Existence Results for Singular Integral Equations of Fredholm Type

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Abstract—New existence results are presented for the singular integral equation
\[ y(t) = \theta(t) + \int_0^1 k(t, s)[g(y(s)) + h(y(s))] \, ds, \quad t \in [0, 1]. \]
Our nonlinear term \( g + h \) may be singular at \( y = 0 \). © 2000 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

This paper discusses the existence of one (or indeed more) solutions to singular integral equations of the form
\[ y(t) = \theta(t) + \int_0^1 k(t, s)[g(y(s)) + h(y(s))] \, ds, \quad t \in [0, 1]; \quad (1.1) \]
here our nonlinearity \( g + h \) may be singular at \( y = 0 \). In [1], we established the existence of one nonnegative solution to (1.1) using the Leray-Schauder alternative. However, in [1], we had to assume a rather strong lower-growth type assumption, namely that
\[ \text{there exists a } \psi \geq 0 \text{ continuous on } [0, 1] \text{ with } g(u) + h(u) \geq \psi(t) \text{ on } (0, \infty) \text{ and there exists a subset } I \text{ of } [0, 1] \text{ of measure zero with } \theta(t) + \int_0^1 k(t, s)\psi(s) \, ds \geq 0 \text{ for } t \in [0, 1] \setminus I. \quad (1.2) \]
So for example if
\[ g(u) + h(u) = u^{-\alpha} + u^\beta + A, \quad \alpha > 0, \quad \beta > 0, \]
we need to assume usually that \( A > 0 \). The case \( A = 0 \) was not discussed in [1], and this is the situation that occurs most frequently in applications (for example, in fluid dynamics [2,3]). In this paper, we present results for the case \( A = 0 \) (in particular, we will remove assumption (1.2) and replace it with a concavity type assumption). To do so, we will use Krasnoselskii’s fixed-point theorem in a cone. This paper has two main sections. Section 2 presents results for singular problems, whereas Section 3 discusses nonsingular problems.

For the remainder of this section, we present some results from the literature which will be needed in Sections 2 and 3. First we state Krasnoselskii’s fixed-point theorem in a cone.
THEOREM 1.1. Let \( E = (E, \| \cdot \|) \) be a Banach space and let \( K \subset E \) be a cone in \( E \). Assume \( \Omega_1, \Omega_2 \) are open subsets of \( E \) with \( 0 \in \Omega_1, \), \( \overline{\Omega}_1 \subseteq \Omega_2 \) and let

\[
A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K
\]

be a continuous, completely continuous map such that either

(i) \( \| Au \| \leq \| u \|, u \in K \cap \partial \Omega_1 \) and \( \| Au \| \geq \| u \|, u \in K \cap \partial \Omega_2 \)

or

(ii) \( \| Au \| \geq \| u \|, u \in K \cap \partial \Omega_1 \) and \( \| Au \| \leq \| u \|, u \in K \cap \partial \Omega_2 \)

is true. Then \( A \) has a fixed point in \( K \cap (\overline{\Omega}_2 \setminus \Omega_1) \).

In this paper, \( E = (C[0, 1], | \cdot |_0) \) (here, \( |u|_0 = \sup_{t \in [0, 1]} |u(t)|, u \in C[0, 1] \)) will be our Banach space and

\[
K = \{ y \in C[0, 1] : y(t) \geq 0, \text{ for } t \in [0, 1] \text{ and } y(t) \text{ is concave on } [0, 1] \}. \quad (1.3)
\]

Let \( \tau : [0, 1] \times [0, 1] \to [0, \infty) \) be defined by

\[
\tau(t, s) = \begin{cases} 
\frac{t}{s}, & \text{if } 0 \leq t \leq s, \\
1 - t, & \text{if } s < t < 1.
\end{cases}
\]

The following result is easy to prove and is well known.

THEOREM 1.2. Let \( y \in K \) (as in (1.3)). Then there exists \( t_0 \in [0, 1] \) with \( y(t_0) = |y|_0 \) and

\[
y(t) \geq \tau(t, t_0)|y|_0 \geq t(1 - t)|y|_0, \quad \text{for } t \in [0, 1].
\]

PROOF. The existence of \( t_0 \) is immediate. Now if \( 0 \leq t \leq t_0 \), then since \( y(t) \) is concave on \([0, 1]\), we have

\[
y(t) = y \left( \left( 1 - \frac{t}{t_0} \right) 0 + \frac{t}{t_0} y(t_0) \right) \geq \left( 1 - \frac{t}{t_0} \right) y(0) + \frac{t}{t_0} y(t_0).
\]

That is,

\[
y(t) \geq \frac{t}{t_0} y(t_0) = \tau(t, t_0)|y|_0 \geq t(1 - t)|y|_0.
\]

A similar argument establishes the result if \( t_0 < t \leq 1 \). \( \Box \)

2. SINGULAR PROBLEMS

In this section, we are interested in proving the existence of one (or more) nonnegative solutions to the nonlinear integral equation

\[
y(t) = \theta(t) + \int_0^1 k(t, s)[g(y(s)) + h(y(s))] ds, \quad \text{for } t \in [0, 1]. \quad (2.1)
\]

THEOREM 2.1. Choose \( a \in (0, (1/2)) \) and fix it. Suppose the following conditions are satisfied:

\[
0 \leq k_t(s) = k(t, s) \in L^p[0, 1], \quad \text{for each } t \in [0, 1]; \quad \text{here } 1 \leq p \leq \infty \text{ is a constant}, \quad (2.2)
\]

the map \( t \to k_t \) is continuous from \([0, 1]\) to \( L^p[0, 1], \quad (2.3)\)

\[
g > 0 \text{ is continuous and nonincreasing on } (0, \infty), \quad (2.4)
\]

\[
h \geq 0 \text{ continuous on } [0, \infty) \text{ with } \frac{h}{g} \text{ nondecreasing on } (0, \infty), \quad (2.5)
\]

there exists a constant \( K_0 > 0 \) with \( g(ab) \leq K_0 g(a)g(b), \quad \text{for all } a \geq 0, b \geq 0, \quad (2.6)\)

\[
\theta \in C[0, 1] \text{ with } \theta(t) \geq 0, \quad \text{for } t \in [0, 1]. \quad (2.7)
\]
there exists $m_0 \in \{1, 2, \ldots \}$ such that for any $y \in C[0,1]$ with $y(t) \geq 0$ for $t \in [0, 1]$ and $y(t)$ concave on $[0, 1]$, we have that $F_m y(t) = \frac{1}{m} + \theta(t) + \int_0^1 k(t, s) [g^*(y(s)) + h(y(s))] \, ds$ is concave on $[0, 1]$ for every $m \in \{m_0, m_0 + 1, \ldots \}$; here $g^*(u) = g(u)$ if $u \geq \frac{1}{m}$ and $g^*(u) = g \left( \frac{1}{m} \right)$ if $0 \leq u \leq \frac{1}{m}$.

\[ \int_0^1 [g(s(1 - s))]^q \, ds < \infty; \text{ here } \frac{1}{p} + \frac{1}{q} = 1 \text{ (i.e., } q \text{ is the conjugate of } p), \]  

(2.8)

and

\[ \text{there exists a constant } r > 0 \text{ with } \frac{r}{\theta_0 + a_0 K_0 \{g(r) + h(r)\}} > 1; \]  

(2.9)

here $|\theta_0| = \sup_{t \in [0,1]} |\theta(t)|$ and $a_0 = \sup_{t \in [0,1]} \int_0^1 k(t, s) g(s(1 - s)) \, ds$. Finally assume there exists $R > r$ and $\sigma \in [0, 1]$ with

\[ \frac{R g(a(1-a)R)}{[g(R)g(a(1-a)R) + g(R)h(a(1-a)R)] \int_0^1 k(\sigma, s) \, ds + \theta_0 g(a(1-a)R)} \leq 1. \]  

(2.10)

Then (2.1) has a solution $y \in C[0,1]$ with $y > 0$ on $(0, 1)$ and $r < |y_0| \leq R$.

**Proof.** To show the existence of the solution to (2.1), we will apply Theorem 1.1. First choose $\epsilon > 0$ and $\epsilon < r$ with

\[ \epsilon + |\theta_0| + a_0 K_0 \{g(r) + h(r)\} > 1. \]  

(2.11)

Let $m_0 \in \{1, 2, \ldots \}$ be chosen so that $1/m_0 < \epsilon$, $1/m_0 < a(1-a)R$ and that (2.8) is true for $m \in \{m_0, m_0 + 1, \ldots \}$. Let $N_0 = \{m_0, m_0 + 1, \ldots \}$. We first show that

\[ y(t) = \frac{1}{m} + \theta(t) + \int_0^1 k(t, s) [g(y(s)) + h(y(s))] \, ds, \quad \text{for } t \in [0,1] \]  

(2.12)

has a solution $y_m$ for each $m \in N_0$ with $y_m \geq 1/m$ on $[0,1]$ and $r \leq |y_m|_0 \leq R$. To show (2.12) has such a solution for each $m \in N_0$, we will look at

\[ y(t) = \frac{1}{m} + \theta(t) + \int_0^1 k(t, s) \, g^*(y(s)) + h(y(s)) \, ds, \quad \text{for } t \in [0,1] \]  

(2.13)

with

\[ g^*(u) = \begin{cases} g(u), & u \geq \frac{1}{m}, \\ g \left( \frac{1}{m} \right), & 0 \leq u \leq \frac{1}{m}. \end{cases} \]

**Remark 2.1.** Notice $g^*(u) \leq g(u)$ for $u \geq 0$.

Fix $m \in N_0$. Let $E = (C[0,1], |.|_0)$ and

\[ K = \{ u \in C[0,1] : u(t) \geq 0, \text{ for } t \in [0,1] \text{ and } u(t) \text{ is concave on } [0,1] \}. \]  

(2.14)

Clearly, $K$ is a cone of $E$. Let $A : K \rightarrow C[0,1]$ be defined by

\[ Ay(t) = \frac{1}{m} + \theta(t) + \int_0^1 k(t, s) \, g^*(y(s)) + h(y(s)) \, ds. \]  

(2.15)

A standard argument [4, Chapter 4] implies $A : K \rightarrow C[0,1]$ is continuous and completely continuous. Next we show $A : K \rightarrow K$. If $u \in K$, then clearly $Au(t) \geq 0$ for $t \in [0,1]$ (see
(2.2), (2.4), (2.5), and (2.7)) and $Au(t)$ is concave on $[0, 1]$ from (2.8). Consequently, $Au \in K$, so $A : K \to K$. Let

$$
\Omega_1 = \{ u \in C[0, 1] : |u|_0 < r \} \quad \text{and} \quad \Omega_2 = \{ u \in C[0, 1] : |u|_0 < R \}.
$$

We first show

$$|Ay| \leq |y|, \quad \text{for } y \in K \cap \partial \Omega_1. \tag{2.17}$$

Let $y \in K \cap \partial \Omega_1$. Then $|y|_0 = r$ and $y \in K$. This together with Theorem 1.2 implies $y(t) \geq t(1 - t)|y|_0 = t(1 - t)r$, for $t \in [0, 1]$. Also notice

$$g^*(y(t)) + h(y(t)) \leq g(y(t)) + h(y(t)), \quad \text{for } t \in (0, 1),$$

since $g$ is nonincreasing on $(0, \infty)$. Now for $t \in [0, 1]$, we have, using (2.4) and (2.5),

$$|Ay|_0 \leq \varepsilon + |\theta|_0 + a_0 K_0 \{ g(r) + h(r) \} < r = |y|_0.$$

Hence, (2.17) holds. Next we show

$$|Ay|_0 \geq |y|_0, \quad \text{for } y \in K \cap \partial \Omega_2. \tag{2.18}$$

To see this, let $y \in K \cap \partial \Omega_2$, so $|y|_0 = R$. Also from Theorem 1.2, we have $y(t) \geq t(1 - t)|y|_0 \geq t(1 - t)R$ for $t \in [0, 1]$. In addition, for $s \in [a, 1 - a]$, we have

$$g^*(y(s)) + h(y(s)) = g(y(s)) + h(y(s))$$

since $y(s) \geq a(1 - a)R > 1/m_0$ for $s \in [a, 1 - a]$. Note in particular that

$$y(x) \in [a(1 - a)R, R], \quad \text{for } x \in [a, 1 - a]. \tag{2.19}$$

Now with $\sigma$ as defined in (2.11), we have, using (2.19) and (2.11),

$$Ay(\sigma) \geq \theta(\sigma) + \int_a^{1-a} k(\sigma, s) \left[ g^*(y(s)) + h(y(s)) \right] ds$$

$$\geq \theta(\sigma) + \int_a^{1-a} k(\sigma, s) g(y(s)) \left[ 1 + \frac{h(y(s))}{g(y(s))} \right] ds$$

$$\geq \theta(\sigma) + g(R) \left[ 1 + \frac{h(a(1 - a)R)}{g(a(1 - a)R)} \right] \int_a^{1-a} k(\sigma, s) ds$$

$$\geq R = |y|_0,$$

and so $|Ay|_0 \geq |y|_0$. Hence, (2.18) is true.
Now Theorem 1.1 implies \( A \) has a fixed point \( y_m \in K \cap (\overline{\Omega_2 \setminus \Omega_1}) \), i.e., \( r \leq |y_m|_0 \leq R \). Also, \( y_m(t) \geq 1/m \) for \( t \in [0,1] \), and since \( y_m \in K \), we have \( y_m(t) \geq \tau(t)(1-t)|y_m|_0 \geq \tau(t)(1-t)r \) for \( t \in [0,1] \). In fact, \( |y_m|_0 > r \). To see this, notice if \( |y_m|_0 = r \), then

\[
|y_m(t)| \leq \epsilon + |\theta|_0 + \left\{ 1 + \frac{h(r)}{g(r)} \right\} \int_0^1 k(t,s)g(s(1-s)r) \, ds \leq \epsilon + |\theta|_0 + a_0 K_0 \left\{ g(r) + h(r) \right\},
\]

for each \( t \in [0,1] \) and so

\[
\frac{r}{\epsilon + |\theta|_0 + a_0 K_0 \left\{ g(r) + h(r) \right\}} \leq 1.
\]

This contradicts (2.12). Consequently, \((2.14)^m\) (and also \((2.13)^m\)) has a solution \( y_m \in C[0,1] \), \( y_m \in K \), with

\[
\frac{1}{m} \leq y_m(t), \quad \text{for } t \in [0,1], \quad r < |y_m|_0 \leq R
\] (2.20)

and

\[
y_m(t) \geq t(1-t)r, \quad \text{for } t \in [0,1].
\] (2.21)

We will obtain a solution to (2.1) by means of the Arzela-Ascoli theorem. To this end, we will show

\[
\{y_m\}_{m \in N_0} \text{ is a bounded, equicontinuous family on } [0,1].
\] (2.22)

Now \( y_m, m \in N_0 \) satisfies

\[
y_m(t) = \frac{1}{m} + \theta(t) + \int_0^1 k(t,s) \left[ g(y_m(s)) + h(y_m(s)) \right] \, ds, \quad \text{for } t \in [0,1].
\] (2.23)

Also

\[
k(t,s) \left[ g(y_m(s)) + h(y_m(s)) \right] \leq \left\{ 1 + \frac{h(R)}{g(R)} \right\} k(t,s)g(s(1-s)r).
\]

Next notice for \( t, x \in [0,1] \) that

\[
|y_m(t) - y_m(x)| \leq |\theta(t) - \theta(x)| + K_0 g(r) \left\{ 1 + \frac{h(R)}{g(R)} \right\} \int_0^1 |k(t,s) - k(x,s)| g(s(1-s)) \, ds
\]

\[
\leq K_0 g(r) \left\{ 1 + \frac{h(R)}{g(R)} \right\} \left( \int_0^1 \left| k_t(s) - k_x(s) \right|^p \, ds \right)^{1/p} \left( \int_0^1 \left[ g(s(1-s)) \right]^q \, ds \right)^{1/q} + |\theta(t) - \theta(x)|.
\]

This together with (2.9) immediately guarantees that (2.22) is true. The Arzela-Ascoli theorem guarantees the existence of a subsequence \( N \) of \( N_0 \) and a function \( y \in C[0,1] \) with \( y_m \) converging uniformly on \([0,1]\) to \( y \) as \( m \to \infty \) through \( N \). Also, \( r \leq |y|_0 \leq R \) and \( y(t) \geq \tau(t)(1-t)r \) for \( t \in [0,1] \). In particular, \( y > 0 \) on \((0,1)\). Fix \( t \in (0,1) \). Now \( y_m, m \in N \) satisfies the integral equation (2.23). Let \( m \to \infty \) through \( N \) in (2.23) (here we use the Lebesgue dominated convergence theorem) to obtain

\[
y(t) = \theta(t) + \int_0^1 k(t,s)\left[ g(y(s)) + h(y(s)) \right] \, ds.
\] (2.24)

Finally, it is easy to see that \( |y|_0 > r \) (note if \( |y|_0 = r \), then its easy to check, as before, that we get a contradiction).

**Remark 2.2.** If in (2.11), we have \( R < r \), then (2.1) has a solution \( y \in C[0,1] \) with \( y > 0 \) on \((0,1)\) and \( R \leq |y|_0 < r \). The argument is essentially the same as that in Theorem 2.1, except here we use the other half of Theorem 1.1.

It is easy to use Theorem 2.1 and Remark 2.2 to obtain the existence of multiple solutions to (2.1). For completeness, we next present a result which guarantees the existence of two solutions.
THEOREM 2.2. Choose $a \in (0, (1/2))$ and fix it. Suppose (2.2)–(2.11) hold. In addition, assume there exists a constant $L$, $0 < L < r$, and $\tau \in [0, 1]$ with

$$\frac{Lg(a(1-a)L)}{|g(L)g(a(1-a)L) + g(L)h(a(1-a)L)|} \int_a^{1-a} k(\tau, s) \, ds + \theta(\tau)g(a(1-a)L) \leq 1.$$  \hspace{1cm} (2.25)

Then (2.1) has two solutions $y_1, y_2 \in C[0, 1]$ with $y_1 > 0, y_1 > 0$ on $(0, 1)$ and $L \leq |y_1|_0 < r < |y_2|_0 \leq R$.

PROOF. The existence of $y_2$ follows from Theorem 2.1 and the existence of $y_1$ follows from Remark 2.1.

EXAMPLE 2.1. Consider the singular integral equation

$$y(t) = \int_0^1 k(t, s) \left( |y(s)|^{-\alpha} + |y(s)|^\beta \right) \, ds, \quad \text{for } t \in [0, 1],$$  \hspace{1cm} (2.26)

with $0 < \alpha < 1 < \beta$ and

$$k(t, s) = \begin{cases} \frac{(1-\alpha)s}{3}, & 0 \leq s \leq t, \\ \frac{(1-\alpha)t}{3}, & t \leq s \leq 1. \end{cases}$$

Then (2.26) has a solution $y \in C[0, 1]$ with $y > 0$ on $(0, 1)$ and $|y|_0 < 1$.

To see this, we will apply Theorem 2.1 with $\theta = 0$, $g(u) = u^{-\alpha}$, $h(u) = u^\beta$, $p = \infty$, $q = 1$, and $a = 1/4$. Clearly, (2.2)–(2.6) with $K_0 = 1$, (2.7) and (2.9) (since $0 < a < 1$ and $q = 1$) hold.

Fix $m_0 \in \{1, 2, \ldots\}$ and let $m \in \{m_0, m_0 + 1, \ldots\}$. Let

$$g^*(u) = \begin{cases} u^{-\alpha}, & u \geq \frac{1}{m}, \\ m^\alpha, & 0 \leq u \leq \frac{1}{m}, \end{cases}$$

and for any $y \in C[0, 1]$ with $y(t) \geq 0$ for $t \in [0, 1]$ and $y(t)$ concave on $[0, 1]$ let us look at

$$F_m y(t) = \frac{1}{m} + \int_0^1 k(t, s) [g^*(y(s)) + h(y(s))] \, ds.$$

It is easy to check that

$$(F_m y)'(t) = -\frac{(1-\alpha)}{3} |g^*(y(t)) + h(y(t))| < 0,$$ \hspace{1cm} \text{for } t \in (0, 1),

so $F_m y(t)$ is concave on $[0, 1]$. Thus, (2.8) holds. Next notice

$$a_0 = \frac{(1-\alpha)}{3} \sup_{t \in [0, 1]} \left( \int_0^t s^{-\alpha} (1-s)^{-\alpha} \, ds + t \int_t^1 s^{-\alpha} (1-s)^{-\alpha} \, ds \right) \leq \frac{(1-\alpha)}{3} \left( \frac{1}{1-\alpha} \right) = \frac{1}{3},$$

and with $r = 1$,

$$\frac{r}{|\theta|_0 + a_0 K_0 \{g(r) + h(r)\}} = \frac{r}{a_0 [r^{-\alpha} + r^\beta]} \geq \frac{3r}{r^{-\alpha} + r^\beta} = \frac{3}{2} > 1.$$  \hspace{1cm} \text{As a result, (2.9) holds with } r = 1. \text{ Finally, note since } \beta > 1, \text{ that}

$$\lim_{R \to \infty} \frac{Rg(3R/16)}{g(R)g(3R/16) + g(R)h(3R/16)} = \lim_{R \to \infty} \left( \frac{R^{-\alpha+1}(3/16)^{-\alpha}}{(3/16)^{-\alpha} + (3/16)^\beta R^{\alpha+\beta}} \right) = 0.$$  \hspace{1cm} \text{Thus, there exists } R > 1 \text{ so that (2.11) holds for any } \sigma \in (0, 1). \text{ The result now follows from Theorem 2.1.}

REMARK 2.3. In fact, we can guarantee the existence of two nonnegative solutions to (2.26) from Theorem 2.2, since

$$\lim_{L \to 0} \frac{Lg(3L/16)}{g(L)g(3L/16) + g(L)h(3L/16)} = 0.$$
3. NONSINGULAR PROBLEMS

This section discusses nonsingular integral equations of Fredholm type (i.e., \( (2.1) \) with \( g = 0 \)). In particular, we discuss

\[
y(t) = \theta(t) + \int_0^1 k(t, s)h(y(s)) \, ds, \quad \text{for } t \in [0, 1].
\]

**Theorem 3.1.** Choose \( a \in (0, (1/2)) \) and fix it. Suppose the following conditions are satisfied:

1. \( 0 \leq k_t(s) = k(t, s) \in L^1[0, 1], \quad \text{for each } t \in [0, 1], \)
2. \( h \geq 0 \) is continuous and nondecreasing on \([0, 1]\) with \( h(u) > 0 \) for \( u > 0 \),
3. \( \theta \in C[0, 1] \) with \( \theta(t) \geq 0 \), \( \text{for } t \in [0, 1], \)
4. for any \( y \in C[0, 1] \) with \( y(t) \geq 0 \), for \( t \in [0, 1] \) and \( y(t) \) concave on \([0, 1]\), we have that \( Fy(t) = \theta(t) + \int_0^1 k(t, s)h(y(s)) \, ds \) is concave on \([0, 1]\),
5. there exists a constant \( r > 0 \) with \( \frac{r}{|\theta|_0 + b_0 h(r)} > 1 \); here \( |\theta|_0 = \sup_{t \in [0,1]} |\theta(t)| \) and \( b_0 = \sup_{t \in [0,1]} \int_0^1 k(t, s) \, ds, \)
6. there exists \( R > r \) and \( \sigma \in [0, 1] \) with \( \frac{R}{\theta(\sigma) + h(a(1-a)R) \int_0^{1-a} k(\sigma, s) \, ds} \leq 1. \)

Then \( (3.1) \) has a solution \( y \in C[0, 1] \) with \( y > 0 \) on \((0, 1)\) and \( r < |y|_0 < R \).

**Remark 3.1.** From Theorem 2.2 in [5], we know that there exists a solution \( y_1 \) to \( (3.1) \) with \( y_1 \geq 0 \) on \([0, 1]\) and \( |y_1|_0 < r \).

**Proof.** Let \( E = (C[0,1], |.|_0) \) and let \( K \) be as in (2.15). Let \( A : K \to C[0, 1] \) be defined by

\[
Ay(t) = \theta(t) + \int_0^1 k(t, s)h(y(s)) \, ds.
\]

Now (3.6) implies \( A : K \to K \). Let

\[
\Omega_1 = \{ u \in C[0,1] : |u|_0 < r \} \quad \text{and} \quad \Omega_2 = \{ u \in C[0,1] : |u|_0 < R \}.
\]

First we show

\[
|Ay| \leq |y|, \quad \text{for } y \in K \cap \partial\Omega_1.
\]

Let \( y \in K \cap \partial\Omega_1 \). Then \( |y|_0 = r \) and \( y \in K \). Now for \( t \in [0, 1] \), we have

\[
|Ay(t)| \leq |\theta|_0 + h(r) \int_0^1 k(t, s) \, ds \leq |\theta|_0 + b_0 h(r),
\]

and as a result

\[
|Ay|_0 \leq |\theta|_0 + b_0 h(r) < r = |y|_0.
\]

Hence, (3.10) is true. Next we show

\[
|Ay|_0 \geq |y|_0, \quad \text{for } y \in K \cap \partial\Omega_2.
\]
To see this, let $y \in K \cap \partial \Omega_2$, so $|y|_0 = R$ and so $y(t) \geq t(1-t)|y|_0 \geq t(1-t)R$ for $t \in [0,1]$. In particular,
\[ y(x) \in [a(1-a)R, R], \quad \text{for } x \in [a, 1-a]. \]

Now with $\sigma$ as defined in (3.8), we have
\[
Ay(\sigma) \geq \theta(\sigma) + \int_{a}^{1-a} k(\sigma, s)h(y(s)) \, ds \geq \theta(\sigma) + h(a(1-a)R) \int_{a}^{1-a} k(\sigma, s) \, ds \geq R = |y|_0,
\]
and so $|Ay|_0 \geq |y|_0$. Hence, (3.11) is true.

Now Theorem 1.1 implies $A$ has a fixed point $y \in K \cap (\bar{\Omega}_2 \setminus \Omega_1)$, i.e., $r \leq |y|_0 \leq R$. Also, $y(t) \geq t(1-t)|y|_0 \geq t(1-t)r$ for $t \in [0,1]$ and, in fact, $|y|_0 > r$ from a standard argument. □

REMARK 3.2. If in (3.8), we have $R < r$, then (3.1) has a solution $y \in C[0,1]$ with $y > 0$ on $(0,1)$ and $R \leq |y|_0 < r$. The argument is essentially the same as that in Theorem 3.1, except here we use the other half of Theorem 1.1.

REMARK 3.3. It is also possible to state a multiple solution result for (3.1). We leave the details to the reader.

REFERENCES