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# Examples of dispersive effects in non-viscous rotating fluids

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**Abstract**

Strichartz estimates for rotating fluids have already been used to show that the velocity fields converge, as the Rossby number goes to zero, to a solution of a nearly two-dimensional Navier–Stokes system.

Using a similar method, it is possible to get results of convergence also in the non-viscous case—to solutions of a nearly two-dimensional Euler system. The initial data do not need to be well prepared, and the limit can be as singular as a vortex patch or a Yudovich solution.


**Résumé**

Des estimations de Strichartz pour les fluides tournants ont déjà été utilisées pour montrer que les champs de vitesses convergent, lorsque le nombre de Rossby tend vers zéro, vers une solution d'un système de Navier–Stokes quasi bidimensionnel.

En utilisant une méthode analogue, il est possible d'obtenir des résultats de convergence aussi dans le cas non visqueux—vers des solutions d'un système d'Euler quasi bidimensionnel. Il n'est pas nécessaire que les données initiales soient bien préparées, et la limite peut être aussi singulière qu'une poche de tourbillon ou une solution de Yudovich.


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## 1. Introduction

We consider the system of rapidly rotating, non-viscous, incompressible fluids extended in the whole space,

$$\begin{cases} \partial_t u_\varepsilon(t,x) + u_\varepsilon(t,x)\cdot\nabla u_\varepsilon(t,x) + u_\varepsilon(t,x)\times e_3/\varepsilon = -\nabla p_\varepsilon(t,x), \\ \operatorname{div} u_\varepsilon(t,x) = 0, \\ u_\varepsilon(0,x) = u_{0,\varepsilon}(x), \end{cases} \tag{1}$$

for $(t,x) \in [0, T_\varepsilon[ \times \mathbb{R}^3$, where:

- $e_3$ is the unit vector along the vertical direction,
- $\varepsilon$ is the Rossby number, a small positive parameter ($0 < \varepsilon < 1$),

and where, for each $\varepsilon$,

- $u_{0,\varepsilon}$ is a smooth divergence-free initial datum,
- $T_\varepsilon$ is the maximal time of existence of the corresponding smooth solution $(u_\varepsilon, p_\varepsilon)$.

The system (1) is the Euler system of perfect incompressible fluids with an additional term, $u_\varepsilon \times e_3/\varepsilon$, which appears when Coriolis forces are taken into account. For a justification of this model and further physical motivations, we refer to books of geophysical fluid dynamics [16,19].

We are interested here in proving estimates on $T_\varepsilon$ and convergence properties of $u_\varepsilon$ as $\varepsilon \to 0$, when $u_{0,\varepsilon}$ tends to a field whose regularity is critical. This problem has already been studied in the viscous case [5,6] using Strichartz, dispersive estimates, and we follow the same method. In the first version of this paper, we used another dispersive inequality, which we still include because its proof may be found interesting (see Section 6). However, as was pointed out by the referee, the method of Chemin, Desjardins, Gallagher and Grenier [5] does not depend on a non-zero viscosity—we will see in Section 5 that it is much simpler and finally gives a better estimate.

In the sequel of this introduction, we explain at length, before stating the results, in which spaces of the initial data are supposed to belong, because these spaces are a bit peculiar. The general ideas of the proof are given in Section 2. The remaining sections contain the technical details.

### 1.1. *A case for mixed initial data*

First assume that, for each $\varepsilon$, $u_{0,\varepsilon} \in H^r(\mathbb{R}^3)^3$ for some large $r$. The local existence and uniqueness of smooth solutions of (1) can be proved without difficulty after elimination of the pressure. If $u$ is a divergence-free vector field and $P = I - \nabla\Delta^{-1}\operatorname{div}$ is Leray's projector onto divergence-free vector fields, then

$$P(u \times e_3) = -\Delta^{-1}\nabla \times \partial_3 u.$$

Hence, the system (1) is equivalent to:

$$\begin{cases} \partial_t u_\varepsilon + P(u_\varepsilon \cdot \nabla u_\varepsilon) + \frac{1}{\varepsilon} A(D) u_\varepsilon = 0, \\ u_\varepsilon|_{t=0} = u_{0,\varepsilon} = P u_{0,\varepsilon}, \end{cases} \tag{2}$$

with $A$ defined by:

$$A(D) f = -\frac{D}{|D|} \times \left( \frac{D_3}{|D|} f \right) = \mathcal{F}^{-1} \left( -\frac{\xi}{|\xi|} \times \left( \frac{\xi_3}{|\xi|} \hat{f}(\xi) \right) \right)$$

for all $f \in L^2(\mathbb{R}^3)^3$. When $\varepsilon$ tends to zero, the solutions of (2) are expected to converge towards some divergence-free vector field that belongs to the kernel of the penalization operator $A$. In $L^2(\mathbb{R}^3)^3$, there is no such field other than zero, so investigating the limit for initial data $u_{0,\varepsilon} \in H^r(\mathbb{R}^3)^3$ is not very appealing.

However, any divergence-free vector field $v_\varepsilon$ depending only on the first two variables $(x_1, x_2) = x_h$ ("$h$" will stand for "horizontal" throughout the paper) may be written:

$$v_\varepsilon = \begin{pmatrix} -\partial_2 \psi_\varepsilon \\ \partial_1 \psi_\varepsilon \\ v_\varepsilon^3 \end{pmatrix}$$

for some scalar function $\psi_\varepsilon = \psi_\varepsilon(x_h)$, so that

$$\frac{v_\varepsilon \times e_3}{\varepsilon} = \frac{1}{\varepsilon} \begin{pmatrix} \partial_1 \psi_\varepsilon \\ \partial_2 \psi_\varepsilon \\ 0 \end{pmatrix} = \frac{1}{\varepsilon} \nabla \psi_\varepsilon.$$

Therefore $(u_\varepsilon, p_\varepsilon) = (v_\varepsilon, \tilde{p}_\varepsilon - \psi_\varepsilon/\varepsilon)$ satisfies the evolution equation in (1) if and only if $(v_\varepsilon, \tilde{p}_\varepsilon)$ satisfies:

$$\partial_t v_\varepsilon + v_\varepsilon \cdot \nabla v_\varepsilon = -\nabla \tilde{p}_\varepsilon.$$

So we will suppose that the initial data $u_{0,\varepsilon}$ are made up of two smooth parts—a two-dimensional one $v_{0,\varepsilon}$ and a three-dimensional one $\phi_{0,\varepsilon}$. Then the solutions of (1) can be split accordingly: $u_\varepsilon = v_\varepsilon + \phi_\varepsilon$, where

$$\begin{cases} \partial_t v_\varepsilon^h + v_\varepsilon^h \cdot \nabla_h v_\varepsilon^h = -\nabla_h \tilde{p}_\varepsilon, \\ \operatorname{div} v_\varepsilon^h = 0, \\ \partial_t v_\varepsilon^3 + v_\varepsilon^h \cdot \nabla_h v_\varepsilon^3 = 0, \\ v_\varepsilon|_{t=0} = v_{0,\varepsilon} \end{cases} \tag{3}$$

and

$$\begin{cases} \partial_t \phi_\varepsilon + P(v_\varepsilon \cdot \nabla \phi_\varepsilon + \phi_\varepsilon \cdot \nabla \phi_\varepsilon + \phi_\varepsilon \cdot \nabla v_\varepsilon) + \frac{1}{\varepsilon} A(D) \phi_\varepsilon = 0, \\ \phi_\varepsilon|_{t=0} = \phi_{0,\varepsilon} = P \phi_{0,\varepsilon}. \end{cases} \tag{4}$$

If $v_{0,\varepsilon} \to v_0$, then $v_\varepsilon$ should tend to a solution $v$ of:

$$\begin{cases} \partial_t v^h + v^h \cdot \nabla_h v^h = -\nabla_h p, \\ \operatorname{div} v^h = 0, \\ \partial_t v^3 + v^h \cdot \nabla_h v^3 = 0, \\ v|_{t=0} = v_0, \end{cases} \tag{5}$$

while $\phi_\varepsilon$, on the other hand, should in some sense tend to zero.

Our goal is to prove such results of convergence when $v$ is a solution of (5) such that $v^h = (v^1, v^2)$ is a vortex patch (that is, when $\operatorname{rot} v_0^h$ is the characteristic function of a bounded $C^{1+s}$ domain, with $0 < s < 1$) or even a Yudovich solution (when $\operatorname{rot} v_0^h$ is only supposed to be bounded, with some decrease at infinity) of the underlying two-dimensional Euler system.

## *1.2. Functional spaces*

The estimates that we get on $\phi_\varepsilon$ in Section 4 are uniform with respect to $\varepsilon$; this is possible only in functional spaces constructed on $L^2$. For the sake of simplicity, we have chosen the classical Sobolev spaces, although slightly sharper results can be expressed with Besov spaces [10].

There is a complication with the two-dimensional parts of the velocity fields. Indeed, a two-dimensional divergence-free velocity field whose curl is bounded and compactly supported does not, in general, belong to $L^2$. It can be shown [4] that appropriate spaces for Yudovich solutions are the affine spaces $\sigma + L^2(\mathbb{R}^2)^2$, where

$$\sigma\left(x^1, x^2\right) = \frac{1}{|x_h|^2} \begin{pmatrix} -x^2 \\ x^1 \end{pmatrix} \int_0^{|x_h|} \tau g(\tau)\, \mathrm{d}\tau,$$

with $g \in C_0^\infty(\mathbb{R}_0^+)$. Such $\sigma$ are smooth stationary solutions of the two-dimensional Euler system—that is,

$$\partial_t \sigma = P(\sigma \cdot \nabla \sigma) = 0,$$

where $P$ still denotes Leray's projector, but in dimension two—and behave like $1/|x_h|$ at infinity; moreover, $\nabla\sigma$ belongs to $H^r(\mathbb{R}^2)$ for all $r \in \mathbb{R}$.

So we will look for solutions of (3) and (5) such that

$$v_\varepsilon = \begin{pmatrix} \sigma \\ 0 \end{pmatrix} + \tilde{v}_\varepsilon \quad \text{and} \quad v = \begin{pmatrix} \sigma \\ 0 \end{pmatrix} + \tilde{v},$$

with $\tilde{v}_\varepsilon$ and $\tilde{v}$ in spaces included in $L^2$, and rewrite the systems (3) and (5) as follows:

$$\begin{cases} \partial_t \tilde{v}_\varepsilon^h + P(v_\varepsilon^h \cdot \nabla_h \tilde{v}_\varepsilon^h + \tilde{v}_\varepsilon^h \cdot \nabla_h \sigma) = 0, \\ \tilde{v}_\varepsilon^h|_{t=0} = \tilde{v}_{0,\varepsilon}^h = P\tilde{v}_{0,\varepsilon}^h, \\ \partial_t v_\varepsilon^3 + v_\varepsilon^h \cdot \nabla_h v_\varepsilon^3 = 0, \\ v_\varepsilon^3|_{t=0} = v_{0,\varepsilon}^3, \end{cases} \tag{6}$$

and

$$\begin{cases} \partial_t \tilde{v}^h + P(v^h \cdot \nabla_h \tilde{v}^h + \tilde{v}^h \cdot \nabla_h \sigma) = 0, \\ \tilde{v}^h|_{t=0} = \tilde{v}_0^h = P\tilde{v}_0^h, \\ \partial_t v^3 + v^h \cdot \nabla_h v^3 = 0, \\ v^3|_{t=0} = v_0^3. \end{cases} \tag{7}$$

*1.3. Results*

The two theorems stated below actually follow from a more general but quite technical proposition (see Section 2). Each time, the initial data are regular but their regularity degenerates, as their two-dimensional parts converge either to a vortex patch (Theorem 1) or to a field in $\sigma + L^2(\mathbb{R}^2)^2$ whose curl is essentially bounded (Theorem 2). No assumption of convergence (or even boundedness) is made on $\phi_{0,\varepsilon}$, so the data may be very ill prepared.

In our statements, $\rho_k = k^2\rho(k\,\cdot)$ denote the usual mollifiers, for all $k \in \mathbb{R}^+$. From now on, we will also make abuses of notation like writing $\sigma + L^2$ instead of $(\sigma + L^2(\mathbb{R}^2)^2) \times L^2(\mathbb{R}^2)$.

**Theorem 1** (vortex patches). *Let $D$ be a bounded domain of class $C^{1+s}$, with $s \in ]0,1[$. Let $a < +\infty$. Let $v_0 \in \sigma + L^2$ be a divergence-free vector field such that*

$$v_0^3 \in \mathrm{Lip} \quad \textit{and} \quad \mathrm{rot}\, v_0^h = \Omega_{0,i}\mathbf{1}_D + \Omega_{0,e}\mathbf{1}_{\mathbb{R}^2\setminus D},$$

*where $\Omega_{0,i} \in C^s(\overline{D})$ and $\Omega_{0,e} \in C^s \cap L^a(\mathbb{R}^2 \setminus D)$.*

*Suppose that*

$$u_{0,\varepsilon} = \rho_{\varepsilon^{-1/156}} * v_0 + \phi_{0,\varepsilon},$$

*with*

$$\|\phi_{0,\varepsilon}\|_{H^{s+7/2}} \lesssim \varepsilon^{-\alpha}$$

*for some $\alpha < 1/24$.*

*Then solutions $u_\varepsilon = v_\varepsilon + \phi_\varepsilon$ of* (1) *exist in $L^\infty_{\mathrm{loc}}(\mathbb{R}^+; \sigma + H^{s+11/2}(\mathbb{R}^2)) \oplus L^\infty_{\mathrm{loc}}([0, T_\varepsilon[; H^{s+7/2})$, with*

$$T_\varepsilon \gtrsim \ln\ln\ln\varepsilon^{-1} \quad \textit{as } \varepsilon \to 0; \tag{8}$$

*moreover, $v_\varepsilon \to v$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}^+; \sigma + L^2)$, where $v$ is the vortex patch solution of* (5)*, and $\phi_\varepsilon \to 0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{Lip})$.*

The system (3) is a two-dimensional Euler system on $v_\varepsilon^h$ and a linear transport equation on $v_\varepsilon^3$. Therefore, its solutions are global. So the maximal times of existence of (1) and (4) coincide; they tend to infinity, as we will see, owing to a dispersive phenomenon. The growth of the lifespans, however, is very slow.

In the case of Yudovich solutions, we have to reduce the rate of regularization and also assume that $v_0^3 = 0$.

**Theorem 2** (Yudovich solutions). *Let $a < +\infty$. Let $v_0 \in \sigma + L^2(\mathbb{R}^2)$ be a divergence-free vector field such that*

$$v_0^3 = 0 \quad \textit{and} \quad \operatorname{rot} v_0^h \in L^\infty \cap L^a.$$

*Choose $\beta \in ]0, 1[$, then $k(\varepsilon) \leqslant \exp((\ln \varepsilon^{-1})^\beta)$, and suppose that*

$$u_{0,\varepsilon} = \rho_{k(\varepsilon)} * v_0 + \phi_{0,\varepsilon},$$

*with*

$$\|\phi_{0,\varepsilon}\|_{H^{s+7/2}} \lesssim \varepsilon^{-\alpha}$$

*for some $\alpha < 1/24$ and $s \in ]0, 1[$.*

*Then solutions $u_\varepsilon = v_\varepsilon + \phi_\varepsilon$ of* (1) *exist in $L^\infty_{\mathrm{loc}}(\mathbb{R}^+; \sigma + H^{s+11/2}(\mathbb{R}^2)) \oplus L^\infty_{\mathrm{loc}}([0, T_\varepsilon[; H^{s+7/2})$, with*

$$T_\varepsilon \gtrsim \ln \ln \varepsilon^{-1} \quad \textit{as } \varepsilon \to 0;$$

*moreover, $v_\varepsilon \to v$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}^+; \sigma + L^2)$, where $v$ is the Yudovich solution of* (5)*, and $\phi_\varepsilon \to 0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{Lip})$.*

Note that a “ln” could have been suppressed in (8), had we supposed $v_0^3 = 0$. So the speed of convergence has not really improved.

These results are similar to the ones that we have already got about the quasigeostrophic and incompressible limits [10,11], and their proof relies on the same techniques. The decomposition $u_\varepsilon = v_\varepsilon + \phi_\varepsilon$, indeed, is analogous to decompositions of solutions of the Boussinesq system into a quasigeostrophic part and an oscillating part, and of solutions of the compressible Euler system into an incompressible part, a compressible part and an acoustic part. For fluids extended on the whole space, three-dimensional, oscillating, compressible and acoustic parts are all subject to dispersive effects, so that Strichartz estimates can be used to show that they all tend to zero.

In the case of non-viscous compressible fluids, the Strichartz estimates are just those available on solutions of the wave equation [15]. They were first exploited by Ukai [22] to deal with ill-prepared initial data, and afterwards the method has been extended to viscous fluids [7,8]. In the other two cases, the possibility of getting suitable dispersive inequalities was not so obvious; previous works either dealt with well-prepared initial data [3,18] or concerned viscous fluids [5,6].

Recall finally that methods and results are different on the torus, where there is no dispersion but resonances [1,2,13].

## 2. Sketch of the proof

An important role will be played by the curl of $v^h_\varepsilon$,

$$\Omega_\varepsilon \stackrel{\text{def}}{=} \partial_1 v^2_\varepsilon - \partial_2 v^1_\varepsilon, \tag{9}$$

and its derivatives in the directions indicated by a collection of vector fields. A system of continuous vector fields $W = \{w^\mu;\ \mu = 1, \ldots, M\}$ on $\mathbb{R}^2$ is *admissible* [4,14] if the function $[W] \stackrel{\text{def}}{=} \min(|w^\mu|;\ \mu = 1, \ldots, M)$ is bounded away from zero. Given $s \in ]0, 1[$, we denote by $X_s(W, \Omega)$ the quantity:

$$\|\Omega\|_{L^\infty} + \big\|[W]^{-1}\big\|_{L^\infty} + \sum_{\mu=1}^{M} \|w^\mu\|_{C^s} + \sum_{\mu=1}^{M} \big\|\operatorname{div}(w^\mu \Omega)\big\|_{C^{s-1}}. \tag{10}$$

If $W$ is admissible and $X_s(W, \Omega) < \infty$, then $\Omega$ is in some way $C^s$ in at least one direction (typically, the direction tangent to a vortex patch of class $C^{1+s}$). We suppose below that $\Omega_{0,\varepsilon}$ has this sort of regularity.

**Proposition 1.** *Let $W_0 = \{w^\mu_0; \mu = 1, \ldots, M\}$ be an admissible system of $C^s$ vector fields on $\mathbb{R}^2$, with $s \in ]0, 1[$. Suppose that*

$$u_{0,\varepsilon} = v_{0,\varepsilon} + \phi_{0,\varepsilon} = \begin{pmatrix} \sigma \\ 0 \end{pmatrix} + \tilde{v}_{0,\varepsilon} + \phi_{0,\varepsilon},$$

*where, for some constant $C_0 \geqslant 1$, independent of $\varepsilon$,*

$$\|\tilde{v}_{0,\varepsilon}\|_{L^2(\mathbb{R}^2)} + \|\Omega_{0,\varepsilon}\|_{L^\infty} \leqslant C_0,$$

*and*

$$\|\tilde{v}_{0,\varepsilon}\|_{H^{s+11/2}(\mathbb{R}^2)} + \|\phi_{0,\varepsilon}\|_{H^{s+7/2}} \leqslant C_0 \varepsilon^{-\alpha}$$

*with $\alpha < 1/24$. Suppose also*

$$C_0 \geqslant \|\sigma\|_{L^\infty} + \|\nabla\sigma\|_{H^{s+11/2}(\mathbb{R}^2)} + \big\|[W_0]^{-1}\big\|_{L^\infty} + \sum_{\mu=1}^{M} \big\|w^\mu_0\big\|_{C^s}$$

*and denote by $C_\varepsilon$ and $\widetilde{C}_\varepsilon$ constants such that*

$$\big\|\operatorname{div}\big(w^\mu_0 \Omega_{0,\varepsilon}\big)\big\|_{C^{s-1}} \leqslant C_\varepsilon$$

*and*

$$\big\|v^3_{0,\varepsilon}\big\|_{\mathrm{Lip}} \leqslant \widetilde{C}_\varepsilon.$$

*Then for all* $p \in ]4, 1/(6\alpha)[$, *there is a constant* $C$ (*independent of* $W_0$, $u_{0,\varepsilon}$, $C_0$, $C_\varepsilon$ *and* $\widetilde{C}_\varepsilon$) *such that the lifespans of the smooth solutions of* (1) *are bounded from below as follows: for all* $\gamma \in ]0, 1/p - 6\alpha[$ *and for all* $\varepsilon < 1$ *such that* $\ln \varepsilon^{-\gamma} \geqslant \widetilde{C}_\varepsilon$,

$$T_\varepsilon \geqslant T_\varepsilon^{(\gamma)} \stackrel{\text{def}}{=} \frac{1}{CC_0} \ln \frac{\ln m(\varepsilon, \gamma)}{\ln(\mathrm{e} + C_\varepsilon)} - 1 \tag{11}$$

*with*

$$m(\varepsilon, \gamma) \stackrel{\text{def}}{=} \min\left(\varepsilon^{-\gamma}, \frac{\ln \varepsilon^{-\gamma}}{\widetilde{C}_\varepsilon}\right).$$

*Moreover, solutions of* (3) *and* (4) *satisfy*:

$$\left\| v_\varepsilon^h(t) \right\|_{\mathrm{Lip}} \leqslant \ln(\mathrm{e} + C_\varepsilon) \mathrm{e}^{CC_0(t+1)} \tag{12}$$

*and*

$$\left\| v_\varepsilon^3(t) \right\|_{\mathrm{Lip}} \leqslant \widetilde{C}_\varepsilon (\mathrm{e} + C_\varepsilon)^{\mathrm{e}^{CC_0(t+1)}} \tag{13}$$

*for all* $t \geqslant 0$, *and*

$$\int_0^t \left\| \phi_\varepsilon(t') \right\|_{\mathrm{Lip}} \mathrm{d}t' \leqslant \varepsilon^{1/p - 6\alpha - \gamma} \tag{14}$$

*for all* $t \in [0, T_\varepsilon^{(\gamma)}]$.

If $C_\varepsilon$ and $\widetilde{C}_\varepsilon$ are such that $T_\varepsilon^{(\gamma)} \to +\infty$ as $\varepsilon \to 0$, so that $\phi_\varepsilon \to 0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{Lip})$, this proposition can be exploited to show convergence results on arbitrarily large intervals of time.

First let us place ourselves under the assumptions of Theorem 1. For a field like $\Omega_0 = \operatorname{rot} v_0^h$, one can construct an admissible system of (two) $C^s$ vector fields, $W_0$, such that $X_s(W_0, \Omega_0) < +\infty$. Then we have for all $w_0^\mu \in W_0$:

$$\begin{aligned} \left\| \operatorname{div}\left(w_0^\mu \operatorname{rot}\left(\rho_{\varepsilon^{-1/156}} * v_0^h\right)\right) \right\|_{C^{s-1}} &= \left\| \operatorname{div}\left(w_0^\mu \left(\rho_{\varepsilon^{-1/156}} * \Omega_0\right)\right) \right\|_{C^{s-1}} \\ &\lesssim \left\| \operatorname{div}\left(w_0^\mu \Omega_0\right) \right\|_{C^{s-1}}, \end{aligned}$$

independently of $\varepsilon$. So Proposition 1 applies with $C_\varepsilon + \widetilde{C}_\varepsilon \leqslant C_0$. Hence, $v_\varepsilon$ is bounded in $\mathrm{Lip}(\mathbb{R}^2)$ on any finite time-interval, uniformly with respect to $\varepsilon$, and therefore the solutions of (3) tend in $L^\infty_{\mathrm{loc}}(\mathbb{R}^+; \sigma + L^2)$ to a solution of (5). Indeed, the difference $\delta$ of two solutions of (6), $\tilde{v}_{\varepsilon_1}$ and $\tilde{v}_{\varepsilon_2}$, satisfies:

$$\begin{cases} \partial_t \delta^h + P(v_{\varepsilon_1}^h \cdot \nabla \delta^h + \delta^h \cdot \nabla \tilde{v}_{\varepsilon_2}^h + \delta^h \cdot \nabla \sigma^h) = 0, \\ \partial_t \delta^3 + v_{\varepsilon_1}^h \cdot \nabla \delta^3 + \delta^h \cdot \nabla v_{\varepsilon_2}^3 = 0, \\ \delta|_{t=0} = v_{0,\varepsilon_1} - v_{0,\varepsilon_2}; \end{cases}$$

since $v_{0,\varepsilon} = \rho_{\varepsilon^{-1/156}} * v_0 \to v_0$ in $\sigma + L^2$, it is easy to see that $\delta \to 0$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}^+; L^2)$ as $\varepsilon_1, \varepsilon_2 \to 0$, by Gronwall's lemma.

Under the assumptions of Theorem 2, $\operatorname{rot} v_0^h$ has no tangential regularity, but we may always take $W_0 = \{w_0^1, w_0^2\}$ with $w_0^1 = (1, 0)$ and $w_0^2 = (0, 1)$ and then write, for $\mu = 1, 2$,

$$\begin{aligned}\big\|\operatorname{div}\big(w_0^\mu \operatorname{rot}\big(\rho_{k(\varepsilon)} * v_0^h\big)\big)\big\|_{C^{s-1}} &= \big\|\operatorname{div}\big(w_0^\mu \big(\rho_{k(\varepsilon)} * \operatorname{rot} v_0^h\big)\big)\big\|_{C^{s-1}} \\ &\lesssim \big\|\rho_{k(\varepsilon)} * \operatorname{rot} v_0^h\big\|_{C^s} \\ &\lesssim k(\varepsilon)^s \big\|\operatorname{rot} v_0^h\big\|_{L^\infty} \\ &\lesssim C_0 \mathrm{e}^{s(\ln \varepsilon^{-1})^\beta}.\end{aligned}$$

So this time Proposition 1 applies with $C_\varepsilon = C_0 \exp(s(\ln \varepsilon^{-1})^\beta)$ and, since $v_{0,\varepsilon}^3 = 0$, with $\widetilde{C}_\varepsilon$ arbitrarily small. Therefore

$$T_\varepsilon \geqslant \frac{1}{CC_0} \ln \frac{\ln \varepsilon^{-\gamma}}{\ln(\mathrm{e} + C_0 \mathrm{e}^{s(\ln \varepsilon^{-1})^\beta})} - 1,$$

which means $T_\varepsilon \gtrsim \ln \ln \varepsilon^{-1}$ for $\varepsilon$ sufficiently small. After that the convergence in $L^\infty_{\mathrm{loc}}(\mathbb{R}^+; \sigma + L^2)$ can be proved exactly as Yudovich's theorem itself [4, Chapter 5].

Now we present the main lines of the proof of Proposition 1; the details are spread in the remaining sections.

First (in Section 3), we recall how the striated regularity is propagated; there is nothing new in this part, only the standard two-dimensional theory of $C^s$ vortex patches [4] written with the notation of the three-dimensional ones [14]. This gives the estimate (12) in a straightforward manner.

Let us note:

$$V_\varepsilon(t) = \int_0^t \big\|v_\varepsilon^h(t')\big\|_{\mathrm{Lip}} \,\mathrm{d}t'$$

and

$$H_\varepsilon(t) = (\mathrm{e} + C_\varepsilon)^{\mathrm{e}^{CC_0(t+1)}}.$$

Since, in view of (12),

$$\mathrm{e}^{CV_\varepsilon(t)} \leqslant H_\varepsilon(t),$$

we have:

$$\big\|v_\varepsilon^3(t)\big\|_{\mathrm{Lip}} \lesssim \big\|v_{0,\varepsilon}^3\big\|_{\mathrm{Lip}} \mathrm{e}^{\int_0^t \|\nabla v_\varepsilon^h(t')\|_{L^\infty} \,\mathrm{d}t'} \lesssim \widetilde{C}_\varepsilon \mathrm{e}^{V_\varepsilon(t)} \lesssim \widetilde{C}_\varepsilon H_\varepsilon(t). \tag{15}$$

This proves (13). In Section 4, we will use (12) and (13) to get:

$$\left\|\tilde{v}_\varepsilon(t)\right\|_{H^{s+11/2}(\mathbb{R}^2)} + \left\|\phi_\varepsilon(t)\right\|_{H^{s+7/2}} \leqslant \varepsilon^{-3\alpha} H_\varepsilon(t) \mathrm{e}^{\widetilde{C}_\varepsilon H_\varepsilon(t)} \mathrm{e}^{C\int_0^t \|\phi_\varepsilon(t')\|_{\mathrm{Lip}}\,\mathrm{d}t'}, \qquad (16)$$

by classical energy methods. (From what follows, it could seem that an estimate of $\tilde{v}_\varepsilon$ in $H^{s+9/2}$ should suffice, but in fact we have to estimate $\tilde{v}_\varepsilon$ in $H^{s+11/2}$ in order to estimate $\phi_\varepsilon$ in $H^{s+7/2}$.)

Finally we define the projectors $P_1$ and $P_{-1}$ on $L^2(\mathbb{R}^3)^3$ by:

$$P_{\pm 1}(D)u = \frac{1}{2}\left(Pu \pm \mathrm{i}\frac{D}{|D|} \times u\right)$$

for all $u \in L^2(\mathbb{R}^3)^3$. Remark that

$$-\frac{\xi}{|\xi|} \times P_{\pm 1}(\xi)\hat{u}(\xi) = \pm \mathrm{i} P_{\pm 1}(\xi)\hat{u}(\xi)$$

for all $\xi \in \mathbb{R}^3 \setminus \{0\}$; therefore,

$$A(D)\phi_\varepsilon = A(D)(P_1\phi_\varepsilon + P_{-1}\phi_\varepsilon) = \mathrm{i}\frac{D_3}{|D|}(P_1\phi_\varepsilon - P_{-1}\phi_\varepsilon).$$

As $P_{\pm 1}P_{\pm 1} = P_{\pm 1}$ and $P_{\pm 1}P_{\mp 1} = 0$, we deduce from (4) that

$$\begin{cases} \partial_t P_{\pm 1}\phi_\varepsilon \pm \frac{\mathrm{i}}{\varepsilon}\frac{D_3}{|D|} P_{\pm 1}\phi_\varepsilon = -P_{\pm 1}(v_\varepsilon \cdot \nabla\phi_\varepsilon + \phi_\varepsilon \cdot \nabla\phi_\varepsilon + \phi_\varepsilon \cdot \nabla v_\varepsilon), \\ P_{\pm 1}\phi_\varepsilon|_{t=0} = P_{\pm 1}\phi_{0,\varepsilon}. \end{cases}$$

So, by Corollary 3—the dispersive result, proved in Section 5—and the tame estimate (29), we get:

$$\begin{aligned} \int_0^t \left\|\phi_\varepsilon(t')\right\|_{\mathrm{Lip}} \mathrm{d}t' &\lesssim \varepsilon^{1/p} t^{1-1/p}\Bigg(\|\phi_{0,\varepsilon}\|_{H^{s+5/2}} + \int_0^t \left\|v_\varepsilon(t') \cdot \nabla\phi_\varepsilon(t')\right\|_{H^{s+5/2}} \mathrm{d}t' \\ &\quad + \int_0^t \left\|\phi_\varepsilon(t') \cdot \nabla\phi_\varepsilon(t')\right\|_{H^{s+5/2}} \mathrm{d}t' + \int_0^t \left\|\phi_\varepsilon(t') \cdot \nabla v_\varepsilon(t')\right\|_{H^{s+5/2}} \mathrm{d}t'\Bigg) \\ &\lesssim \varepsilon^{1/p} t^{1-1/p}\Bigg(\|\phi_{0,\varepsilon}\|_{H^{s+5/2}} + \int_0^t \left\|\phi_\varepsilon(t')\right\|_{H^{s+7/2}}^2 \mathrm{d}t' \\ &\quad + \int_0^t \left(\left\|v_\varepsilon(t')\right\|_{L^\infty} + \left\|\nabla v_\varepsilon(t')\right\|_{H^{s+7/2}(\mathbb{R}^2)}\right)\left\|\phi_\varepsilon(t')\right\|_{H^{s+7/2}} \mathrm{d}t'\Bigg) \end{aligned}$$

for all $\varepsilon > 0$ and all $t < T_\varepsilon$.

Therefore, using (16), we see that

$$\int_0^t \left\|\phi_\varepsilon(t')\right\|_{\mathrm{Lip}}\,\mathrm{d}t' \leqslant \varepsilon^{1/p-6\alpha} H_\varepsilon(t)\mathrm{e}^{\widetilde{C}_\varepsilon H_\varepsilon(t)}\mathrm{e}^{C\int_0^t \|\phi_\varepsilon(t')\|_{\mathrm{Lip}}\,\mathrm{d}t'}$$

$$\leqslant \varepsilon^{1/p-6\alpha} H_\varepsilon(t)\mathrm{e}^{\widetilde{C}_\varepsilon H_\varepsilon(t)}$$

as long as $\int_0^t \|\phi_\varepsilon(t')\|_{\mathrm{Lip}}\,\mathrm{d}t' \leqslant 1$, for some possibly new constant $C$ inside $H_\varepsilon$. This proves (14), because

$$H_\varepsilon(t)\mathrm{e}^{\widetilde{C}_\varepsilon H_\varepsilon(t)} \leqslant \varepsilon^{-\gamma}$$

is equivalent to (11), and so concludes the proof of Proposition 1.

## 3. Propagation of the striated regularity

Taking the rotational of $\partial_t v_\varepsilon^h + P(v_\varepsilon^h \cdot \nabla_h v_\varepsilon^h) = 0$, it is readily seen that $\Omega_\varepsilon$ is preserved along the lines of flow:

$$\begin{cases} \partial_t \Omega_\varepsilon + v_\varepsilon^h \cdot \nabla_h \Omega_\varepsilon = 0, \\ \Omega_\varepsilon|_{t=0} = \Omega_{0,\varepsilon} = \operatorname{rot} v_{0,\varepsilon}. \end{cases} \tag{17}$$

Define $w_\varepsilon^\mu$ as the solution of

$$\begin{cases} \partial_t w_\varepsilon^\mu + v_\varepsilon^h \cdot \nabla w_\varepsilon^\mu = w_\varepsilon^\mu \cdot \nabla v_\varepsilon^h, \\ w_\varepsilon^\mu|_{t=0} = w_0^\mu. \end{cases} \tag{18}$$

As $W_0$ is admissible, $W_\varepsilon(t)$ is admissible at any time $t$, with the estimate:

$$\left\|\left[W_\varepsilon(t)\right]^{-1}\right\|_{L^\infty} \lesssim \left\|[W_0]^{-1}\right\|_{L^\infty}\mathrm{e}^{CV_\varepsilon(t)} \lesssim C_0\mathrm{e}^{CV_\varepsilon(t)}, \tag{19}$$

and, consequently, the static estimate stays also valid for all times:

$$\begin{aligned} \left\|v_\varepsilon^h(t)\right\|_{\mathrm{Lip}} &\lesssim \|\sigma\|_{L^\infty} + \left\|\tilde{v}_\varepsilon^h(t)\right\|_{L^2} + \left\|\Omega_\varepsilon(t)\right\|_{L^\infty}\ln\left(\mathrm{e} + \frac{X_s(W_\varepsilon(t), \Omega_\varepsilon(t))}{\|\Omega_\varepsilon(t)\|_{L^\infty}}\right) \\ &\lesssim C_0 + C_0\mathrm{e}^{C_0 t} + C_0\ln\left(\mathrm{e} + \frac{X_s(W_\varepsilon(t), \Omega_\varepsilon(t))}{C_0}\right) \\ &\lesssim C_0 + C_0\mathrm{e}^{C_0 t} + C_0\ln\left(\mathrm{e} + X_s\left(W_\varepsilon(t), \Omega_\varepsilon(t)\right)\right). \end{aligned} \tag{20}$$

We have used

$$\left\|\tilde{v}_\varepsilon^h(t)\right\|_{L^2} \leqslant \left\|\tilde{v}_{0,\varepsilon}^h\right\|_{L^2}\mathrm{e}^{\|\nabla\sigma\|_{L^\infty} t} \lesssim C_0\mathrm{e}^{C_0 t}, \tag{21}$$

which follows directly from (6), and

$$\left\|\Omega_\varepsilon(t)\right\|_{L^\infty} \lesssim C_0, \tag{22}$$

which is due to (17). Owing to (17) and (18), $\operatorname{div}(w_\varepsilon^\mu \Omega_\varepsilon)$ is preserved along the lines of flow, so

$$\left\|\operatorname{div}\bigl(w_\varepsilon^\mu(t)\Omega_\varepsilon(t)\bigr)\right\|_{C^{s-1}} \lesssim C_\varepsilon \mathrm{e}^{CV_\varepsilon(t)}. \tag{23}$$

The remaining term in (10) is estimated thanks to the inequality,

$$\left\|w_\varepsilon^\mu \cdot \nabla v_\varepsilon^h\right\|_{C^s} \lesssim \left\|v_\varepsilon^h\right\|_{\mathrm{Lip}}\left\|w_\varepsilon^\mu\right\|_{C^s} + \left\|\operatorname{div}(w_\varepsilon^\mu \Omega_\varepsilon)\right\|_{C^{s-1}},$$

which shows that the velocity inherits the striated regularity of its curl [9, Section 2.4.1] and leads to:

$$\left\|w_\varepsilon^\mu(t)\right\|_{C^s} \lesssim C_0 + \int_0^t \left\|v_\varepsilon^h(t')\right\|_{\mathrm{Lip}}\left\|w_\varepsilon^\mu(t')\right\|_{C^s}\,\mathrm{d}t' + \int_0^t \left\|\operatorname{div}\bigl(w_\varepsilon^\mu(t')\Omega_\varepsilon(t')\bigr)\right\|_{C^{s-1}}\,\mathrm{d}t',$$

so that

$$\left\|w_\varepsilon^\mu(t)\right\|_{C^s} \lesssim \bigl(C_0 + C_\varepsilon \mathrm{e}^{CV_\varepsilon(t)} t\bigr)\mathrm{e}^{CV_\varepsilon(t)}. \tag{24}$$

Putting (19), (22)–(24) together gives:

$$X_s\bigl(W_\varepsilon(t), \Omega_\varepsilon(t)\bigr) \lesssim (C_0 + C_\varepsilon + C_\varepsilon t)\mathrm{e}^{CV_\varepsilon(t)};$$

hence, substituting in (20), we get:

$$\left\|v_\varepsilon^h(t)\right\|_{\mathrm{Lip}} \lesssim C_0 + C_0\mathrm{e}^{C_0 t} + C_0\bigl(\ln(\mathrm{e} + C_0 + C_\varepsilon + C_\varepsilon t) + V_\varepsilon(t)\bigr),$$

from which we deduce (12) by Gronwall's lemma.

## 4. Energy estimates

We begin with estimates on $\phi_\varepsilon$.

Since $A(\xi)\hat{u}(\xi) \cdot \hat{u}(\xi) = 0$ for all $\xi \neq 0$, we have by Plancherel's formula:

$$\bigl(A(D)u, u\bigr)_{L^2} = 0$$

for all $u \in L^2$. Therefore, we get from (4):

$$\left\|\phi_\varepsilon(t)\right\|_{L^2} \leqslant \|\phi_{0,\varepsilon}\|_{L^2}\mathrm{e}^{\int_0^t \|\nabla v_\varepsilon(t')\|_{L^\infty}\,\mathrm{d}t'}, \tag{25}$$

and, by derivation of (4),

$$\|\nabla\phi_\varepsilon(t)\|_{L^2} \lesssim \|\nabla\phi_{0,\varepsilon}\|_{L^2} + \int_0^t \big(\|\nabla v_\varepsilon(t')\|_{L^\infty} + \|\nabla\phi_\varepsilon(t')\|_{L^\infty}\big)\|\nabla\phi_\varepsilon(t')\|_{L^2}\,\mathrm{d}t' + \int_0^t \|\phi_\varepsilon(t')\|_{L^2}\|\nabla^2 v_\varepsilon(t')\|_{L^\infty}\,\mathrm{d}t'. \tag{26}$$

A consequence of (25) is that

$$\begin{aligned}\|\phi_\varepsilon(t)\|_{L^2} &\leqslant C_0\varepsilon^{-\alpha}\mathrm{e}^{\int_0^t \|\nabla v_\varepsilon^h(t')\|_{L^\infty}\,\mathrm{d}t'}\mathrm{e}^{\int_0^t \|\nabla v_\varepsilon^3(t')\|_{L^\infty}\,\mathrm{d}t'}\\ &\leqslant C_0\varepsilon^{-\alpha}\mathrm{e}^{V_\varepsilon(t)}\mathrm{e}^{\widetilde{C}_\varepsilon H_\varepsilon(t)}\\ &\leqslant \varepsilon^{-\alpha}H_\varepsilon(t)\mathrm{e}^{\widetilde{C}_\varepsilon H_\varepsilon(t)}.\end{aligned} \tag{27}$$

More generally, in order to obtain estimates in $H^r$ for $r > 0$, we use Littlewood–Paley theory [4,20,21]. The operators $\Delta_m$ are defined by:

$$\Delta_m f = \begin{cases}\varphi(2^{-m}D)f & \text{if } m \geqslant 0,\\ \chi(|D|)f & \text{if } m = -1,\\ 0 & \text{if } m \leqslant -2,\end{cases}$$

where $\chi \in C_0^\infty(\mathbb{R}^+)$ is equal to 1 near the origin, so that

$$\varphi(\xi) \stackrel{\text{def}}{=} \chi\big(|\xi|/2\big) - \chi\big(|\xi|\big) \tag{28}$$

is supported in an annulus. For $m \geqslant -1$, we write:

$$\begin{aligned}&\frac{1}{2}\partial_t\|\Delta_m\phi_\varepsilon\|_{L^2}^2 + \big(v_\varepsilon\cdot\nabla\Delta_m\phi_\varepsilon + \phi_\varepsilon\cdot\nabla\Delta_m\phi_\varepsilon + \Delta_m(\phi_\varepsilon\cdot\nabla v_\varepsilon), \Delta_m\phi_\varepsilon\big)_{L^2}\\ &\quad= \big([v_\varepsilon\cdot\nabla, \Delta_m]\phi_\varepsilon, \Delta_m\phi_\varepsilon\big)_{L^2} + \big([\phi_\varepsilon\cdot\nabla, \Delta_m]\phi_\varepsilon, \Delta_m\phi_\varepsilon\big)_{L^2},\end{aligned}$$

which implies

$$\begin{aligned}\|\Delta_m\phi_\varepsilon(t)\|_{L^2} &\lesssim \|\Delta_m\phi_{0,\varepsilon}\|_{L^2} + \int_0^t \big\|\Delta_m\big(\phi_\varepsilon(t')\cdot\nabla v_\varepsilon(t')\big)\big\|_{L^2}\,\mathrm{d}t'\\ &\quad+ \int_0^t \big\|\big[v_\varepsilon(t')\cdot\nabla, \Delta_m\big]\phi_\varepsilon(t')\big\|_{L^2}\,\mathrm{d}t' + \int_0^t \big\|\big[\phi_\varepsilon(t')\cdot\nabla, \Delta_m\big]\phi_\varepsilon(t')\big\|_{L^2}\,\mathrm{d}t';\end{aligned}$$

then we multiply both sides by $2^{m(s+7/2)}$, take the $l^2$-norm and use the following lemmas:

**Lemma 1.** *If $a \in \sigma + H^r(\mathbb{R}_h^2) \cap L^\infty(\mathbb{R}_h^2)$, with $\nabla a \in H^r(\mathbb{R}_h^2)$, and $b \in H^r(\mathbb{R}^3) = H^r(\mathbb{R}_h^2 \times \mathbb{R})$, then*

$$\|ab\|_{H^r} \lesssim \|a\|_{L^\infty}\|b\|_{H^r} + \|b\|_{L^2}\|\nabla a\|_{H^r(\mathbb{R}^2)} \tag{29}$$

*and*

$$\left\|\left(2^{mr}\left\|[a\cdot\nabla, \Delta_m]b\right\|_{L^2}\right)_{m=-1}^{+\infty}\right\|_{l^2} \lesssim \|\nabla a\|_{L^\infty}\|b\|_{H^r} + \|\nabla b\|_{L^2}\|\nabla a\|_{H^r(\mathbb{R}^2)}. \tag{30}$$

**Lemma 2.** *If $a, b \in H^r(\mathbb{R}^d) \cap \mathrm{Lip}(\mathbb{R}^d)$, then*

$$\left\|\left(2^{mr}\left\|[a\cdot\nabla, \Delta_m]b\right\|_{L^2}\right)_{m=-1}^{+\infty}\right\|_{l^2} \lesssim \|\nabla a\|_{L^\infty}\|b\|_{H^r} + \|\nabla b\|_{L^\infty}\|a\|_{H^r}.$$

Once these two lemmas will have been proved, we will have:

$$\begin{aligned}\left\|\phi_\varepsilon(t)\right\|_{H^{s+7/2}} \lesssim \|\phi_{0,\varepsilon}\|_{H^{s+7/2}} &+ \int_0^t \left\|\phi_\varepsilon(t')\right\|_{H^1}\left\|\nabla v_\varepsilon(t')\right\|_{H^{s+9/2}(\mathbb{R}^2)}\,\mathrm{d}t' \\ &+ \int_0^t \left(\left\|\nabla v_\varepsilon(t')\right\|_{L^\infty} + \left\|\nabla\phi_\varepsilon(t')\right\|_{L^\infty}\right)\left\|\phi_\varepsilon(t')\right\|_{H^{s+7/2}}\,\mathrm{d}t'.\end{aligned} \tag{31}$$

**Proof of Lemma 1.** The product $ab$ is decomposed as follows:

$$ab = \sum_{m'=-1}^{+\infty} S_{m'+2}a\,\Delta_{m'}b + \sum_{m'=1}^{+\infty} S_{m'-1}b\,\Delta_{m'}a,$$

with the notation:

$$S_n = \chi\left(2^{-n}|D|\right) = \sum_{m\leqslant n-1}\Delta_m.$$

So we see that

$$\Delta_m(ab) = \sum_{m'=m-N}^{+\infty}\Delta_m(S_{m'+2}a\,\Delta_{m'}b) + \sum_{m'=\min(1,\,m-N)}^{m+N}\Delta_m(S_{m'-1}b\,\Delta_{m'}a)$$

for some integer $N$ depending only on $\chi$. Therefore, we have:

$$
\begin{aligned}
2^{mr}\left\|\Delta_m(ab)\right\|_{L^2} &\lesssim 2^{mr}\sum_{m'=m-N}^{+\infty}\|a\|_{L^\infty}\|\Delta_{m'}b\|_{L^2} \\
&\quad + 2^{mr}\sum_{m'=\min(1,m-N)}^{m+N}\|S_{m'-1}b\|_{L^{\infty,2}_{x_h,x_3}}\|\Delta_{m'}a\|_{L^2(\mathbb{R}^2)} \\
&\lesssim \|a\|_{L^\infty}\sum_{m'=m-N}^{+\infty}2^{(m-m')r}2^{m'r}\|\Delta_{m'}b\|_{L^2} \\
&\quad + \|b\|_{L^2}\sum_{m'=\min(1,m-N)}^{m+N}2^{(m-m')r}2^{m'r}\|\Delta_{m'}\nabla a\|_{L^2(\mathbb{R}^2)},
\end{aligned}
$$

thanks to a classical Bernstein's inequality [4, Lemma 2.1.1],

$$
\|\Delta_{m'}a\|_{L^2(\mathbb{R}^2)} \lesssim 2^{-m'}\|\Delta_{m'}\nabla a\|_{L^2(\mathbb{R}^2)}, \tag{32}
$$

and one of its anisotropic generalizations [12, Lemma 5.2],

$$
\|S_{m'-1}b\|_{L^{\infty,2}_{x_h,x_3}} \lesssim 2^{m'}\|S_{m'-1}b\|_{L^2}, \tag{33}
$$

where the norm $L^{\infty,2}_{x_h,x_3}$ is defined on the functions $f = f(x_h, x_3)$ by:

$$
\|f\|_{L^{\infty,2}_{x_h,x_3}} = \left\|\|f\|_{L^2_{x_3}}\right\|_{L^\infty_{x_h}} = \left\|\left(\int_{\mathbb{R}}\left|f(\cdot,x_3)\right|^2\,\mathrm{d}x_3\right)^{1/2}\right\|_{L^\infty(\mathbb{R}^2_h)}.
$$

Then (29) follows by Young's inequality:

$$
\begin{aligned}
\|ab\|_{H^r} &= \left\|\left(2^{mr}\left\|\Delta_m(ab)\right\|_{L^2}\right)_{m=-1}^{+\infty}\right\|_{l^2} \\
&\lesssim \|a\|_{L^\infty}\left\|(2^{-mr})_{m=N}^{+\infty}\right\|_{l^1}\left\|\left(2^{mr}\|\Delta_m b\|_{L^2}\right)_{m=-1}^{+\infty}\right\|_{l^2} \\
&\quad + \|b\|_{L^2}\left\|(2^{mr})_{m=-N}^{+N}\right\|_{l^1}\left\|\left(2^{mr}\|\Delta_m\nabla a\|_{L^2(\mathbb{R}^2)}\right)_{m=-1}^{+\infty}\right\|_{l^2} \\
&\lesssim \|a\|_{L^\infty}\|b\|_{H^r} + \|b\|_{L^2}\|\nabla a\|_{H^r(\mathbb{R}^2)}.
\end{aligned}
$$

The proof of (30) goes along the same lines:

$$
\begin{aligned}
[a\cdot\nabla,\Delta_m]b &= \sum_{m'=m-N}^{m+N}\left[(S_{m'-1}a)\cdot\nabla,\Delta_m\right]\Delta_{m'}b \\
&\quad + c_m + \sum_{m'=\min(0,m-N)}^{+\infty}\left[(\Delta_{m'}a)\cdot\nabla,\Delta_m\right]S_{m'+2}b,
\end{aligned}
$$

where:

$$c_m \stackrel{\text{def}}{=} \begin{cases} [(\Delta_{-1}a)\cdot\nabla,\Delta_m]S_1 b & \text{if } m-N\leqslant -1,\\ 0 & \text{otherwise.}\end{cases}$$

Firstly, we have:

$$\begin{aligned}
&\left\|\left(2^{mr}\left\|\sum_{m'=m-N}^{m+N}\big[(S_{m'-1}a)\cdot\nabla,\Delta_m\big]\Delta_{m'}b\right\|_{L^2}\right)_{m=-1}^{+\infty}\right\|_{l^2}\\
&\quad\lesssim\left\|\left(2^{mr}\sum_{m'=m-N}^{m+N}2^{-m}\|\nabla S_{m'-1}a\|_{L^\infty}\|\nabla\Delta_{m'}b\|_{L^2}\right)_{m=-1}^{+\infty}\right\|_{l^2}\\
&\quad\lesssim\|\nabla a\|_{L^\infty}\left\|\left(\sum_{m'=m-N}^{m+N}2^{(m-m')(r-1)}2^{m'(r-1)}\|\Delta_{m'}\nabla b\|_{L^2}\right)_{m=-1}^{+\infty}\right\|_{l^2}\\
&\quad\lesssim\|\nabla a\|_{L^\infty}\|\nabla b\|_{H^{r-1}};
\end{aligned}$$

here we have used the inequality,

$$\big\|[f,\Delta_m]g\big\|_{L^2}\lesssim 2^{-m}\|\nabla f\|_{L^\infty}\|g\|_{L^2},\tag{34}$$

which is true for any $f\in\mathrm{Lip}$, $g\in L^2$ and $m\geqslant -1$. Secondly, using (34) on $c_m$, we get:

$$\big\|\big(2^{mr}\|c_m\|_{L^2}\big)_{m=-1}^{+\infty}\big\|_{l^2}\lesssim\|\nabla a\|_{L^\infty}\|b\|_{L^2}.$$

And thirdly, we have:

$$\begin{aligned}
&\left\|\left(2^{mr}\left\|\sum_{m'=\min(0,m-N)}^{+\infty}\big[(\Delta_{m'}a)\cdot\nabla,\Delta_m\big]S_{m'+2}b\right\|_{L^2}\right)_{m=-1}^{+\infty}\right\|_{l^2}\\
&\quad\lesssim\left\|\left(2^{mr}\sum_{m'=\min(0,m-N)}^{+\infty}\|\Delta_{m'}a\|_{L^2(\mathbb{R}^2)}\|S_{m'+2}\nabla b\|_{L^{\infty,2}_{x_h,x_3}}\right)_{m=-1}^{+\infty}\right\|_{l^2}\\
&\quad\lesssim\left\|\left(\sum_{m'=\min(0,m-N)}^{+\infty}2^{(m-m')r}2^{m'r}\|\Delta_{m'}\nabla a\|_{L^2(\mathbb{R}^2)}\|\nabla b\|_{L^2}\right)_{m=-1}^{+\infty}\right\|_{l^2}\\
&\quad\lesssim\|\nabla b\|_{L^2}\|\nabla a\|_{H^r(\mathbb{R}^2)},
\end{aligned}$$

owing to (32) and (33) again. □

**Proof of Lemma 2.** The proof may safely be omitted, as it is similar to (and easier than) that of the second part of Lemma 1. □

Now we turn to the estimates of higher orders on $\tilde v_\varepsilon$.

**Lemma 3.** *If $a \in \sigma + H^r(\mathbb{R}^2) \cap \mathrm{Lip}(\mathbb{R}^2)$ and $b \in H^r(\mathbb{R}^2) \cap \mathrm{Lip}(\mathbb{R}^2)$, then*

$$\left\| \left(2^{mr} \left\| [a \cdot \nabla, \Delta_m] b \right\|_{L^2}\right)_{m=-1}^{+\infty} \right\|_{l^2} \lesssim \|\nabla a\|_{L^\infty} \|b\|_{H^r} + \|\nabla b\|_{L^\infty} \|\nabla a\|_{H^{r-1}}.$$

**Proof.** Again, the proof is very much the same as the proof of (30). □

Thanks to this lemma and Lemma 2 (with $d = 2$) and proceeding like we have just done for $\phi_\varepsilon$, we get from (6) the estimate:

$$\begin{aligned}\left\| \tilde{v}_\varepsilon(t) \right\|_{H^{s+11/2}} \lesssim \|\tilde{v}_{0,\varepsilon}\|_{H^{s+11/2}} &+ \int_0^t \left(\left\| v_\varepsilon(t') \right\|_{\mathrm{Lip}} + \|\sigma\|_{\mathrm{Lip}}\right) \left\| \tilde{v}_\varepsilon(t') \right\|_{H^{s+11/2}} \mathrm{d}t' \\ &+ \int_0^t \left\| v_\varepsilon(t') \right\|_{\mathrm{Lip}} \|\nabla \sigma\|_{H^{s+11/2}} \mathrm{d}t',\end{aligned}$$

so

$$\begin{aligned}\left\| \tilde{v}_\varepsilon(t) \right\|_{H^{s+11/2}} &\lesssim \left( C_0 \varepsilon^{-\alpha} + C_0 \int_0^t \left\| v_\varepsilon(t') \right\|_{\mathrm{Lip}} \mathrm{d}t' \right) \mathrm{e}^{\int_0^t \|v_\varepsilon(t')\|_{\mathrm{Lip}} \mathrm{d}t' + C_0 t} \\ &\lesssim \varepsilon^{-\alpha} H_\varepsilon(t) \mathrm{e}^{\widetilde{C}_\varepsilon H_\varepsilon(t)}.\end{aligned} \tag{35}$$

Introducing this in (26) yields:

$$\left\| \phi_\varepsilon(t) \right\|_{H^1} \leqslant \varepsilon^{-2\alpha} H_\varepsilon(t) \mathrm{e}^{\widetilde{C}_\varepsilon H_\varepsilon(t)} \mathrm{e}^{C \int_0^t \|\phi_\varepsilon(t')\|_{\mathrm{Lip}} \mathrm{d}t'},$$

which gives, substituted in (31) with (35),

$$\left\| \phi_\varepsilon(t) \right\|_{H^{s+7/2}} \leqslant \varepsilon^{-3\alpha} H_\varepsilon(t) \mathrm{e}^{\widetilde{C}_\varepsilon H_\varepsilon(t)} \mathrm{e}^{C \int_0^t \|\phi_\varepsilon(t')\|_{\mathrm{Lip}} \mathrm{d}t'}.$$

## 5. Dispersive estimates

Here is the main dispersive inequality.

**Proposition 2.** *If $\widetilde{\mathcal{C}} \subset \mathbb{R}^3$ is an annulus centered at the origin and $\psi \in C^\infty(\mathbb{R}^3)$ is supported in $\widetilde{\mathcal{C}}$, the function $K$ defined by*:

$$K(\tau, z) = \int_{\mathbb{R}^3_\xi} \psi(\xi) \mathrm{e}^{\mathrm{i}\tau \frac{\xi_3}{|\xi|} + \mathrm{i} z \cdot \xi} \, \mathrm{d}\xi, \quad \tau \geqslant 1, \ z \in \mathbb{R}^3, \tag{36}$$

*satisfies*

$$\left\|K(\tau,\cdot)\right\|_{L^\infty} \leqslant C\tau^{-1/2}\ln(\mathrm{e}+\tau). \tag{37}$$

**Proof.** If $\varphi$ is the function defined by (28) for $\xi\in\mathbb{R}$, the series

$$\sum_{m=-\infty}^{M}\varphi(2^{-m}\xi_3)\psi(\xi)$$

converges to $\psi(\xi)$ in $L^1$ for some $M\geqslant 0$. So

$$\left\|K(\tau,\cdot)\right\|_{L^\infty} \leqslant \sum_{m=-\infty}^{M}\left\|K_m(\tau,\cdot)\right\|_{L^\infty}$$

with

$$K_m(\tau,z)\stackrel{\text{def}}{=}\int_{\mathbb{R}^3_\xi}\varphi(2^{-m}\xi_3)\psi(\xi)\mathrm{e}^{\mathrm{i}\tau\frac{\xi_3}{|\xi|}+\mathrm{i}z\cdot\xi}\,\mathrm{d}\xi.$$

The proof of estimate (3.4) in the preprint of Chemin, Desjardins, Gallagher and Grenier [5, p. 9] gives readily:

$$\left\|K_m(\tau,\cdot)\right\|_{L^\infty} \leqslant C'\|\psi\|_{L^1}\min\left(2^m,\tau^{-1/2}\right), \tag{38}$$

the constant $C'$ being independent of $m$, $\tau$ and $\psi$. Hence:

$$\left\|K(\tau,\cdot)\right\|_{L^\infty} \leqslant C'\|\psi\|_{L^1}\sum_{m=-\infty}^{\lfloor\log_2\tau^{-1/2}\rfloor}2^m + C'\|\psi\|_{L^1}\tau^{-1/2}\left(M-\left\lfloor\log_2\tau^{-1/2}\right\rfloor\right),$$

which gives (37). □

The idea of the proof of (38) is that integrating by parts in one horizontal direction gives:

$$\left\|K_m(\tau,\cdot)\right\|_{L^\infty} \lesssim \tau^{-1/2}\|\psi\|_{L^1},$$

independently of $m$ (it does not give directly a dispersive estimate on $K$ because the first derivative of $\xi_3/|\xi|$ with respect to any horizontal direction vanishes when $\xi_3=0$), while

$$\left\|K_m(\tau,\cdot)\right\|_{L^\infty} \lesssim 2^m\|\psi\|_{L^1}$$

by a Bernstein's inequality. Alternatively, a dispersive inequality for low vertical frequencies can be proved thanks to the variation of $\xi_3/|\xi|$ with respect to $\xi_3$ (see Lemma 4 in

Section 6). This method leads to Proposition 3 which, though usable, is less good than Proposition 2: one finally gets:

$$\left\| K(\tau,\cdot) \right\|_{L^\infty} \leqslant C\tau^{-1/3}$$

instead of (37).

Starting from Proposition 2, the Strichartz estimates can be deduced in the following classical way.

**Corollary 1.** *Let $\mathcal{C} \subset \mathbb{R}^3$ be an annulus centered at the origin. Let $r \in [2,+\infty]$ and let $f_m$ be $C^\infty$ functions such that* $\operatorname{supp} \hat{f}_m \subset 2^m\mathcal{C}$ *for all $m \in \mathbb{Z}$.*

*Then*

$$\left\| e^{\pm i \frac{t}{\varepsilon} \frac{D_3}{|D|}} f_m \right\|_{L^r} \lesssim \left( 2^{3m} \max\left(1, \frac{t}{\varepsilon}\right)^{-1/2} \ln\left(e + \frac{t}{\varepsilon}\right)\right)^{1-2/r} \| f_m \|_{L^{\bar{r}}} \tag{39}$$

*for all $t \in \mathbb{R}^+$ and all $\varepsilon > 0$.*

**Proof.** Plancherel's formula implies:

$$\left\| e^{\pm i \frac{t}{\varepsilon} \frac{D_3}{|D|}} f_m \right\|_{L^2} = \| f_m \|_{L^2} \tag{40}$$

for all $t$ and $\varepsilon$. Below we show that

$$\left\| e^{\pm i \frac{t}{\varepsilon} \frac{D_3}{|D|}} f_m \right\|_{L^\infty} \lesssim 2^{3m} \max\left(1, \frac{t}{\varepsilon}\right)^{-1/2} \ln\left(e + \frac{t}{\varepsilon}\right) \| f_m \|_{L^1}, \tag{41}$$

so (39) will follow from (40) and (41) by interpolation.

Let $\psi \in C_0^\infty(\mathbb{R}^3)$ equal to 1 on $\mathcal{C}$ and vanishing near the origin. Then

$$\begin{aligned} e^{\pm i \frac{t}{\varepsilon} \frac{D_3}{|D|}} f_m &= e^{\pm i \frac{t}{\varepsilon} \frac{D_3}{|D|}} \psi(2^{-m}D) f_m \\ &= e^{\pm i \frac{t}{\varepsilon} \frac{D_3}{|D|}} \psi(D)\big(f_m(2^{-m}\cdot)\big), \end{aligned}$$

thanks to the homogeneity of $\xi_3/|\xi|$, so

$$\left\| e^{\pm i \frac{t}{\varepsilon} \frac{D_3}{|D|}} f_m \right\|_{L^\infty} \leqslant 2^{3m} \left\| e^{\pm i \frac{t}{\varepsilon} \frac{D_3}{|D|}} \mathcal{F}^{-1}\psi \right\|_{L^\infty} \| f_m \|_{L^1}.$$

Now (41) follows either from Proposition 2 (if $t \geqslant \varepsilon$) or by integrations by parts (if $t \leqslant \varepsilon$, in which case deriving the phase costs nothing). □

**Corollary 2.** *For all $p > 4r/(r-2)$, with the same notation as in Corollary 1, we have:*

$$\left\| e^{\pm i \frac{t}{\varepsilon} \frac{D_3}{|D|}} f_m \right\|_{L_t^p(L_x^r)} \lesssim \varepsilon^{1/p} 2^{3m(1/2-1/r)} \| f_m \|_{L^2}. \tag{42}$$

**Proof.** The proof is a $TT^\star$ argument [5,6], which we recall only for the convenience of the reader.

Let $\bar{p}$ such that $1/p + 1/\bar{p} = 1$ and

$$\mathcal{B} \stackrel{\text{def}}{=} \left\{\Psi \in C_0^\infty\left(\mathbb{R}^+ \times \mathbb{R}^3\right); \|\Psi\|_{L_t^{\bar{p}}(L_x^{\bar{r}})} \leqslant 1\right\}.$$

Then

$$\begin{aligned}
\left\|\mathrm{e}^{\pm \mathrm{i}\frac{t}{\varepsilon}\frac{D_3}{|D|}} f_m\right\|_{L_t^p(L_x^r)} &= \sup_{\Psi\in\mathcal{B}} \int_0^\infty \left(\mathrm{e}^{\pm \mathrm{i}\frac{t}{\varepsilon}\frac{D_3}{|D|}} f_m, \Psi(t)\right)_{L^2} \mathrm{d}t \\
&= \sup_{\Psi\in\mathcal{B}} \int_0^\infty \left(f_m, \mathrm{e}^{\mp \mathrm{i}\frac{t}{\varepsilon}\frac{D_3}{|D|}} \psi(2^{-m}D)\Psi(t)\right)_{L^2} \mathrm{d}t \\
&\leqslant \|f_m\|_{L^2}\alpha,
\end{aligned}$$

with $\psi$ as in the proof of Corollary 1 and

$$\alpha \stackrel{\text{def}}{=} \sup_{\Psi\in\mathcal{B}} \left\| \int_0^\infty \mathrm{e}^{\mp \mathrm{i}\frac{t}{\varepsilon}\frac{D_3}{|D|}} \psi(2^{-m}D)\Psi(t)\,\mathrm{d}t \right\|_{L^2}.$$

We have

$$\begin{aligned}
\alpha^2 &= \sup_{\Psi\in\mathcal{B}} \left( \int_0^\infty \mathrm{e}^{\mp \mathrm{i}\frac{t}{\varepsilon}\frac{D_3}{|D|}} \psi(2^{-m}D)\Psi(t)\,\mathrm{d}t, \int_0^\infty \mathrm{e}^{\mp \mathrm{i}\frac{s}{\varepsilon}\frac{D_3}{|D|}} \psi(2^{-m}D)\Psi(s)\,\mathrm{d}s \right)_{L^2} \\
&= \sup_{\Psi\in\mathcal{B}} \int_0^\infty \int_0^\infty \left(\psi(2^{-m}D)\Psi(t), \mathrm{e}^{\pm \mathrm{i}\frac{t-s}{\varepsilon}\frac{D_3}{|D|}} \psi(2^{-m}D)\Psi(s)\right)_{L^2} \mathrm{d}s\,\mathrm{d}t \\
&\leqslant \sup_{\Psi\in\mathcal{B}} \int_0^\infty \int_0^\infty \left\|\psi(2^{-m}D)\Psi(t)\right\|_{L^{\bar{r}}} \left\|\psi(2^{-m}D)\Psi(s)\right\|_{L^{\bar{r}}} \\
&\quad \times \left(2^{3m} \max\left(1, \frac{|t-s|}{\varepsilon}\right)^{-1/2} \ln\left(\mathrm{e} + \frac{|t-s|}{\varepsilon}\right)\right)^{1-2/r} \mathrm{d}s\,\mathrm{d}t,
\end{aligned}$$

applying Corollary 1 with $f_m = \psi(2^{-m}D)\Psi(s)$. Hence, by the Hölder inequality,

$$\begin{aligned}
\alpha^2 \lesssim 2^{3m(1-2/r)} \Bigg\| \int_0^\infty &\left\|\psi(2^{-m}D)\Psi(s)\right\|_{L^{\bar{r}}} \\
&\times \left(\max\left(1, \frac{|t-s|}{\varepsilon}\right)^{-1/2} \ln\left(\mathrm{e} + \frac{|t-s|}{\varepsilon}\right)\right)^{1-2/r} \mathrm{d}s \Bigg\|_{L_t^p},
\end{aligned} \tag{43}$$

because

$$\left\|\psi(2^{-m}D)\Psi(t)\right\|_{L_t^{\bar p}(L_x^{\bar r})} \leqslant \left\|\mathcal{F}^{-1}\psi\right\|_{L^1}\left\|\Psi(t)\right\|_{L_t^{\bar p}(L_x^{\bar r})}.$$

Finally (42) follows from (43) by Young's inequality, as the norm of the function

$$s \mapsto \left(\max\left(1, \frac{|s|}{\varepsilon}\right)^{-1/2} \ln\left(\mathrm{e} + \frac{|s|}{\varepsilon}\right)\right)^{1-2/r}$$

in $L^{p/2}$ is bounded by $C\varepsilon^{2/p}$. □

**Corollary 3.** *For all* $p > 4$, *there is a constant* $C$ *such that if* $f, g$ *satisfy*:

$$\begin{cases} \partial_t f \pm \frac{\mathrm{i}}{\varepsilon}\frac{D_3}{|D|} f = g, \\ f|_{t=0} = f_0, \end{cases}$$

*then*

$$\|f\|_{L_t^p(C^{s+1})} \leqslant C\varepsilon^{1/p}\left(\|f_0\|_{H^{s+5/2}} + \|g\|_{L_t^1(H^{s+5/2})}\right). \tag{44}$$

**Proof.** Duhamel's formula gives

$$\dot\Delta_m f(t) = \mathrm{e}^{\mp\mathrm{i}\frac{t}{\varepsilon}\frac{D_3}{|D|}}\dot\Delta_m f_0 + \int_0^t \mathrm{e}^{\mp\mathrm{i}\frac{t-t'}{\varepsilon}\frac{D_3}{|D|}}\dot\Delta_m g(t')\,\mathrm{d}t',$$

with $\dot\Delta_m \stackrel{\text{def}}{=} \varphi(2^{-m}D)$ for all $m \in \mathbb{Z}$, so

$$2^{m(s+1)}\|\dot\Delta_m f\|_{L_t^p(L_x^\infty)} \lesssim \varepsilon^{1/p}2^{m(s+5/2)}\left(\|\dot\Delta_m f_0\|_{L^2} + \int_0^\infty \left\|\dot\Delta_m g(t')\right\|_{L^2}\mathrm{d}t'\right).$$

So we get the part of (44) corresponding to high frequencies:

$$\begin{aligned} \left\|\left\|\left(2^{m(s+1)}\|\Delta_m f\|_{L^\infty}\right)_{m=0}^{+\infty}\right\|_{l^\infty}\right\|_{L_t^p} &\leqslant \left\|\left\|\left(2^{m(s+1)}\|\Delta_m f\|_{L^\infty}\right)_{m=0}^{+\infty}\right\|_{l^2}\right\|_{L_t^p} \\ &\leqslant \left\|\left\|\left(2^{m(s+1)}\|\Delta_m f\|_{L^\infty}\right)_{m=0}^{+\infty}\right\|_{L_t^p}\right\|_{l^2} \\ &\lesssim \varepsilon^{1/p}\left(\|f_0\|_{\dot H^{s+5/2}} + \|g\|_{L_t^1(\dot H^{s+5/2})}\right). \end{aligned}$$

Since $f_0 \in L^2$ and $g \in L_t^1(L_x^2)$ imply $f(t) \in L^2$ for all $t$, the low frequencies can be treated as follows:

$$\begin{aligned}\|\Delta_{-1} f\|_{L_t^p(L_x^\infty)} &\leqslant \big\| \big\| \big(\|\dot{\Delta}_m f\|_{L^\infty}\big)_{m=-\infty}^{-2} \big\|_{l^1} \big\|_{L_t^p} \\ &\leqslant \big\| \big\| \big(\|\dot{\Delta}_m f\|_{L^\infty}\big)_{m=-\infty}^{-2} \big\|_{L_t^p} \big\|_{l^1} \\ &\lesssim \varepsilon^{1/p}\big(\|f_0\|_{L^2} + \|g\|_{L_t^1(L^2)}\big).\end{aligned}$$

This completes the proof of (44). □

## 6. Another dispersive inequality

To conclude, here is the alternative method which we spoke about in the introduction and after Proposition 2 in the previous section.

**Lemma 4.** *Let $\widetilde{\mathcal{C}} \subset \mathbb{R}^3$ be an annulus centered at the origin and of radii $a$ and $b$, with $0 < a < b$. Then there is a constant $r > 0$ such that, for any $\psi \in C^\infty(\mathbb{R}^3)$ supported in*:

$$\widetilde{\mathcal{C}}_r \stackrel{\text{def}}{=} \widetilde{\mathcal{C}} \cap \big\{\xi \in \mathbb{R}^3;\ |\xi_3| \leqslant r\big\},$$

*the function $K$ defined by* (36) *satisfies*

$$\big\|K(\tau,\cdot)\big\|_{L^\infty} \leqslant C\tau^{-1/3}. \tag{45}$$

The constant $C$ may depend on $a$, $b$, $r$ and $\psi$, but not on $\tau$.

**Proof.** Set $\xi_h = (\xi_1, \xi_2)$. Choosing eventually $r < a/4$, we may suppose that $a\sqrt{15}/4 \leqslant |\xi_h| \leqslant b$ if $\xi \in \operatorname{supp}\psi$; let $\widetilde{\mathcal{C}}_h \subset \mathbb{R}^2$ be the (closed) annulus centered at the origin and of radii $a\sqrt{15}/4$ and $b$.

Integrating first with respect to $\xi_3$ gives

$$K(\tau, z) = \int_{\widetilde{\mathcal{C}}_h} \widetilde{K}(\tau;\xi_h, z_3)\mathrm{e}^{\mathrm{i}z_1\xi_1 + \mathrm{i}z_2\xi_2}\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2,$$

with

$$\widetilde{K}(\tau;\xi_h, z_3) \stackrel{\text{def}}{=} \int_{\mathbb{R}} \psi(\xi_h,\zeta)\mathrm{e}^{\mathrm{i}\tau\frac{\zeta}{\sqrt{|\xi_h|^2+\zeta^2}} + \mathrm{i}z_3\zeta}\,\mathrm{d}\zeta.$$

Therefore, thanks to the compactness of $\widetilde{\mathcal{C}}_h$, we only have to show that, for all $\xi_h^0 \in \widetilde{\mathcal{C}}_h$, there is a neighborhood of $\xi_h^0$, noted $V(\xi_h^0)$, and a constant $r = r(a, b, \xi_h^0) > 0$ such that, for any $\psi \in C^\infty(\mathbb{R}^3)$ supported in $\widetilde{\mathcal{C}}_r$,

$$\big\|\widetilde{K}(\tau;\cdot,\cdot)\big\|_{L^\infty(V(\xi_h^0)\times\mathbb{R})} \leqslant C_0\tau^{-1/3}, \tag{46}$$

the constant $C_0$ possibly depending on $a$, $b$, $r$, $\psi$ and $\xi_h^0$.

Fix $\xi_h^0 \in \widetilde{\mathcal{C}}_h$. Let $\tilde{\psi}:\mathbb{R}^+ \to \mathbb{R}$ be a smooth function equal to 1 in $[0,1]$ and vanishing outside $[0,2]$. Then set, using a new variable $y = (y_h, y_3) \in \mathbb{R}^2 \times \mathbb{R}$,

$$u_r(\zeta, y) = \tilde{\psi}\big(|y_h|/r\big)\psi\big(\xi_h^0 + y_h, \zeta\big),$$

where $r > 0$ is to be determined, and

$$f(\zeta, y) = \zeta\left(\frac{1}{\sqrt{|\xi_h^0 + y_h|^2 + \zeta^2}} - \frac{1}{|\xi_h^0 + y_h|} + y_3\right).$$

Finally let:

$$\overline{K}_r^{(1)}(\tau, y) = \int_{\mathbb{R}} \tilde{\psi}\big(|y_3|/r\big)u_r(\zeta, y)\mathrm{e}^{\mathrm{i}\tau f(\zeta,y)}\,\mathrm{d}\zeta$$

and

$$\overline{K}_r^{(2)}(\tau, y) = \int_{\mathbb{R}} \big(1 - \tilde{\psi}\big(|y_3|/r\big)\big)u_r(\zeta, y)\mathrm{e}^{\mathrm{i}\tau f(\zeta,y)}\,\mathrm{d}\zeta.$$

Remark that

$$\widetilde{K}(\tau; \xi_h, z_3) = \overline{K}_r^{(1)}(\tau, y) + \overline{K}_r^{(2)}(\tau, y) \tag{47}$$

at

$$y = \left(\xi_h - \xi_h^0, \frac{z_3}{\tau} + \frac{1}{|\xi_h|}\right),$$

provided $|\xi_h - \xi_h^0| \leqslant r$.

The ordinary methods of non-stationary or stationary phase do not yield any time-decay on $\overline{K}_r^{(1)}(t, y)$ for small $y$. Indeed, it is elementary to check (see (51) and (52) below) that

$$\frac{\partial f}{\partial \zeta}(0,0) = \frac{\partial^2 f}{\partial \zeta^2}(0,0) = 0.$$

However, since

$$\frac{\partial^3 f}{\partial \zeta^3}(0,0) = -3/\big|\xi_h^0\big|^3 \neq 0,$$

more refined methods, described by Hörmander [17, p. 234], yield the following: there exist $C^\infty$ real valued functions $\alpha(y)$ and $\beta(y)$ near 0, depending only on $\xi_h^0$, such that

$$\big\|\overline{K}_r^{(1)}(\tau, \cdot) - \mathrm{e}^{\mathrm{i}\tau\beta}\big(u_{0,r}Ai\big(\alpha\tau^{2/3}\big)\tau^{-1/3} + u_{1,r}Ai'\big(\alpha\tau^{2/3}\big)\tau^{-2/3}\big)\big\|_{L^\infty} \leqslant C_{1,r}\tau^{-1}, \tag{48}$$

where $Ai$ denotes the Airy function [17, p. 213–215] and $u_{0,r}(y), u_{1,r}(y) \in C_0^\infty$, providing that the support of $(\zeta, y) \mapsto \tilde{\psi}(|y_3|/r)u_r(\zeta, y)$ is sufficiently close to 0—which means that (48) is valid if $0 < r \leqslant r_0$, for some constant $r_0$ depending only on $\xi_h^0$. From this we conclude that

$$\left\| \overline{K}_r^{(1)}(\tau, \cdot) \right\|_{L^\infty} \leqslant \widetilde{C}_{1,r} \tau^{-1/3} \tag{49}$$

if $0 < r \leqslant r_0$, because

$$Ai(x) \lesssim |x|^{-1/4} \quad \text{and} \quad Ai'(x) \lesssim |x|^{1/4}$$

when $|x|$ is large.

On the other hand,

$$\left\| \overline{K}_r^{(2)}(\tau, \cdot) \right\|_{L^\infty} \leqslant C_{2,r} \tau^{-1} \tag{50}$$

if $0 < r \leqslant \min(a/4, a^3/100)$. Indeed, for each $(\zeta, y)$ such that

$$\left(1 - \tilde{\psi}\left(|y_3|/r\right)\right) u_r(\zeta, y) \neq 0,$$

we then have:

$$\left|\xi_h^0 + y_h\right| \geqslant a\frac{\sqrt{15}}{4} - 2r \geqslant \frac{a}{4}\left(\sqrt{15} - 2\right) > \frac{a}{4}$$

in addition of $|y_3| \geqslant r$ and $|\zeta| \leqslant r$. Since

$$\left| \frac{1}{\sqrt{c^2 + \zeta^2}} - \frac{1}{c} \right| = \left| \frac{-\zeta^2}{c\sqrt{c^2 + \zeta^2}(c + \sqrt{c^2 + \zeta^2})} \right| \leqslant \frac{\zeta^2}{2c^3},$$

for all $c > 0$, this implies:

$$\begin{aligned} \left| \frac{\partial f}{\partial \zeta}(\zeta, y) \right| &= \left| \frac{1}{\sqrt{|\xi_h^0 + y_h|^2 + \zeta^2}} - \frac{1}{|\xi_h^0 + y_h|} + y_3 - \frac{\zeta^2}{(|\xi_h^0 + y_h|^2 + \zeta^2)^{3/2}} \right| \\ &\geqslant r - \frac{3r^2}{2|\xi_h^0 + y_h|^3} \\ &\geqslant r\left(1 - \frac{96r}{a^3}\right) \\ &\geqslant r/25. \end{aligned} \tag{51}$$

So (50) follows from a simple integration by parts ($' = \partial_\zeta$):

$$\overline{K}_r^{(2)}(\tau, y) = \int_{\mathbb{R}} \big(1 - \tilde{\psi}\big(|y_3|/r\big)\big) u_r(\zeta, y) \left( \frac{(e^{i\tau f})'}{i\tau f'} \right)(\zeta, y)\, d\zeta$$
$$= -\frac{1}{i\tau} \int_{\mathbb{R}} \big(1 - \tilde{\psi}\big(|y_3|/r\big)\big) \left( \frac{u_r'}{f'} - \frac{u_r f''}{(f')^2} \right)(\zeta, y)\, e^{i\tau f(\zeta, y)}\, d\zeta,$$

and

$$\frac{\partial^2 f}{\partial \zeta^2}(\zeta, y) = -\frac{3\zeta}{(|\xi_h^0 + y_h|^2 + \zeta^2)^{3/2}} + \frac{3\zeta^3}{(|\xi_h^0 + y_h|^2 + \zeta^2)^{5/2}} \tag{52}$$

is clearly bounded on $\operatorname{supp} u_r$, once $r$ is fixed so that $\left|\xi_h^0 + y_h\right| > a/4$.

So we get (46) from (47), (49) and (50), choosing $r = \min(r_0, a/4, a^3/100)$ and $V(\xi_h^0) = B(\xi_h^0, r)$. □

**Lemma 5.** *For any $r > 0$, let*

$$\widetilde{\mathcal{C}}_r' \stackrel{\text{def}}{=} \widetilde{\mathcal{C}} \cap \big\{\xi \in \mathbb{R}^3;\ |\xi_3| \geqslant r\big\},$$

*and suppose this time that $\psi \in C^\infty(\mathbb{R}^3)$ is supported in $\widetilde{\mathcal{C}}_r'$.*

*Then $K$, always defined by* (36), *satisfies*:

$$\big\| K(\tau, \cdot) \big\|_{L^\infty} \leqslant C\tau^{-1/2}.$$

**Proof.** This is a particular case of Lemma 1 in the preprint of Chemin, Desjardins, Gallagher and Grenier [5, p. 5]. □

**Proposition 3.** *For any $\psi \in C^\infty(\mathbb{R}^3)$ supported in $\widetilde{\mathcal{C}}$, there is a constant $C$ such that* (45), *with $K$ defined by* (36), *is valid.*

**Proof.** Let $\tilde{\psi} : \mathbb{R}^+ \to \mathbb{R}$ be, again, a smooth function equal to 1 in $[0, 1]$ and vanishing outside $[0, 2]$. First, choose $r_0$ such that (45) is true if $\operatorname{supp} \psi \subset \widetilde{\mathcal{C}}_{r_0}$ (Lemma 4), then write

$$\psi(\xi) = \tilde{\psi}\left( \frac{2|\xi_3|}{r_0} \right) \psi(\xi) + \left( 1 - \tilde{\psi}\left( \frac{2|\xi_3|}{r_0} \right) \right) \psi(\xi)$$

and apply Lemma 5 with $r = r_0/2$. □

This proposition may then be used instead of Proposition 2, with slightly weaker results.

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## References

[1] A. Babin, A. Mahalov, B. Nicolaenko, Regularity and integrability of 3D Euler and Navier–Stokes equations for rotating fluids, Asymptot. Anal. 15 (2) (1997) 103–150.
[2] A. Babin, A. Mahalov, B. Nicolaenko, Global regularity of 3D rotating Navier–Stokes equations for resonant domains, Appl. Math. Lett. 13 (4) (2000) 51–57.
[3] J.-Y. Chemin, À propos d'un problème de pénalisation de type antisymétrique, J. Math. Pures Appl. (9) 76 (9) (1997) 739–755.
[4] J.-Y. Chemin, Perfect Incompressible Fluids, Oxford Lecture Series in Mathematics and its Applications, vol. 14, Clarendon Press, Oxford University Press, New York, 1998. Translated from the 1995 French original by Isabelle Gallagher and Dragos Iftimie.
[5] J.-Y. Chemin, B. Desjardins, I. Gallagher, E. Grenier, Anisotropy and dispersion in rotating fluids, in: Nonlinear Partial Differential Equations and Their Applications. Collège de France Seminar, vol. XIV (Paris, 1997/1998), in: Stud. Math. Appl., vol. 31, North-Holland, Amsterdam, 2002, pp. 171–192.
[6] J.-Y. Chemin, B. Desjardins, I. Gallagher, E. Grenier, Fluids with anisotropic viscosity, M2AN Math. Model. Numer. Anal. 34 (2) (2000) 315–335. Special issue for R. Temam's 60th birthday.
[7] R. Danchin, Zero Mach number limit in critical spaces for compressible Navier–Stokes equations, Ann. Sci. École Norm. Sup. (4) 35 (1) (2002) 27–75.
[8] B. Desjardins, E. Grenier, Low Mach number limit of viscous compressible flows in the whole space, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. 455 (1986) (1999) 2271–2279.
[9] A. Dutrifoy, On 3-D vortex patches in bounded domains, Comm. Partial Differential Equations 28 (7–8) (2003) 1237–1263.
[10] A. Dutrifoy, Slow convergence to vortex patches in quasigeostrophic balance, Arch. Rational Mech. Anal. 171 (3) (2004) 417–449.
[11] A. Dutrifoy, T. Hmidi, The incompressible limit of solutions of the two-dimensional compressible Euler system with degenerating initial data, Comm. Pure Appl. Math. 57 (9) (2004) 1159–1177.
[12] I. Gallagher, The tridimensional Navier–Stokes equations with almost bidimensional data: stability, uniqueness, and life span, Internat. Math. Res. Notices 18 (1997) 919–935.
[13] I. Gallagher, Applications of Schochet's methods to parabolic equations, J. Math. Pures Appl. (9) 77 (10) (1998) 989–1054.
[14] P. Gamblin, X. Saint Raymond, On three-dimensional vortex patches, Bull. Soc. Math. France 123 (3) (1995) 375–424.
[15] J. Ginibre, G. Velo, Generalized Strichartz inequalities for the wave equation, J. Funct. Anal. 133 (1) (1995) 50–68.
[16] H. Greenspan, The Theory of Rotating Fluids, Cambridge Monographs on Mechanics and Applied Mathematics, Cambridge Univ. Press, Cambridge, UK, 1980.
[17] L. Hörmander, The Analysis of Linear Partial Differential Operators, vol. I, Springer-Verlag, Berlin/New York, 1983.
[18] D. Iftimie, The approximation of the quasigeostrophic system with the primitive systems, Asymptot. Anal. 21 (2) (1999) 89–97.
[19] J. Pedlosky, Geophysical Fluid Dynamics, second ed. (Study ed.), Springer-Verlag, New York, 1987.
[20] T. Runst, W. Sickel, Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations, de Gruyter Series in Nonlinear Analysis and Applications, vol. 3, de Gruyter, Berlin, 1996.
[21] H. Triebel, Theory of Function Spaces, Monographs in Mathematics, vol. 78, Birkhäuser-Verlag, Basel, 1983.
[22] S. Ukai, The incompressible limit and the initial layer of the compressible Euler equation, J. Math. Kyoto Univ. 26 (2) (1986) 323–331.