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A Phenomenon in the Theory of Sorting

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1. THE PHENOMENON

The proof that a proposed sorting method actually works frequently involves the observation that certain operations preserve the order introduced by previous operations. The present paper gives a unified treatment of this order-preservation phenomenon. A number of results about sorting networks and other sorting methods, previously proved in an *ad hoc* manner, are immediate consequences of our theorems.

The following examples will serve to introduce the phenomenon.

EXAMPLE 1. Consider a rectangular array of locations, with a number placed in each location. To *sort* a row of locations means to take the numbers occurring in that row and permute them so that they are in ascending order from left to right. Similarly, sorting a column places the numbers in that column in ascending order from top to bottom. Figure 1 shows the effect of sorting the rows, and then the columns, of a 3×3 array.

The reader will note that, in array (c), the rows are in ascending order. In fact, whenever we sort the rows of an array, and then sort the columns, the rows remain in order. Thus, we say that sorting the columns preserves the order in the rows.

4	8	6		4	6	8		1	3	5
9	7	1		1	7	9		2	6	8
5	2	3		2	3	5		4	7	9
	(a)				(b)				(c)	

FIG. 1. Column sorting preserves row sorting.

EXAMPLE 2. Suppose each of the locations $0, 1,..., n - 1$ contains a number. Let p be a positive integer. For $i = 0, 1,..., p - 1$, let S_i be the sequence $\{kp + i \mid k$ is a nonnegative integer and $kp + i < n\}$. Thus, the p subsequences S_i partition the sequence $0, 1,..., n - 1$ into "residue classes." To p-sort means to sort each of the sequences S_i. Figure 2 shows the effect of 3-sorting a sequence, and then 2-sorting the result. In general, if a p-sorted sequence is q-sorted, it remains p-sorted. Thus, for any positive integers p and q, q-sorting preserves p-sorting. This result, which was pointed out to us by D. E. Knuth, is fundamental to the analysis of Shell's method of sorting [4].

7	4	2	8	0	9	6	1	3	5
5	0	2	6	1	3	7	4	9	8
1	0	2	3	5	4	7	6	9	8

FIG. 2. 2-Sorting preserves 3-sorting.

The foregoing examples illustrate a phenomenon which we generalize as follows. Suppose we are given a finite set A (the *locations*). Let $\phi : A \to Z$ map A into the integers. For $a \in A$, $\phi(a)$ is the value in location a. Let P be partial ordering relation over A. The function ϕ is *consistent* with P if

$$xPy \Rightarrow \phi(x) \leqslant \phi(y).$$

Let F be a $1 - 1$ partial function from A into A. Let F^k denote k-fold iteration of F; i.e., $y = F^k(x)$ if there exist $u_1, \ldots, u_{k-1}$ such that $u_1 = F(x)$, $u_{i+1} = F(u_i)$, $i = 1, 2, \ldots, k-2$, and $y = F(u_{k-1})$. The partial function F is called a *shift* if the following "acyclic property" holds: for all $x \in A$ and $k > 0$, $F^k(x) \neq x$. A maximal sequence of locations $u_0, u_1, \ldots, u_l$ such that $F(u_i) = u_{i+1}$, $i = 0, 1, \ldots, l-1$, is called a *chain* of the shift F. Observe that each location is in exactly one chain of F, and that the number of such chains is $|\{x \in A \mid \text{for all } y \in A, F(y) \neq x\}|$[1].

We now give a formal definition which corresponds to the process of sorting the chains of F. Given $\phi : A \to Z$ and the shift F, a unique function $\phi_F : A \to Z$ is determined by

(i) $\phi_F(x) \leqslant \phi_F(F(x))$, $x \in A$

(ii) if X is the set of elements of a chain of F and $a \in Z$, then

$$| X \cap \phi^{-1}(a)| = | X \cap \phi_F^{-1}(a)|.$$

The following property is the object of study in this paper.

DEFINITION 1. Let F be a shift and P, a partial ordering. *F preserves P* (abbreviated F pr P) if, whenever ϕ is consistent with P, ϕ_F is also consistent with P.

EXAMPLE 3. Let A be the set of vertices of a uniform binary tree in the plane. Define xPy if vertex y is in the subtree rooted at x, and define F by $F(x) = y$ if x and y

[1] If S is a finite set, $| S |$ denotes the number of elements in S.

are at the same distance from the root, and x is immediately to the left of y. Then F pr P. Figure 3 gives a function ϕ consistent with P, and also gives ϕ_F, the result of sorting along the chains of F.

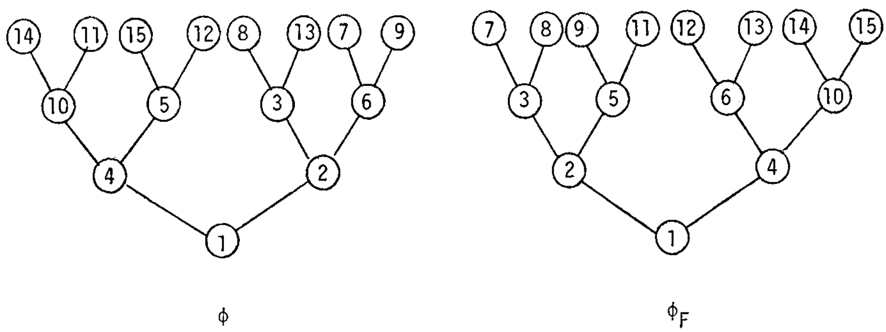

FIG. 3. Sorting the levels of a uniform binary tree.

2. Characterizations of the Property F pr P

The object of this section is to determine necessary and sufficient conditions for the property F pr P to hold.

It will be convenient to establish a notation for certain elementary operations on binary relations over A. If

$$R \subseteq A \times A, \quad \not R = \{(x, y) \mid (x, y) \notin R\} \qquad \text{and} \qquad R^{-1} = \{(y, x) \mid (x, y) \in R\}.$$

If $B \subseteq A$, $R(B) = \{y \mid \text{for some } x \in B, (x, y) \in R\}$. The expression xRy is synonymous with $(x, y) \in R$. The *composition* $R \circ S$ of relations R and S is defined by $x(R \circ S)y$ if, for some z, xRz and zSy. The relations R and S *commute* if $R \circ S$ and $S \circ R$ are equal. The *k-th power* R^k of R is determined by

$$R^0 = \{(x, x) \mid x \in A\},$$

$$R^{k+1} = R^k \circ R, \qquad k = 0, 1, 2, \ldots .$$

We further define $R^+ = \bigcup_{k=1}^{\infty} R^k$ and $R^* = \bigcup_{k=0}^{\infty} R^k$. R^+ is called the *transitive closure* of R, and R^* is called the *reflexive transitive closure* of R.

If P is a partial ordering relation, define the relation C_P by

$$C_P = \{(x, y) \in P \mid x \neq y, \qquad \text{and}$$

$$(x, z) \in P \qquad \text{and} \qquad (z, y) \in P \Rightarrow x = z \qquad \text{or} \qquad y = z\}.$$

For $xC_P\,y$ read "y covers x in P." Note that, if $P = F^*$, where F is a shift, then $C_P = F$.

If F is a shift then F^{-1} is also a shift. If P is a partial ordering relation then P^{-1} is also a partial ordering relation. It is an easy exercise to verify the following *duality principle.*

LEMMA 1. *F pr P if and only if F^{-1} pr P^{-1}.*

The following lemma prepares the way for our first characterization of the property F pr P. Let F be a shift having the sequence $y_1, y_2, \ldots, y_n$ as a chain. Let $Y = y_1, y_2, \ldots, y_n$. Let ϕ be an assignment.

LEMMA 2. *For any integer a,*

(i) $| Y \cap \{x \mid \phi(x) \leqslant a\}| = | Y \cap \{x \mid \phi_F(x) \leqslant a\}|$

(ii) $| Y \cap \{x \mid \phi_F(x) \leqslant a\}| \geqslant j$ *if and only if* $\phi_F(y_j) \leqslant a$.

THEOREM 1. *Let F be a shift and let P be a partial ordering. Then F pr P if and only if*

(a) $xF^+y \Rightarrow y\not{P}x$

and

(b) *i, j condition. Let the sequences $x_1, x_2, \ldots, x_m$, and $y_1, y_2, \ldots, y_n$ be distinct chains of F. Let $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. Then, for any i, j and $\overline{Y} \subseteq Y$, x_iPy_j and $|\overline{Y}| = j \Rightarrow |P^{-1}(\overline{Y} \cap X)| \geqslant i$.*

Put less formally, the i, j condition states that, if the i-th element of one chain precedes the j-th element of another in P, then any j elements of the second chain are preceded in P by at least i elements of the first chain.

Proof. Assume F pr P.

Proof of (a). Let $\phi : A \to Z$ be $1 - 1$ and consistent with P. Let F be a shift. If xF^+y, then $\phi_F(x) < \phi_F(y)$ (equality is ruled out since ϕ is $1 - 1$). If yPx then, since F pr P, $\phi_F(y) < \phi_F(x)$. These conditions are mutually exclusive, so $xF^+y \Rightarrow y\not{P}x$.

Proof of (b). Suppose x_iPy_j, $|\overline{Y}| = j$ and $\overline{Y} \subseteq Y$. The following assignment is consistent with P:

$$\phi(x) = \begin{cases} 0 & \text{if } x \in P^{-1}(\overline{Y}) \\ 1 & \text{otherwise.} \end{cases}$$

Then $| Y \cap \{y \mid \phi(y) = 0\}| \geqslant j$ so (by Lemma 2) $\phi_F(y_j) = 0$. Since F pr P, $\phi_F(x_i) = 0$. Thus (again by Lemma 2) $| X \cap \{y \mid \phi_F(y) = 0\}| \geqslant i$. Hence $| X \cap \{y \mid \phi(y) = 0\}| = | X \cap P^{-1}(\overline{Y})| \geqslant i$, and the i, j condition is established.

Now, assuming (a) and (b) we prove F pr P by showing that, if ϕ is consistent with P, then ϕ_F is consistent with P. Suppose $x \neq y$ and xPy. We prove $\phi_F(x) \leqslant \phi_F(y)$.

(i) Suppose x and y are in the same chain of F.

By condition (a) of the theorem, $y\not{F}^+x$, so xF^+y and, since ϕ_F is consistent with F (and hence consistent with F^+), $\phi_F(x) \leqslant \phi_F(y)$.

(ii) Suppose $x = x_i$ occurs in the chain $x_1, x_2, ..., x_m$ and $y = y_j$ occurs in the chain $y_1, y_2, ..., y_n$.

$$| Y \cap \{y \mid \phi_F(y) \leqslant \phi_F(y_j)\}| \geqslant j.$$

Hence $| Y \cap \{y \mid \phi(y) \leqslant \phi_F(y_j)\}| \geqslant j$. Let $\bar{Y} = Y \cap \{y \mid \phi(y) \leqslant \phi_F(y_j)\}$. By (b), $| P^{-1}(\bar{Y}) \cap X | \geqslant i$. Since ϕ is consistent with P, $x \in P^{-1}(\bar{Y}) \Rightarrow \phi(x) \leqslant \phi_F(y_j)$. Thus $| X \cap \{y \mid \phi(y) \leqslant \phi_F(y_j)\}| = | X \cap \{y \mid \phi_F(y) \leqslant \phi_F(y_j)\}| \geqslant i$ and (by Lemma 2) $\phi_F(x_i) \leqslant \phi_F(y_j)$.

The proof of Theorem 1 suggests the following method of constructing examples to show that F pr P is false in particular cases. Simply choose a set $\bar{Y}$ involved in a violation of the i, j condition, and set

$$\phi(y) = \begin{cases} 0 & \text{if } y \in P^{-1}(\bar{Y}) \\ 1 & \text{otherwise.} \end{cases}$$

Then ϕ is consistent with P, but ϕ_F is not.

To prove that ϕ_F is consistent with P, it suffices to prove that ϕ_F is consistent with C_P, i.e., $xC_Py \Rightarrow \phi_F(x) \leqslant \phi_F(y)$. Accordingly, a weakened form of the i, j condition suffices to prove F pr P, and we obtain the following corollary, which proves useful in applications.

COROLLARY 1. *Let F be a shift and let P be a partial ordering. Then F pr P if and only if*

(a) $xF^+y \Rightarrow y\not{P}x$

(b) *Weak i, j condition. Let the sequences $x_1, x_2, ..., x_m$ and $y_1, y_2, ..., y_n$ be distinct chains of F. Let $X = \{x_1, x_2, ..., x_m\}$ and let $Y = \{y_1, y_2, ..., y_n\}$. Then, for any i, j and $\bar{Y} \subseteq Y$, $x_iC_Py_j$ and $| \bar{Y} | = j \Rightarrow | P^{-1}(\bar{Y}) \cap X | \geqslant i$.*

COROLLARY 2. *If F pr P then F^* commutes with P.*

Proof. We must show the following:

(i) if xPz and zF^*y then, for some w, xF^*w and wPy.

(ii) if yF^*z and zPx then, for some w, yPw and wF^*x.

Proof of (i). If xF^*z then the result follows by taking $w = z$, since the partial ordering P is reflexive. zF^*x violates (a) of Theorem 1. Thus we may assume that x and z lie in distinct chains of F. Let $x = x_i$ in the chain $x_1, x_2, ..., x_m$ and let $z = y_j$ and $y = y_k$, $j < k$, in the chain $y_1, y_2, ..., y_n$. Let $X = \{x_1, ..., x_m\}$ and $Y = \{y_1, y_2, ..., y_n\}$. Then we must prove the existence of $l \geqslant i$ such that $x_l P y_k$. The proof is by induction on j.

Basis ($j = 1$). By Theorem 1, $x_i P y_1$ implies that for every $\{y_k\}$,

$$| P^{-1}(\{y_k\}) \cap X | \geqslant i,$$

so there exists $x_l \in P^{-1}(\{y_k\})$ such that $l \geqslant i$. Thus $x_l P y_k$, as desired.

Induction step. Assume the result for all $j' < j$. Set $\bar{Y} = \{y_1, ..., y_{j-1}, y_k\}$. Since $x_i P y_j$, the i, j condition tells us that $| P^{-1}(\bar{Y}) \cap X | \geqslant i$, so there exists x_l, $l \geqslant i$, such that $x_l \in P^{-1}(\bar{Y})$. Then either $x_l P y_k$ and we are done, or $x_l P y_j'$ for some $j' \in \{y_1, ..., y_{j-1}\}$. In the latter case, the result follows by induction hypothesis.

Proof of (ii). We use the duality principle to accomplish a reduction to (i). Since F pr P, F^{-1} pr P^{-1}. By the result just proven, if $xP^{-1}z$ and $z(F^*)^{-1}y$ then, for some w, $x(F^*)^{-1} w$ and $w(P^{-1})y$. Equivalently, if yF^*z and zPx then, for some w, yPw and wF^*x, and the proof is complete.

Corollary 2 can be restated as follows: Construct a diagram with a node for each element of A, a solid arrow from x to y if xFy, and a wiggly arrow if $xC_P y$. If F pr P, then it is possible to reach y from x by a (possibly null) walk along solid lines followed by a (possibly null) wiggly walk if and only if it is possible to reach y from x by a wiggly walk followed by a solid walk.

EXAMPLE 4. Figure 4 specifies F by solid edges and C_P by wiggly edges.

In this case F^* commutes with P, but it is not true that F pr P, since the weak i, j condition is violated: $x_2 C_P y_2$ but $P^{-1}(\{y_2, y_3\}) \cap \{x_1, x_2\} = \{x_2\}$, so that

$$| P^{-1}(\{y_2, y_3\}) \cap \{x_1, x_2\}| < 2.$$

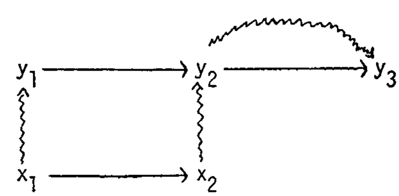

FIG. 4. Two shifts.

COROLLARY 3. *If F pr P then $F^* \circ P$ is a partial ordering relation. In fact, it is the largest partial ordering Q with the following property: If ϕ is consistent with P then ϕ_F is consistent with Q.*

We turn next to the important special case where P is the reflexive transitive closure of a shift. Call the shifts F and G *compatible* if there is no x such that $x(F \cup G)^+x$; i.e., there is no directed cycle of solid and wiggly edges. Clearly compatibility of F and G is a necessary condition for F pr G^*, since no 1–1 assignment is consistent with both F and G^* unless F and G are compatible.

COROLLARY 4. *Let F and G be compatible shifts. If F commutes with G then F* pr *G^*.*

Proof. The proof will consist of a verification of the conditions of Corollary 1. Condition (a) is a special case of the compatibility condition. We proceed to the weak i, j condition. In this case $P = G^*$, so C_P is G. Let $x_1, x_2, \ldots, x_m$ and $y_1, y_2, \ldots, y_n$ be distinct chains of F, and suppose x_iGy_j. Let $X = \{x_1, x_2, \ldots, x_m\}$ and let $Y = \{y_1, y_2, \ldots, y_n\}$. By commutativity

$$x_{i+1}Gy_{j+1} \quad \text{if} \quad i+1 \leqslant m \quad \text{and} \quad j+1 \leqslant n$$

and $x_{i-1}Gy_{j-1}$ if $i \geqslant 1$ and $j \geqslant 1$. Let $t = j - i$. By induction, x_lGy_{l+t} provided $1 \leqslant l \leqslant m$ and $1 \leqslant l + t \leqslant n$. If $t < 0$ then $x_{1-t}Gy_1$. Commutativity demands that, for some z, $x_{-t}Gz$ and zFy_1. But, since y_1 is the first element of a chain of F there is no z such that zFy_1, and a contradiction is reached. Hence $t \geqslant 0$. Thus, for any set $\bar{Y} \subseteq Y$, $P^{-1}(\bar{Y}) \supset \{x_{j-t} \mid y_j \in \bar{Y} \text{ and } j > t\}$, so that $| P^{-1}(\bar{Y}) \cap X | \geqslant | \bar{Y} | - t = j - t = i$. Hence the weak i, j condition is verified and the proof is complete.

A major simplification of Theorem 1 is possible if we assume one further property. Call the shifts F and G *independent* if

$$(F \cap G^*) \cup (F^* \cap G) = \phi.$$

The following theorem often makes it possible to determine by inspection whether F pr G^* when F and G are compatible, independent shifts.

THEOREM 2. *Let F and G be compatible, independent shifts. Then the following are equivalent:*

(a) F pr G^*,

(b) G pr F^*,

(c) *F^* commutes with G^*,*

(d) *F commutes with G.*

Proof. Corollary 2 yields (a) $\Rightarrow$ (c) and (b) $\Rightarrow$ (c). From Corollary 4 we have (d) $\Rightarrow$ (a) and (d) $\Rightarrow$ (b). Hence we can complete the proof by showing (c) $\Rightarrow$ (d). The proof uses the following observations.

(1) If H is a shift such that xH^*y, $y \neq x$ and $x \not{H} y$, then there exists a $z \neq y$ such that xHz and zH^*y.

(2) If H and K are compatible independent shifts such that xKy, then there is no z such that xH^*z or zH^*x and yH^*z or zH^*y.

Now we must prove x_1Fx_2 and $x_2Gx_4 \Rightarrow$ there exists z such that x_1Gz and zFx_4. Since F^* and G^* commute, there exists x_3 such that $x_1G^*x_3$ and $x_3F^*x_4$; $x_1 \neq x_3$ and $x_4 \neq x_3$, since independence would otherwise be violated (cf. observation (2)).

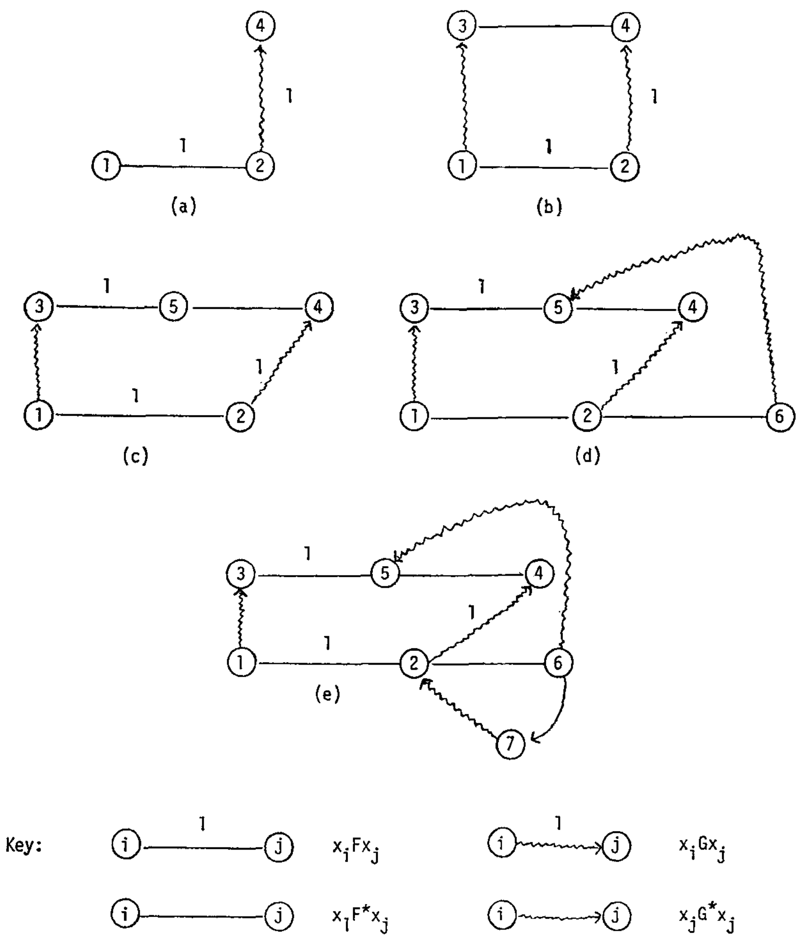

FIG. 5. Movie of a proof.

If $x_1 G x_3$ and $x_3 F x_4$, then x_3 is the required z. We show that any other situation produces a contradiction. Assume $x_3 \not{F} x_4$ (the case $x_1 \not{G} x_3$ is reduced to this case using the duality principle, with F^{-1} corresponding to G, and G^{-1} to F). Then there exists $x_5 \neq x_4$ such that $x_3 F x_5$ and $x_5 F^* x_4$.

Since $x_1 G^* x_3$ and $x_3 F x_5$, there exists x_6 such that $x_1 F^* x_6$ and $x_6 G^* x_5$. $x_1 = x_6$ violates independence, and $x_2 = x_6$ implies $x_4 G^* x_5$ (since $x_2 G x_4$ and $x_6 G^* x_5$), which violates compatibility.

Since $x_6 G^* x_5$ and $x_5 F^* x_4$, there exists x_7 such that $x_6 F^* x_7$ and $x_7 G^* x_4$. $x_7 = x_4$ violates independence. Hence, since $x_2 G x_4$, $x_7 G^* x_2$.

But $x_2 F^* x_6$, $x_6 F^* x_7$ and $x_7 G^* x_2$. This is a violation of compatibility, and a contradiction is reached. Thus the proof is complete.

Figure 5 gives a "movie" of the proof.

COROLLARY 5. *Let F and G be independent shifts. Then $F \operatorname{pr} G^*$ if and only if $G \operatorname{pr} F^*$.*

It is an open question whether the hypothesis of independence is required in Corollary 5.

EXAMPLE 5. Let $A = \{1, 2, 3\}$. Let F and G be the following sets of ordered pairs:

$$F = \{(1, 2), (2, 3)\}$$

$$G = \{(1, 2)\}.$$

Then the shifts F and G are not independent. Clearly $F \operatorname{pr} G^*$ and $G \operatorname{pr} F^*$, but $F \circ G \neq G \circ F$.

3. Applications

Our first application is a result which generalizes Examples 1 and 2. Let $A = A(m_1, m_2, \ldots, m_k)$ be a k-dimensional rectangular array of locations, i.e.,

$$A(m_1, m_2, \ldots, m_k) = \{1, 2, \ldots, m_1\} \times \{1, 2, \ldots, m_2\} \times \cdots \times \{1, 2, \ldots, m_k\},$$

where $m_1, m_2, \ldots, m_k$ are positive integers. For any integer k-vector $a = (a_1, a_2, \ldots, a_k)$ except the zero vector, define the shift F_a as follows: let $x = (x_1, x_2, \ldots, x_k)$ and $y = (y_1, y_2, \ldots, y_k)$ be elements of $A(m_1, m_2, \ldots, m_k)$. Then

$$x F_a y \Leftrightarrow x_i + a_i = y_i, \qquad i = 1, 2, \ldots, k.$$

Now consider two fixed k-vectors $a = (a_1, a_2, ..., a_k)$, and $b = (b_1, b_2, ..., b_k)$, and let the dimensions $(m_1, m_2, ..., m_k)$ of the array vary.

THEOREM 3. *The following are equivalent*:

(a) *For all arrays* $A(m_1, m_2, ..., m_k)$, $F_a \operatorname{pr} F_b{}^*$, *and* $F_b \operatorname{pr} F_a{}^*$;

(b) *the vectors a and b lie in the same orthant*; *i.e.*, $a_i b_i \geqslant 0$, $i = 1, 2, ..., k$.

Proof. If there is a positive integer λ such that $a = \lambda b$ then $F_a{}^* \subseteq F_b{}^*$ and, clearly, $F_a \operatorname{pr} F_b{}^*$ and $F_b \operatorname{pr} F_a{}^*$. The conclusion follows similarly if b is a positive integer multiple of a. Otherwise, the shifts F_a and F_b are independent. F_a and F_b are incompatible if and only if $a = \lambda b$, $\lambda < 0$. In that case, $F_a \operatorname{pr} F_b$ is false and, of course, a and b lie in different orthants. Otherwise, F_a and F_b are independent and compatible. Thus Theorem 2 applies, and $F_a \operatorname{pr} F_b{}^*$ if and only if F_a commutes with F_b. In this case commutativity means $(x \in A(m_1, m_2, ..., m_k)$ and $x + a + b \in A(m_1, m_2, ..., m_k)) \Rightarrow (x + a \in A(m_1, m_2, ..., m_k) \Leftrightarrow x + b \in A(m_1, m_2, ..., m_k))$. This is true for all $m_1, m_2, ..., m_k$ if a and b lie in the same orthant, and otherwise is false for all sufficiently large choices of $m_1, m_2, ..., m_k$.

COROLLARY 6 (Example 2). *p-sorting preserves q-sorting.*

In the two-dimensional case, we abbreviate $F_{(0,1)}$ by R, $F_{(1,0)}$ by C, $F_{(1,1)}$ by D and $F_{-(1,1)}$ by T. The chains of R correspond to rows of the array, the chains of C, to columns, the chains of D to diagonals and the chains of T to "ascending diagonals."

COROLLARY 7. *In the following table, a x is present in the F, G position if and only if F* pr G^* *for all rectangular arrays.*

$F \setminus G$	R	C	D	T
R	x	x	x	x
C	x	x	x	
D	x	x	x	
T	x			x

In the following 2×2 array, the assignment ϕ is consistent with C, while ϕ_T, the result of T-sorting, is not.

$$\begin{array}{cc} 1 & 0 \\ 1 & 0 \end{array} \qquad \begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array}$$

$$\phi \qquad\qquad \phi_T$$

In the following 3×3 array, the assignment ϕ is consistent with D while ϕ_T, the result of T-sorting, is not.

$$\begin{array}{ccc} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ & \phi & \end{array} \qquad \begin{array}{ccc} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ & \phi_T & \end{array}$$

Suppose we sort the rows, and then the columns, of a two-dimensional rectangular array. Since $C \operatorname{pr} R^*$, the resulting assignment is consistent with the partial ordering $Q = C^* \circ R^*$ given by $(u_1, u_2) Q (v_1, v_2)$ if $u_1 \leqslant v_1$ and $u_2 \leqslant v_2$. Suppose we then sort the chains of T (the ascending diagonals). Since, for sufficiently large arrays, $T \operatorname{pr} C^*$ is false, one might expect that $T \operatorname{pr} Q$ is also false. Indeed $T \operatorname{pr} Q$ is false (for arrays with more than one row) if the number of columns exceeds the number of rows. But, surprisingly, if the number of columns is less than or equal to the number of rows, $T \operatorname{pr} Q$.

THEOREM 4.[2] *Let $A = A(d, e)$, $d > 1$. Then $T \operatorname{pr} Q$ if and only if $d \geqslant e$.*

Proof. We apply Corollary 1. Condition (a) of that corollary clearly holds; so it remains only to consider the weak i, j condition. The relation C_Q consists of all pairs of the form $((u_1, u_2), (u_1 + 1, u_2))$ or $((u_1, u_2), (u_1, u_2 + 1))$. Thus, if $x_1, x_2, \ldots, x_m$ and $y_1, y_2, \ldots, y_n$ are chains of T, either

$$\text{(a)} \qquad x_1 C_Q y_1 \quad \text{and} \quad x_i Q y_j \Rightarrow x_i C_Q y_j \qquad \begin{array}{l} i = 1, 2, \ldots, m \\ j = 1, 2, \ldots, n, \end{array}$$

or

$$\text{(b)} \qquad x_i \not C_Q y_j \qquad \begin{array}{l} i = 1, 2, \ldots, m \\ j = 1, 2, \ldots, n. \end{array}$$

Therefore it is only necessary to check the weak i, j condition between "consecutive" chains of T, since it holds vacuously otherwise. Let $x_1, x_2, \ldots, x_m$ and $y_1, y_2, \ldots, y_n$ be consecutive, i.e., $x_1 C_Q y_1$. Let $X = \{x_1, x_2, \ldots, x_m\}$ and let $Y = \{y_1, y_2, \ldots, y_n\}$. Then the sequence $Q^{-1}(y_1), Q^{-1}(y_2), \ldots, Q^{-1}(y_n)$ is of one of the following forms (cf. Fig. 6):

I. $\{x_1\}, \{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_{m-1}, x_m\}, \{x_m\}$

II. $\{x_1\}, \{x_1, x_2\}, \ldots, \{x_{m-1}, x_m\}$

III. $\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_{m-1}, x_m\}$

IV. $\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_{m-1}, x_m\}, \{x_m\}$.

[2] This result has been proved independently by D. E. Knuth.

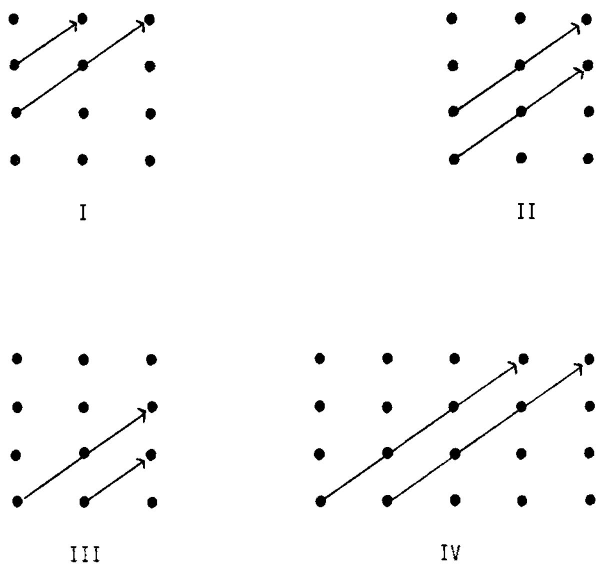

FIG. 6. Four cases in the proof of Theorem 4.

Moreover, Case IV can arise only in the following way: $x_1 = (d, 1)$, $y_1 = (d, 2)$ and $d < e$. In Case IV, $x_2 C_Q y_1$, but $Q^{-1}(y_n) = \{x_m\}$. Thus, taking $i = 2$, $j = 1$ and $\bar{Y} = \{y_n\}$, the weak i, j condition is violated.

In the other cases, the weak i, j condition is easily seen to be satisfied. In Case I, $x_i C_P y_j \Rightarrow i \leqslant j$ and, if $|\bar{Y}| = j$, then $|Q^{-1}(\bar{Y}) \cap X| \geqslant j$. Case II follows similarly. In Case III, $x_i C_Q y_j \Rightarrow i \leqslant j + 1$ and, for $j \leqslant m - 1$, $|\bar{Y}| = j \Rightarrow |Q^{-1}(\bar{Y}) \cap X| \geqslant j + 1$. Thus the proof is complete.

We next show how Theorems 1 and 2 can be used to prove the validity of certain methods for the construction of sorting networks. A sequence of shifts $F_1, F_2, \ldots, F_h$ is called a *plan* if

(a) $F_g \operatorname{pr} F_1^* \circ F_2^* \circ \cdots \circ F_{g-1}^*$, $g = 2, 3, \ldots, h$

and

(b) $F_1^* \circ F_2^* \circ \cdots \circ F_h^*$ is a total ordering of A.

The motivation for this definition is the following: Given a plan, we can sort the set of locations A by successively F_1-sorting, F_2-sorting,..., F_h-sorting. Moreover, the chains of each F_g can themselves be sorted according to plans, so that recursively defined sorting algorithms based on plans are possible. The algorithms of Batcher [1] and Bose and Nelson [2] are of this type, and the following theorem can be viewed as a justification of these algorithms.

THEOREM 5. *The following are plans:*

(a) $A = 2q \quad h = 3$.

F_1 *has the two chains* $1, 3, 5, 7, \ldots, 2q - 1$ *and* $2, 4, 6, 8, \ldots, 2q$.

F_2 *has the two chains* $1, 2, 5, 6, 9, 10, 13, 14, \ldots,$ *and* $3, 4, 7, 8, 11, 12, 15, 16, \ldots$.

F_3 *has the* $q + 1$ *chains* $1\ 2, 3\ 4, 5\ 6, 7 \cdots 2q - 2, 2q - 1\ 2q$.

(b) $A = 4q \quad h = 3$.

F_1 *has the two chains* $1, 2, \ldots, q, 2q + 1, 2q + 2, \ldots, 3q$ *and* $q + 1, q + 2, \ldots, 2q, 3q + 1, 3q + 2, \ldots, 4q$.

F_2 *has the two chains* $1, 2, \ldots, 2q$ *and* $2q + 1, \ldots, 4q$.

F_3 *has* $2q + 1$ *chains, of which only the following one has more than one element:* $q + 1, q + 2, \ldots, 3q$.

The proof of Theorem 5 is a simple exercise in the application of Corollary 1 and Theorem 2.

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