined by (1.3). If $A \in \mathcal{H}$ then

$$R_k(A) \geq \left( \frac{n}{k} \right)^2 \frac{k^2 k!}{n^{k+1}}, \quad (k = 1, \ldots, n),$$

with equality if and only if $A = A_0$.

The proofs of these theorems will be traced back one after another to the proofs of the following equivalence theorems.

THEOREM 4.5. Proposition 2.2 and Theorem 4.1 are equivalent.

THEOREM 4.6. Proposition 2.3 and Theorem 4.2 are equivalent.

THEOREM 4.7. Proposition 2.4 and Theorem 4.3 are equivalent.

THEOREM 4.8. Proposition 2.5 and Theorem 4.4 are equivalent.

We give only the proof of Theorem 4.5, partly because the proofs of the other equivalence theorems are similar to that of Theorem 3.2, and partly because the statement of Theorems 4.2–4.4 can be derived from Theorem 4.1 by the same method as Propositions 2.3–2.5 were derived from Proposition 2.2 [3].

PROOF OF THEOREM 4.5. Obviously, if Theorem 4.1 holds then Proposition 2.2 is valid, too. Namely, if $A \in \mathcal{H}$, then by Theorem 4.1

$$R_k \left( (AA^*)^{1/2} \right) \geq \frac{k^2}{n} T_k \left( (AA^*)^{1/2} \right), \quad R_k \left( (A^*A)^{1/2} \right) \geq \frac{k^2}{n} T_k \left( (A^*A)^{1/2} \right),$$

$$R_k(A) \geq \frac{k^2}{n} T_k(A),$$

with equality if and only if

$$(AA^*)^{1/2} = (A^*A)^{1/2} = A = A_0.$$

Consequently,

$$p^2 \left[ R_k \left( (AA^*)^{1/2} \right) - \frac{k^2}{n} T_k \left( (AA^*)^{1/2} \right) \right] + q^2 \left[ R_k \left( (A^*A)^{1/2} \right) - \frac{k^2}{n} T_k \left( (A^*A)^{1/2} \right) \right]$$

$$+ 2pq \left[ R_k(A) - \frac{k^2}{n} T_k(A) \right] \geq 0,$$

with equality if and only if

$$R_k \left( (AA^*)^{1/2} \right) = \frac{k^2}{n} T_k \left( (AA^*)^{1/2} \right), \quad R_k \left( (A^*A)^{1/2} \right) = \frac{k^2}{n} T_k \left( (A^*A)^{1/2} \right),$$

$$R_k(A) = \frac{k^2}{n} T_k(A),$$

i.e., if $A = A_0$ by Theorem 4.1.
Now we show conversely that Theorem 4.1 is a consequence of Proposition 2.2. If $k = n$, then the statement is trivial. We suppose now that $k = 1, \ldots, n - 1$. We show first the second part of Theorem 4.1.

**THEOREM 4.9.** Over $\mathcal{H}$, the only solution of the equation

$$S_k(A) = \frac{k^2}{n}, \quad (k = 1, \ldots, n - 1),$$

is $A = A_0$.

**PROOF.** We start from the second part of Proposition 2.2, which is the following. Over $\mathcal{H}$, the only solution of the equation

$$R_k^{(p)}(A) = \frac{k^2}{n} T_k^{(p)}(A), \quad (k = 1, \ldots, n - 1),$$

is $A = A_0$. Using the definition of $R_k^{(p)}(A)$ and $T_k^{(p)}(A)$, we get the identity

$$p^2 \left[ R_k \left( (AA^*)^{1/2} \right) + R_k \left( (A^*A)^{1/2} \right) - 2R_k(A) 
- \frac{k^2}{n} \left\{ T_k \left( (AA^*)^{1/2} \right) + T_k \left( (A^*A)^{1/2} \right) - 2T_k(A) \right\} \right] 
- 2p \left[ R_k \left( (A^*A)^{1/2} \right) - R_k(A) - \frac{k^2}{n} \left\{ T_k \left( (A^*A)^{1/2} \right) - T_k(A) \right\} \right] 
+ \left[ R_k \left( (A^*A)^{1/2} \right) - \frac{k^2}{n} T_k \left( (A^*A)^{1/2} \right) \right] \equiv 0,$$

which holds if and only if $A = A_0$. In other words, the equalities

$$R_k \left( (AA^*)^{1/2} \right) + R_k \left( (A^*A)^{1/2} \right) - 2R_k(A) 
- \frac{k^2}{n} \left\{ T_k \left( (AA^*)^{1/2} \right) + T_k \left( (A^*A)^{1/2} \right) - 2T_k(A) \right\} = 0,$$

$$R_k \left( (A^*A)^{1/2} \right) - R_k(A) - \frac{k^2}{n} \left\{ T_k \left( (A^*A)^{1/2} \right) - T_k(A) \right\} = 0,$$

$$R_k \left( (A^*A)^{1/2} \right) - \frac{k^2}{n} T_k \left( (A^*A)^{1/2} \right) = 0$$

are satisfied if and only $A = A_0$.

Using Proposition 2.2 we obtain that the equalities

$$R_k \left( (AA^*)^{1/2} \right) - \frac{k^2}{n} T_k \left( (AA^*)^{1/2} \right) = 0,$$

$$R_k \left( (A^*A)^{1/2} \right) - \frac{k^2}{n} T_k \left( (A^*A)^{1/2} \right) = 0$$

are satisfied over $\mathcal{H}$ if and only if $A = A_0$. Consequently, the equation

$$R_k \left( (A^*A)^{1/2} \right) - \frac{k^2}{n} T_k \left( (A^*A)^{1/2} \right) = R_k(A) - \frac{k^2}{n} T_k(A),$$

obtained by (4.2), shows us that

$$R_k(A) - \frac{k^2}{n} T_k(A) = 0$$
Van der Waerden's Conjecture

is satisfied over \( \mathcal{H} \) if and only if \( A = A_0 \). Equality (4.1) holds if and only if \( A = A_0 \) by (4.4)–(4.6). Thus the proof of Theorem 4.9 is finished.

We prove now the first statement of Theorem 4.1. Let us suppose that the first statement is not true; i.e., there exists a matrix \( A \in \mathcal{H} \) such that

\[
S_k(A) < \frac{k^2}{n}.
\]

(4.7)

It is evident that \( A \neq A_0 \). Let \( k \) be a positive integer. Since the matrix \((AA^*)^{2k+1} \neq A_0\) is positive semidefinite,

\[
S_k((AA^*)^{2k+1}) > \frac{k^2}{n},
\]

by [3, Theorem 3.1]. Let us introduce the matrix function

\[
B(x) := (1 - x)A + x(AA^*)^{2k+1},
\]

defined on the interval \([0,1]\). It is obvious that \( B(x) \in \mathcal{H} \) and

\[
S_k(B(0)) < \frac{k^2}{n}, \quad S_k(B(1)) > \frac{k^2}{n},
\]

by (4.7) and (4.8). Since the function \( S_k(B(x)) \) is continuous on the interval \([0,1]\), there exists a number \( x_0, 0 < x_0 < 1 \), such that

\[
S_k(B(x_0)) = \frac{k^2}{n}.
\]

As a consequence of Theorem 4.9, we get that necessarily

\[
(1 - x_0)A + x_0(AA^*)^{2k+1} = A_0.
\]

But this matrix equation has the only solution \( A = A_0 \) over \( \mathcal{H} \) by Lemma 2.1, contradicting the assumption \( A \neq A_0 \). This completes the proof of Theorem 4.1.

Finally we deal with a theorem having a proof different from the earlier ones. We shall use the following inequality.

**Lemma 4.1.** If

\[
a_1 \geq \cdots \geq a_n, \quad b_1 \geq \cdots \geq b_n
\]

are real numbers, then

\[
\frac{1}{n} \sum_{j=1}^{n} a_j b_j \geq \left( \frac{1}{n} \sum_{j=1}^{n} a_j \right) \left( \frac{1}{n} \sum_{j=1}^{n} b_j \right) \geq \left( \frac{1}{n} \sum_{j=1}^{n} a_j b_{n-j+1} \right),
\]

with equality in both inequalities if and only if either \( a_1 = a_n \) or \( b_1 = b_n \).

**Proof.** We prove only the left-hand side inequality, since the proof of the right-hand side is the same.

We use the following known inequality. If \( \{a_j\}_{j=1}^{n}, \{b_j\}_{j=1}^{n} \) are real numbers satisfying (4.9), then [7, Theorem 368]

\[
\sum_{j=1}^{n} a_j b_j \geq \sum_{j=1}^{n} a_j a_{i_j} \geq \sum_{j=1}^{n} a_j b_{n-j+1},
\]

with equality in both inequalities if and only if either \( a_1 = a_n \) or \( b_1 = b_n \).

We may explain inequality (4.11) saying that the maximum corresponds to ‘similar ordering’ of \( a_1, \ldots, a_n \) and \( b_1, \ldots, b_n \), the minimum to ‘opposite ordering.’
Using the left-hand side of (4.11), we have the inequalities

\[ S_1 = a_1b_1 + a_2b_2 + \cdots + a_nb_{n-1} + a_nb_n, \]
\[ S_1 \geq a_1b_n + a_2b_1 + \cdots + a_nb_{n-2} + a_nb_{n-1} = S_2, \]
\[ S_1 \geq a_1b_{n-1} + a_2b_n + \cdots + a_nb_{n-3} + a_nb_{n-2} = S_3, \]
\[ \vdots \]
\[ S_1 \geq a_1b_2 + a_2b_3 + \cdots + a_nb_{n-1} + a_nb_1 = S_n. \]  

(4.12)

By summation of these inequalities, we get the first statement of the left-hand side of (4.10). Now we deal with the second statement of the left-hand side of (4.10). If \( a_1 = a_n \) or \( b_1 = b_n \) the equality holds on the left-hand side of (4.10). Let us suppose that equality holds on the left-hand side of (4.10). In this case necessarily

\[ S_1 = S_2 = \cdots = S_n, \]

by (4.12). Consequently,

\[ S_1 - S_2 = a_1(b_1 - b_n) + \sum_{j=2}^{n} a_j(b_j - b_{j-1}) = 0, \]
\[ S_1 - S_n = a_n(b_n - b_1) + \sum_{j=1}^{n-1} a_j(b_j - b_{j+1}) = 0, \]

and we see that

\[ (S_1 - S_n) + (S_1 - S_2) = (a_1 - a_n)(b_1 - b_n) + \sum_{j=1}^{n-1} (a_j - a_{j+1})(b_j - b_{j+1}) = 0. \]

(4.13)

Since

\[ (a_j - a_k)(b_j - b_k) \geq 0, \quad j < k, \]

by (4.9), all summands of the right-hand side of (4.13) thus equal to zero; i.e.,

\[ (a_1 - a_n)(b_1 - b_n) = 0. \]

From here either \( a_1 = a_n \) or \( b_1 = b_n \) in agreement with statement of the left-hand side of (4.10). □

We need also the following lemma.

**Lemma 4.2.** If \( A \in \mathcal{H} \) and \( k \) is an integer satisfying \( 2 \leq k \leq n - 1 \), then

\[ \sum A_{i_1 \ldots i_k} = \frac{k^2}{n} \]  

(4.14)

holds for all index pairs (1.1) if and only if \( A = A_0 \).

**Proof.** If \( A = A_0 \), then statement (4.14) holds trivially. If \( k = 1 \), the statement is obvious. In this case it is sufficient to assume that \( A \in M \). Let \( 2 \leq k \leq n - 1 \) in agreement with the condition of the theorem. Let \( (i_1, \ldots, i_k) \) be a fixed element of \( \Gamma_k \). Let the elements of \( (j_1, \ldots, j_k) \in \Gamma_k \) and of \( (j'_1, \ldots, j'_k) \in \Gamma_k \) be the same, except one. Let \( j \) and \( j' \) be these ones. Then

\[ a_{i_1j} + \cdots + a_{i_1j'} = a_{i_1j'} + \cdots + a_{i_1j}, \]

by

\[ \sum A_{i_1 \ldots i_k} = \sum A_{i_1 \ldots i_k}, \]
Van der Waerden's Conjecture

using condition (4.14). I.e., we obtained that

\[ a_{i_1j} + \cdots + a_{i_kj} = a, \quad (j = 1, \ldots, n), \]  

(4.15)

where the quantity \( a \) may depend on \((i_1, \ldots, i_k) \in \Gamma_k\). Summing the equations (4.15), we get

\[ a = \frac{k}{n}, \]  

(4.16)

by the condition \( A \in \mathcal{H} \); i.e., \( a \) is independent on the elements of \( \Gamma_k \).

Assume that the combinations

\[(i_1, \ldots, i_k), (i'_1, \ldots, i'_k) \in \Gamma,\]

have the same elements except one. Let \( i \) and \( i' \) be these. Then

\[ a_{ij} = a_{i'j}, \quad (i, i' = 1, \ldots, n), \]  

(4.17)

by (4.15) and (4.16); i.e., the columns of \( A \) have the same entries.

Since \( A \in \mathcal{H} \) we obtain the similar result

\[ a_{ij} = a_{ij'}, \quad (j, j' = 1, \ldots, n). \]  

(4.18)

Let now

\[ 1 \leq i, j \leq n, \quad 1 \leq k, \ell \leq n \]

be two arbitrary pairs of indices. Then

\[ a_{ij} = a_{kj} = a_{k\ell}, \]

by (4.17) and (4.18), which completes the proof of the lemma.

We introduce the following notations. Let \( A \in \mathcal{H} \) and let \( s \) be an arbitrary positive integer. Then let

\[ Q_k^{(s)}(A) := \sum A^{i_1 \cdots i_k}, \quad (k = 1, \ldots, n), \]

where the summations are extended over (1.1). Moreover, let

\[ q_k^{(s)}(A) := \frac{1}{\binom{n}{k}} Q_k^{(s)}(A), \quad (k = 1, \ldots, n). \]

We prove the following statement.

**Theorem 4.10.** If \( A \in \mathcal{H} \), then

\[ q_k^{(s)}(A) \geq \left( \frac{k^2}{n} \right)^s, \quad (k = 1, \ldots, n; \ s = 2, 3, \ldots), \]

with equality if and only if \( A = A_0 \).

**Proof.** Let

\[(i_1, \ldots, i_k, i_{k+1}, \ldots, i_n), \quad (j_1, \ldots, j_k, j_{k+1}, \ldots, j_n) \]

be two permutations of the elements \( 1, \ldots, n \) without repetition, and let

\[(i_1, \ldots, i_k), (j_1, \ldots, j_k) \in \Gamma_k, \]

\[(i_{k+1}, \ldots, i_n), (j_{k+1}, \ldots, j_n) \in \Gamma_{n-k}. \]
If \( A \in \mathcal{H} \), then it is easy to prove that

\[
\begin{align*}
\sum A_{i_1 \ldots i_n}^{j_1 \ldots j_k} + \sum A_{i_1 \ldots i_n}^{j_{k+1} \ldots j_n} &= k, \\
\sum A_{i_1 \ldots i_n}^{j_1 \ldots j_k} + \sum A_{i_1 \ldots i_n}^{j_{k+1} \ldots j_n} &= n - k, \\
\sum A_{i_1 \ldots i_n}^{j_1 \ldots j_k} + \sum A_{i_1 \ldots i_n}^{j_{k+1} \ldots j_n} &= k, \\
\sum A_{i_1 \ldots i_n}^{j_{k+1} \ldots j_n} + \sum A_{i_1 \ldots i_n}^{j_1 \ldots j_k} &= n - k.
\end{align*}
\]

From identities (4.19) and (4.21) we get

\[
\sum A_{i_1 \ldots i_n}^{j_{k+1} \ldots j_n} = \sum A_{i_1 \ldots i_n}^{j_1 \ldots j_k},
\]

which gives us

\[
\sum A_{i_1 \ldots i_n}^{j_{k+1} \ldots j_n} = \sum A_{i_1 \ldots i_n}^{j_1 \ldots j_k} + n - 2k, \tag{4.22}
\]

by (4.19) and (4.20).

If we multiply both sides of the identity (4.22) by

\[
\sum_{s=1}^{s-1} A_{i_1 \ldots i_n}^{j_1 \ldots j_k},
\]

where \( s \geq 2 \) is an integer, then we have

\[
\sum \sum A_{i_1 \ldots i_n}^{j_1 \ldots j_k} \sum_{s=1}^{s-1} A_{i_1 \ldots i_n}^{j_1 \ldots j_k} = Q_k^{(s)}(A) + (n - 2k)Q_{k-1}(n-1)(A), \tag{4.23}
\]

where the summations are extended over (1.1). Let the \((\binom{n}{k})\) \(k\)-dimensional vectors

\[
V_k(A) = \left\{ \sum A_{i_1 \ldots i_n}^{j_1 \ldots j_k} \right\}, \quad V_{n-k}(A) = \left\{ \sum A_{i_1 \ldots i_n}^{j_{k+1} \ldots j_n} \right\},
\]

be given. Suppose that the components

\[
\sum A_{i_1 \ldots i_n}^{j_1 \ldots j_k} \quad \text{and} \quad \sum A_{i_1 \ldots i_n}^{j_{k+1} \ldots j_n}
\]

have the same component index in \( V_k(A) \) and \( V_{n-k}(A) \), respectively. Then the vectors \( V_k(A) \) and \( V_{n-k}(A) \) are of similar ordering by (4.22). Then

\[
\frac{1}{\binom{n}{k}^2} \sum \sum A_{i_1 \ldots i_n}^{j_1 \ldots j_k} \sum_{s=1}^{s-1} A_{i_1 \ldots i_n}^{j_1 \ldots j_k} \geq \frac{(n-k)^2}{n} q_k^{(s-1)}(A), \tag{4.24}
\]

by Lemma 4.1, where the summations are extended over (1.1). Equality holds in (4.24) by Lemma 4.1 if and only if either the components of \( V_{n-k}(A) \) equal to each other or the components of \( V_k(A) \) equal to each other, in both cases if and only if \( A = A_0 \) by Lemma 4.2. Using inequality (4.24) and the identity (4.23), we obtain

\[
q_k^{(s)}(A) + (n - 2k)q_{k-1}(A) = \frac{(n-k)^2}{n} q_k^{(s-1)}(A),
\]

with equality if and only if \( A = A_0 \), i.e.,

\[
q_k^{(s)}(A) \geq \frac{k^2}{n} q_k^{(s-1)}(A), \tag{4.25}
\]

with equality if and only if \( A = A_0 \). Applying inequality (4.25) successively, we get the statement of the theorem by \( q_k^{(1)}(A) = k^2/n \). \( \square \)
5. REFINEMENT OF VAN DER WAERDEN’S THEOREM

The proof of the following theorems are based on Theorem 4.2.

**Theorem 5.1.** Let \( A \in \mathcal{H} \) and \( A \neq A_0 \). Then the Bernstein polynomial \( B(x) \) adjoined to \( A \) is strictly increasing on the interval \( 0 \leq x \leq 1 \) and

\[
B'(0) = B'(1) = 0.
\]

**Proof.** (See the proof of Theorem 1.6 in [3] and Proposition 2.4.) Let \( A \in \mathcal{H} \) and \( A \neq A_0 \). It is easy to see that \( A(x) \neq A_0 \) if \( 0 < x_0 \leq 1 \), where the matrix function \( A(x) \) is defined by (1.7). Let

\[
B(x) = \text{Per } A(x)
\]

be the Bernstein polynomial adjoined to \( A \). By elementary permanent transformation we get that

\[
B(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} c(k),
\]

where

\[
c(k) = \frac{1}{(n)_k} \frac{(n-k)!}{n^{n-k}} T_k(A), \quad (k = 0, 1, \ldots, n).
\]

Using Theorem 4.2 we have

\[
\frac{c(k+1)}{c(k)} = \frac{n(n+1)}{(n-k)^2} P_k(A) > 1, \quad (k = 1, \ldots, n), \tag{5.1}
\]

if \( A \neq A_0 \). It is known [8, p. 179] that

\[
B'(x) = n \sum_{k=0}^{n-1} \binom{n-1}{k} x^k (1-x)^{n-k} \Delta c(k),
\]

with

\[
\Delta c(k) = c(k+1) - c(k), \quad (k = 0, 1, \ldots, n-1).
\]

Using the facts

\[
c(0) = c(1) = \frac{n!}{n^n},
\]

and

\[
\Delta c(k) > 0, \quad (k = 1, \ldots, n),
\]

by (5.1), we obtain the statement of the theorem.

**Theorem 5.2.** Let \( A \in \mathcal{H} \). Let \( t_k(A) \) be defined by (1.6). Then

\[
\text{Per } A = t_n(A) \geq \frac{(n)_1}{n} t_{n-1}(A) \geq \cdots \geq \frac{(n)_k}{n^k} t_{n-k}(A) \geq \cdots \geq \frac{(n)_{n-1}}{n^{n-1}} t_1(A) = \frac{n!}{n^n},
\]

with equality if and only if \( A \neq A_0 \), where

\[
(n)_k := \frac{n!}{k!}, \quad (k = 0, 1, \ldots, n).
\]

**Proof.** The statement can be proved easily by the following inequality based on Theorem 4.2. If \( A \in \mathcal{H} \), then

\[
t_{k+1}(A) \geq \frac{k+1}{n} t_k(A), \quad (k = 1, \ldots, n-1),
\]

with equality if and only if \( A \neq A_0 \).
6. PROBABILISTIC INTERPRETATION

The aim of this section is to give a probabilistic interpretation of Theorems 4.1 and 4.3.

INTERPRETATION A. Let \( A \in \mathcal{H} \). Let \( 1 \leq k \leq n \) be a fixed integer. We define the random variable \( X_k(A) \) as follows:

\[
P \left\{ X_k(A) = \text{Sum } A_{i_1 \ldots i_k}^{j_1 \ldots j_k} \right\} = \frac{\text{Per} \left( A_{i_1 \ldots i_k}^{j_1 \ldots j_k} \right)}{T_k(A)},
\]

where \( T_k(A) \) is defined by (1.2), and \( (i_1, \ldots, i_k), (j_1, \ldots, j_k) \) run over \( \Gamma_k \) independently of one another. Consequently, the expectation \( E(X_k(A)) \) of \( X_k(A) \) is given by

\[
E(X_k(A)) = S_k(A),
\]

where the quantity \( S_k(A) \) is defined by (1.4). We get the following interpretation of Theorem 4.1.

**THEOREM 6.1.** The minimum of the expectations \( E(X_k(A)) \) when \( A \in \mathcal{H} \) equals to \( k^2/n \), and

\[
\frac{k^2}{n}, \quad (k = 0, 1, \ldots, n),
\]

if and only if \( A = A_0 \).

INTERPRETATION B. Denote by \( 1, \ldots, n \) the \( n \) pieces of distinct particles and \( n \) urns, respectively. Let us throw, one after the other, the particles independently into one of the urns. Denote by \( a_{jk} \) the probability that particle \( j \) falls into the urn \( k \). Obviously

\[
\sum_{k=1}^{n} a_{jk} = 1, \quad (j = 1, \ldots, n).
\]

(6.1)

Let us choose the probabilities \( a_{jk} \) in such a way that all urns contain one particle; i.e., let

\[
\sum_{j=1}^{n} a_{jk} = 1, \quad (k = 1, \ldots, n).
\]

(6.2)

Comparing (6.1) and (6.2) we get that \( A = (a_{jk}) \in \mathcal{H} \).

Let \( E_k(A) \) be the event that exactly one particle falls into the \( k \) pieces among the \( n \) urns. The question is what is the probability \( P(E_k(A)) \) of the event \( E_k(A) \)? To solve this problem we choose randomly the particles \( i_1, \ldots, i_k \) and the urns \( j_1, \ldots, j_k \). Then we consider the event \( E_{i_1 \ldots i_k}^{j_1 \ldots j_k} \) that exactly one particle falls into these urns. It is obvious that the events

\[
E_{i_1 \ldots i_k}^{j_1 \ldots j_k}, \quad (i_1, \ldots, i_k), (j_1, \ldots, j_k) \in \Gamma_k,
\]

are disjoint; moreover

\[
P \left( E_{i_1 \ldots i_k}^{j_1 \ldots j_k} \right) = \text{Per} A_{i_1 \ldots i_k}^{j_1 \ldots j_k},
\]

(6.3)

and

\[
E_{i_1 \ldots i_k}^{j_1 \ldots j_k} = \{ E_k(A) \mid (i_1, \ldots, i_k), (j_1, \ldots, j_k) \}. \]

(6.4)

Applying Bayes' theorem,

\[
P(E_k(A)) = \sum P \{ E_k(A) \mid (i_1, \ldots, i_k), (j_1, \ldots, j_k) \} P \{ (i_1, \ldots, i_k), (j_1, \ldots, j_k) \},
\]

where the summation is extended over (1.1). Using (6.4) and (6.3), and assuming that the Bayes principle holds, i.e.,

\[
P \{ i_1, \ldots, i_k), (j_1, \ldots, j_k) \} = \frac{1}{\binom{n}{k}^2},
\]

(6.5)
for all pairs of combinations (1.1), we get that

$$P(E_k(A)) = t_k(A),$$

where $t_k(A)$ is defined by (1.6).

Using Theorem 4.3, we can give the following interpretation.

**Theorem 6.2.** If the Bayes principle (6.5) is satisfied, then the minimum of the probability function $P(E_k(A))$ when $A \in \mathcal{H}$ equals to $k!/n^k$, and the equalities

$$P(E_k(A)) = \frac{k!}{n^k}, \quad (k = 1, \ldots, n),$$

hold if and only if $A = A_0$.

**Remark.** The interpretation of Theorem 6.2 was known in the case $k = n$, i.e., the interpretation of Van der Waerden’s theorem.

It is not difficult to show that

$$\sum \text{Sum} A_{i_1 \ldots i_k}^{j_1 \ldots j_k} = n \binom{n-1}{k-1}^2, \quad (k = 1, \ldots, n),$$

if the summation is extended over the set (1.1). Assume that

$$P\{(i_1, \ldots, i_k), (j_1, \ldots, j_k)\} = \frac{\text{Sum} A_{i_1 \ldots i_k}^{j_1 \ldots j_k}}{n \binom{n-1}{k-1}^2}, \quad (6.6)$$

where the indices run over (1.1). Then we get that

$$P(E_k(A)) = \frac{R_k(A)}{n \binom{n-1}{k-1}^2}, \quad (k = 1, \ldots, n),$$

where $R_k(A)$ is defined by (1.3). We obtain the following interpretation of Theorem 4.4.

**Theorem 6.3.** If assumption (6.6) is satisfied, then the minimum of the probability $P(E_k(A))$ when $A \in \mathcal{M}$ equals to $k!/n^k$, and the equalities

$$P(E_k(A)) = \frac{k!}{n^k}, \quad (k = 1, \ldots, n),$$

hold if and only if $A = A_0$.

**References**