Note

The volume of relaxed Boolean-quadric and cut polytopes

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Abstract

For \( n \geq 2 \), the \textit{boolean quadric polytope} \( Q_n \) is the convex hull in \( d:=\binom{n+1}{2} \) dimensions of the binary solutions \( x_i x_j = y_{ij} \), for all \( i < j \) in \( N := \{1, 2, \ldots, n\} \). The polytope is naturally modeled by a somewhat larger polytope; namely, \( Q_n \) the solution set of \( y_{ij} \leq x_i, \quad y_{ij} \leq x_j, \quad x_i + x_j \leq 1 + y_{ij}, \quad y_{ij} \geq 0, \) for all \( i, j \) in \( N \). In a first step toward seeing how well \( Q_n \) approximates \( P_n \), we establish that the \( d \)-dimensional volume of \( Q_n \) is \( 2^{2n-4n!/(2n)!} \). Using a well-known connection between \( P_n \) and the 'cut polytope' of a complete graph on \( n + 1 \) vertices, we also establish the volume of a relaxation of this cut polytope.

1. Introduction

A natural approach to the unconstrained, quadratic-objective, binary program in \( n (\geq 2) \) variables

\[
\max \left\{ \sum_{i \in N} c_i x_i + \sum_{i < j \in N} d_{ij} x_i x_j : x_i \in \{0, 1\} \forall i \in N \right\},
\]

where \( N := \{1, 2, \ldots, n\} \), is to model the problem as a linearly constrained, linear-objective, binary program, through the use of (2) auxiliary binary variables \( y_{ij} \) which model the quadratic terms \( x_i x_j \). We obtain the equivalent program

\[
\max \sum_{i \in N} c_i x_i + \sum_{i < j \in N} d_{ij} y_{ij}
\]
subject to

\[ y_{ij} \leq x_i \quad \forall i < j \in N, \quad (3) \]
\[ y_{ij} \leq x_j \quad \forall i < j \in N, \quad (4) \]
\[ y_{ij} \geq 0 \quad \forall i < j \in N, \quad (5) \]
\[ x_i + x_j \leq 1 + y_{ij} \quad \forall i < j \in N, \quad (6) \]
\[ x_i \in \{0, 1\} \quad \forall i \in N, \quad (7) \]
\[ y_{ij} \in \{0, 1\} \quad \forall i < j \in N. \quad (8) \]

The boolean quadric polytope \( P_n \) is the convex hull (in real \( d := \binom{n+1}{2} \) space) of the set of solutions of (3)–(8). As the problem of solving (2)–(8) is NP-hard, it is natural to consider branch-and-cut methods based on (2)–(6). The relaxed feasible region (3)–(6) is denoted by \( \mathcal{Z}_n \). Padberg [5] has made a detailed study of \( P_n \) and \( \mathcal{Z}_n \) (also see [3, 6]).

It is natural to consider how good of an approximation \( \mathcal{Z}_n \) is to \( P_n \). The Chvátal–Gomory rank (see [7, 1]) of \( P_n \) with respect to \( \mathcal{Z}_n \) increases with \( n \), so in a certain combinatorial sense, \( \mathcal{Z}_n \) is a poor approximation of \( P_n \). In a different combinatorial sense \( \mathcal{Z}_n \) is quite close to \( P_n \); that is, the 1-skeleton of \( P_n \) is a subset of the 1-skeleton of \( \mathcal{Z}_n \) (the so-called Trubin Property) (see [5]). Another method has been proposed to study the closeness of pairs of nested polytopes, based on the volumes of the polytopes. Lee and Morris [4] have suggested the distance function

\[ \rho_d(\mathcal{Z}_n, P_n) := \left( \frac{\text{vol}_d(\mathcal{Z}_n)}{\text{vol}_d(B^d)} \right)^{1/d} - \left( \frac{\text{vol}_d(P_n)}{\text{vol}_d(B^d)} \right)^{1/d}, \]

where \( B^d \) is the \( d \)-dimensional Euclidean ball, and \( \text{vol}_d \) denotes \( d \)-dimensional Lebesgue measure. For polytope pairs contained in \([0, 1]^d \), \( \rho_d \) is at most \( O(\sqrt{d}) \). In some interesting cases of sets of polytope pairs, \( \rho_d \) may increase more slowly than this upper bound, in other situations the bound is sharp (see [4]). In Section 2, as a step toward determining the asymptotic behavior of \( \rho_d(\mathcal{Z}_n, P_n) \), we calculate \( \text{vol}_d(\mathcal{Z}_n) \).

There is a well-known connection between \( P_n \) and the 'cut polytope' of a complete graph on \( n + 1 \) vertices. In Section 3, we determine the volume of a natural relaxation of this cut polytope.

2. The volume of a relaxed boolean-quadric polytope

Let \( \mathcal{Z}_n := 2 \mathcal{Z}_n \), that is, the polytope \( \mathcal{Z}_n \) magnified by a factor of 2. Clearly, \( \text{vol}_d(\mathcal{Z}_n) = 2^d \text{vol}_d(\mathcal{Z}_n) \). Padberg demonstrated that \( \mathcal{Z}_n \) is a lattice polytope (i.e., its extreme points are lattice points). For simplicity, we work with \( \mathcal{Z}_n \), which is defined by the inequalities

\[ y_{ij} \leq x_i \quad \forall i < j \in N, \quad (9) \]
\[ y_{ij} \leq x_j \quad \forall i < j \in N, \quad (10) \]
\[ y_{ij} \geq 0 \quad \forall i < j \in N, \quad (11) \]
\[ x_i + x_j \leq 2 + y_{ij} \quad \forall i < j \in N. \quad (12) \]
Our first step in calculating $\text{vol}_d(\mathcal{C}_a)$ is to reduce the problem to that of calculating the volume of a subset of $\mathcal{Z}_n$. Points in Euclidean $d$-space will be denoted by $(x, y) = (x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_{n-1})$. For $a \in \{0, 1\}^n$, let

$$C_a := \{(x, y) \in \mathcal{Z}_n : a \leq x \leq a + 1\},$$

where $1$ is the $n$-vector $(1, 1, \ldots, 1)$. Clearly, $\mathcal{Z}_n$ is the union of all such polytopes $C_a$. Furthermore, $\text{vol}_d(C_a \cap C_b) = 0$ for $a \neq b$, so $\text{vol}_d(\mathcal{Z}_n) = \sum_a \text{vol}_d(C_a)$.

**Proposition 1.** $\text{vol}_d(C_a) = \text{vol}_d(C_b)$ for all $a, b \in \{0, 1\}^n$.

**Proof.** It suffices to demonstrate that if binary $n$-vectors $a$ and $b$ differ in precisely one coordinate, then $\text{vol}_d(C_a) = \text{vol}_d(C_b)$. Suppose, without loss of generality, that $a_i = 0, b_i = 1$ for $i \neq j, a_j = 0, b_j = 1$. We define a map $\Phi_i : C_a \mapsto C_b$ as follows: $\Phi_i$ is a composition of coordinate maps $\{\phi_i, \phi_j, \phi_k, \phi_{ij}, \phi_{ki}, \phi_{ij} : 1 \leq k < i < j \leq n\}$, where

$$\phi_i(x_i) := 2 - x_i, \quad \phi_j(x_j) := x_j \quad \text{for} \ j \neq i, \quad \phi_k(x_k) := x_k, \quad \phi_{ij}(y_{ij}) := y_{ij}, \quad \phi_{ij}(y_{ij}) := x_j - y_{ij}, \quad \text{and} \quad \phi_{ki}(y_{ki}) := x_k - y_{ki}.$$

To see that the range of $\Phi_i$ is contained in $C_b$, we only need to consider $\phi_{ij}$; the analysis for $\phi_{ki}$ is similar. Clearly,

$$\phi_{ij}(y_{ij}) = x_j - y_{ij} \leq x_j = \phi_j(x_j),$$

and

$$\phi_{ij}(y_{ij}) = x_j - y_{ij} \leq x_j - x_i - x_j + 2 = 2 - x_i = \phi_i(x_i).$$

Also,

$$\phi_i(x_i) + \phi_j(x_j) = 2 - x_i + x_j \leq 2 - y_{ij} + x_j = 2 + \phi_{ij}(y_{ij}).$$

Thus, we have shown that $\Phi_i$ is, indeed, a map from $C_a$ into $C_b$. It is trivial to check that $\Phi_i$ is an involution. Consequently, $\Phi_i$ is bijective and unimodular, and thus measure preserving, so $\text{vol}_d(C_a) = \text{vol}_d(C_b)$. Now, given an arbitrary binary $n$-vector $a$, the composition of the maps in $\{\Phi_i : a_i = 1\}$ gives a measure preserving bijection from $C_0$ to $C_a$, so $\text{vol}_d(C_a) = \text{vol}_d(C_0)$. □

**Corollary 2.** $\text{vol}_d(\mathcal{Z}_n) = 2^n \text{vol}_d(C_0)$.

Let $(S_n, \prec)$ denote the poset (partially ordered set) on $S_n := \{x_i : 1 \leq i \leq n\} \cup \{y_{ij} : 1 \leq i < j \leq n\}$ having $y_{ij} \prec x_i$ and $y_{ij} \prec x_j$. Let $e(S_n, \prec)$ denote the number of (linear) extensions of $(S_n, \prec)$, i.e., the number of order-preserving bijections from $S_n$ to $D := \{1, 2, \ldots, d\}$, where the order on $D$ is the usual one.

**Proposition 3.** $\text{vol}_d(C_0) = e(S_n, \prec)/d!$

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1 The map $\Phi_i$ is called a 'switching' and is a standard tool in the analysis of the 'cut polytope' (see [2, 6]).
Proof. By definition,

\[ C_0 = \{(x, y) \in \mathcal{P}_n : 0 \leq x_i \leq 1 \ (1 \leq i \leq n)\}. \]

It follows that \( C_0 \) is defined by the inequalities (9)-(11) and

\[ x_i \leq 1, \quad 1 \leq i \leq n, \]

with (12) rendered vacuous. \( C_0 \) is the order polytope (see [8]) of the poset \((S_n, \prec)\). The result follows by Corollary 4.2 of Stanley. \( \square \)

Theorem 4. \( e(S_n, \prec) = n! \cdot d! \cdot 2^n/(2n)! \).

Proof. We regard extensions of \((S_n, \prec)\) as permutations of the set \( S_n \). That is, given a bijection \( \pi : S_n \to D \), we represent \( \pi \) by the permutation \( \pi^{-1}(d) \pi^{-1}(d-1) \cdots \pi^{-1}(1) \). Define an ordered extension of \((S_n, \prec)\) to be an extension of \((S_n, \prec)\) such that \( x_i \) appears to the left of \( x_k \), for \( 1 \leq i < k \leq n \). That is, we regard an ordered extension of \((S_n, \prec)\) as a permutation of \( S_n \) in which \( x_i \) appears to the left of \( x_k \), and \( y_{ik} \) appears to the right of both \( x_i \) and \( x_k \), for \( 1 \leq i < k \leq n \). Clearly, the number of extensions equals \( n! \) times the number of ordered extensions.

Next, we proceed to count the number of ordered extensions of \((S_n, \prec)\). Suppose that \( \{y_{ii} : k + 1 \leq l \leq n\} \) have already been positioned, for some fixed \( i < k + 1 \). We see, now, how to place \( \{y_{ik} : 1 \leq l \leq k - 1\} \). The element \( y_{1k} \) should be placed to the right of \( x_k \). As there are already \( f_k := n - k + 1 + \binom{k}{2} \) elements of \( S_n \) placed after \( x_k \), there are \( f_k \) possible positions for \( y_{1k} \). Then, there are \( f_k + 1 \) possible positions for \( y_{2k} \), up through \( f_k + k - 1 \) possible positions for \( y_{k-1,k} \). In total, the number of ordered extensions is equal to

\[
\prod_{k=2}^{n} \prod_{i=0}^{k-2} (f_k + i) = \prod_{k=2}^{n} \frac{(f_k + k - 2)!}{(f_k - 1)!} = \prod_{k=2}^{n} \frac{(\binom{n-k+1}{2} - \binom{k}{2} - 1)!}{(\binom{n-k+1}{2} - \binom{k}{2})!} = \frac{2^n \cdot d!}{(2n)!}
\]

Hence, we get that \( e(S_n, \prec) = n! \cdot d! \cdot 2^n/(2n)! \). \( \square \)

Corollary 2, Proposition 3, and Theorem 4 now yield

Theorem 5. \( \text{vol}_d(\mathcal{P}_n) = 2^{2n - d} n!/(2n)! \).

We note that by Stirling’s formula, \( \text{vol}_d(\mathcal{P}_n) = 2^{-1/2}(e/n)^d(1 + o(1)) \). Hence, for example, if it could be shown that \( \text{vol}_d(2\mathcal{P}_n) = 2^{-1/2}(e/n)^d(1 + o(1)) \), then we could conclude that \( \rho_d(\mathcal{P}_n) \) behaves like \( \sqrt{d} \).
3. The volume of a relaxed cut-polytope of a complete graph

Let $G$ be a simple undirected graph with vertex set $V(G) := \{0, 1, 2, \ldots, n\} = N \cup \{0\}$ and edge set $E(G)$. A cut of $G$ is any set of edges that crosses a nontrivial partition of $V(G)$; that is, $F \subseteq E(G)$ is a cut if $F$ is the set of edges with exactly one endpoint in some nonempty proper subset $W$ of $V(G)$. Associated with every cut of $G$ is its incidence vector $z \in \{0, 1\}^{E(G)}$. Let the cut polytope of $G$ be the convex hull of the incidence vectors of cuts of $G$. For the complete graph $K_{n+1}$, we denote the cut polytope by $\mathcal{P}_{n+1}$. We immediately notice that $\mathcal{P}_n$ and $\mathcal{P}_{n+1}$ both have dimension $d = (\binom{n+1}{2})$. As has been observed by many authors (see [2]), there is a linear bijective transformation $\tau$ from $\mathcal{P}_n$ to $\mathcal{P}_{n+1}$; namely,

$$x_i := z_{oi} \quad \forall i \in N,$$

$$y_{ij} := \frac{1}{2}(z_{oi} + z_{oj} - z_{ij}) \quad \forall i < j \in N.$$

We may apply this same transformation to the relaxed Boolean-quadric polytope and define $\mathcal{Q}_{n+1} := \tau(\mathcal{Q}_n)$, a natural relaxation of $\mathcal{C}_{n+1}$. The polytope $\mathcal{Q}_{n+1}$ is the solution set of

$$z_{oi} - z_{oj} - z_{ij} \leq 0 \quad \forall i < j \in N,$$

$$- z_{oi} + z_{oj} - z_{ij} \leq 0 \quad \forall i < j \in N,$$

$$- z_{oi} - z_{oj} + z_{ij} \leq 0 \quad \forall i < j \in N,$$

$$z_{oi} + z_{oj} + z_{ij} \leq 2 \quad \forall i < j \in N.$$

Let $x := (x_1, x_2, \ldots, x_n)^T$, $z^0 := (z_{01}, z_{02}, \ldots, z_{0n})^T$, $z := (z^0, z^N)$, and define $y$ and $z^N$ so that $y_{ij}$ occupies the same position in $y$ as $z_{ij}$ does in $z^N$, $i < j \in N$. In matrix terms, we can view the transformation $\tau$ as

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} I & 0 \\ \frac{1}{2}A^T & -\frac{1}{2}I \end{pmatrix} \begin{pmatrix} z^0 \\ z^N \end{pmatrix},$$

where $A$ is a vertex-edge incidence matrix of $K_n$ on vertex set $N$. The absolute value of the determinant of the transformation matrix is $2^{n-d}$, so we can conclude the following result.

**Theorem 6.** $\text{vol}_d(\mathcal{Q}_{n+1}) = 2^n n! / (2n)!$.

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References


