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Oscillations of Retarded Differential Equations of the Neutral and the Mixed Types*

ISTVÁN GYÖRI

Computing Centre of the Szeged University of Medicine, 6720 Szeged, Pécsi u. 4/a, Hungary

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This paper is devoted to the study of the oscillation of solutions of delay differential equations of the neutral and mixed types. Some general results are proved for certain general Volterra type neutral differential equations and many particular cases are discussed. © 1989 Academic Press, Inc.

1. INTRODUCTION

During the last 20 years there has been a great deal of work on the oscillation of solutions of delay equations of the type

$$\dot{x}(t) = -\sum_{i=1}^{n} b_i(t)\, x(t-\tau_i(t)), \qquad t \geqslant t_0, \tag{1.1}$$

where the continuous functions $b_i(t)$ and $\tau_i(t)$ are non-negative for any $1 \leqslant i \leqslant n$ and $t \geqslant t_0$.

Considerably less is known about the behavior of the solutions of (1.1) when it is of mixed type, that is there exist two indexes $1 \leqslant i \neq j \leqslant n$ such that $b_i(t) > 0$ and $b_j(t) < 0$, moreover $\operatorname{sign} b_k(t) = \operatorname{sign} b_k(t_0)$ $(1 \leqslant k \leqslant n,$ $t \geqslant t_0)$. As recent contributions to this study we cite the papers of Arino, Ladas, and Sficas [1] and Ladas and Sficas [12].

The neutral delay differential equations of the type

$$\frac{d}{dt}\left[x(t) - \sum_{j=1}^{m} a_j x(t-\delta_j)\right] = -\sum_{i=1}^{n} b_i x(t-\tau_i), \tag{1.2}$$

where a_j, δ_j $(1 \leqslant j \leqslant n)$, b_i, $\tau_i \geqslant 0$ $(1 \leqslant i \leqslant n)$ are given constants, are interesting in some applications (see [2, 4]). Thus it is understandable that

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recently there also has been a great deal of interest in obtaining results on the oscillatory behavior of solutions of neutral delay differential equations. Such results can be found in the papers of Grammatikopoulos, Grove, and Ladas [6, 7], Kulenovic, Ladas, and Meimaridou [11], and Ladas and Sficas [13].

At first sight—even from the above-mentioned papers—the investigations of the mixed type and of the neutral type equations seem to be independent. But after careful consideration we recognized that if we give some general results for the neutral delay differential equation

$$\frac{d}{dt}\left[x(t)-g(t, x(\cdot)\right]+f(t, x(\cdot))=0, \qquad t \geqslant t_0, \tag{1.3}$$

where $f, g\colon [t_0, \infty)\times C([t_{-1}, \infty), R)\to R$ are continuous Volterra functionals and $-\infty \leqslant t_{-1} \leqslant t_0$, then from these general results we can deduce new statements concerning the mixed and neutral type delay equations, too. Therefore the aim of this paper is to give some general results and principles for the general equation (1.3) and after that to discuss many interesting particular cases.

In Section 2 of this paper we give our general results concerning Eq. (1.3).

In Section 3, we apply our general results to some time dependent neutral equations to generalize some results of Hunt and Yorke [10] and Ladas and Sficas [12]. We also investigate the mixed type equation

$$\dot{x}(t) = -px(t-\tau)+qx(t-\sigma), \tag{1.4}$$

where p, q, τ, and σ are positive constants. Our statement concerning (1.4) is a significantly generalized and sharpened version of a recent theorem of Arino, Ladas, and Sficas [1].

In Section 4, we deal with some integro-differential equations of the neutral type. In that section we also deal with a conjecture of Hunt and Yorke [10] which was raised for non-neutral delay differential equations. That conjecture was proved in many particular cases in [9] and now we show that it is valid for some neutral delay differential equations, too.

2. Some General Results

Let $C([a, b], D)$ be a set of continuous functions mapping the interval $[a, b]$, $-\infty \leqslant a < b \leqslant \infty$, into $D \subset R = (-\infty, \infty)$.

Suppose $-\infty \leqslant t_{-1} \leqslant t_0 < \infty$ and $f\colon [t_0, \infty)\times C([t_{-1}, \infty), R)\to R$ is a given functional. The functional f is called a continuous Volterra functional, if $f(t, x(\cdot))$ is a continuous function of t for $t \in [t_0, \infty)$ for all $x \in C([t_{-1}, \infty), R)$, moreover for any $x, y \in C([t_{-1}, \infty), R)$,

$$f(t, x(\cdot)) = f(t, y(\cdot)),$$

whenever $t \geqslant t_0$ is such that $x(s) = y(s)$, $t_{-1} \leqslant s \leqslant t$.

Consider now the neutral functional differential equation

$$\frac{d}{dt}[x(t) - g(t, x(\cdot))] + f(t, x(\cdot)) = 0, \tag{2.1}$$

where $f, g\colon [t_0, \infty) \times C([t_{-1}, \infty), R) \to R$ are continuous Volterra functionals.

A function x is said to be a solution of Eq. (2.1), if $x \in C([t_{-1}, \infty), R)$, $x(t) - g(t, x(\cdot))$ is continuously differentiable and $x(t)$ satisfies Eq. (2.1) on $[t_0, \infty)$.

We say Eq. (2.1) is oscillatory if every solution x of Eq. (2.1) at t_0 is oscillatory; i.e., there exists a sequence $\{t_k\}_{k=1}^{\infty}$ such that $t_0 \leqslant t_k \to +\infty$ $(k \to +\infty)$, and $x(t_k) = 0$, $k \geqslant 1$. A function $x \in C([t_{-1}, \infty), R)$ is said to be eventually positive (negative) if there exists a $t_1 \geqslant t_0$ such that $x(t) > 0$ $(x(t) < 0)$ for any $t \geqslant t_1$.

PROPOSITION 2.1. *Assume that* $-\infty \leqslant t_{-1} \leqslant t_0 < \infty$ *and* $f, g\colon [t_0, \infty) \times C([t_{-1}, \infty) \to R$ *are continuous Volterra functionals and*

(A_1) *there exists a continuous function* $\tau\colon [t_0, \infty) \to R_+$ *such that* $t - \tau(t) \geqslant t_{-1}$ *and it is increasing on* $t_0 \leqslant t < \infty$, $t - \tau(t) \to +\infty$ $(t \to +\infty)$, *and for any eventually positive function* $x \in C([t_{-1}, \infty), R)$, *there exists* $t_x \geqslant t_0$ *such that* $x(s) \geqslant 0$ $(s \geqslant t_x - \tau(t_x))$, *and*

$$g(t, x(\cdot)) \leqslant \max\{x(s)\colon t - \tau(t) \leqslant s \leqslant t\}, \qquad t \geqslant t_x. \tag{2.2}$$

If $x(t)$ is a solution of Eq. (2.1) on $[t_{-1}, \infty)$ and there exists some $T_x \geqslant t_{-1}$ such that $x(t) > 0$ $(t \geqslant T_x)$, moreover

$$f(t, x(\cdot)) \geqslant 0, \qquad \sup\{f(s, x(\cdot))\colon s \geqslant t\} > 0,$$
$$t \geqslant T_x^1 = \max\{T_x, t_0\}, \tag{2.3}$$

then the function

$$u(t) = \begin{cases} x(t) - g(t, x(\cdot)), & t \geqslant T_x^1, \\ \min\{x(t), x(T_x^1) - g(T_x^1, x(\cdot))\}, & t_{-1} \leqslant t < T_x^1, \end{cases} \tag{2.4}$$

is continuous on $[t_{-1}, \infty)$, $u(t) > 0$ $(t \geqslant T_x)$, and

$$\dot{u}(t) = -f(t, x(\cdot)) \leqslant 0, \qquad t \geqslant T_x^1. \tag{2.5}$$

Proof. Since $x(t)$ is a solution of Eq. (2.1) on $[t_{-1}, \infty)$, we have that $x(t)$ and $g(t, x(\cdot))$ are continuous and so $u(t)$ is also continuous on $[t_{-1}, \infty)$. From (2.1) and (2.3), it follows that

$$\dot{u}(t) = \frac{d}{dt}[x(t) - g(t,, x(\cdot))] = -f(t, x(\cdot)) \leqslant 0, \qquad t \geqslant T_x^1, \tag{2.6}$$

and

$$\inf\{\dot{u}(s) : s \geqslant t\} = -\sup\{f(s, x(\cdot)) : s \geqslant t\} < 0, \qquad t \geqslant T_x^1. \tag{2.7}$$

Now we shall show that $u(t) > 0$, $t \geqslant T_x$. To do this we show that the negation of this assertion leads to a contradiction. If we assume that there exists a $T_1 \geqslant T_x^1$ such that $u(T_1) = 0$, then (2.6) and (2.7) imply that there exists a constant $T_2 \geqslant T_1$ such that $\dot{u}(T_2) < 0$ and

$$u(t) \leqslant u(T_2) < u(T_1) = 0, \qquad t \geqslant T_2.$$

Thus by using the definition of $u(t)$, we obtain

$$x(t) = u(t) + g(t, x(\cdot)) \leqslant g(t, x(\cdot)) + u(T_2), \qquad t \geqslant T_2. \tag{2.8}$$

From Assumption (A_1), we have that there exists a $t_x \geqslant t_0$ such that $x(s) \geqslant 0$ $(s \geqslant t_x - \tau(t_x))$, and (2.2) holds. Combining this with (2.8), we have

$$x(t) < g(t, x(\cdot)) \leqslant \max\{x(s) : t - \tau(t) \leqslant s \leqslant t\},$$
$$t \geqslant \tilde{t}_x = \max\{T_2, t_x\}. \tag{2.9}$$

Now, let us define a constant $M > 0$ such that

$$0 \leqslant x(t) < M, \qquad \tilde{t}_x - \tau(\tilde{t}_x) \leqslant s \leqslant \tilde{t}_x.$$

We shall show that $x(t) < M$, for any $t \geqslant \tilde{t}_x$. Indeed, if we assume that for some $t_1 > \tilde{t}_x$, $x(t_1) = M$, then there exists some $t_2 \in (t_x, t_1]$ such that

$$0 \leqslant x(t) < M, \qquad \tilde{t}_x - \tau(\tilde{t}_x) \leqslant t < t_2, \qquad x(t_2) = M.$$

However, from (2.9), we obtain the contradiction

$$M = x(t_2) < \max\{x(s) : t_2 - \tau(t_2) \leqslant s \leqslant t_2\} \leqslant M;$$

therefore, $0 \leqslant x(t) < M$, for any $t \geqslant \tilde{t}_x - \tau(\tilde{t}_x)$. Using this fact, (2.2) and (2.8) yield

$$\limsup_{t \to +\infty} x(t) \leqslant \limsup_{t \to +\infty} g(t, x(\cdot)) + u(T_2)$$
$$< \limsup_{t \to +\infty} \max \{x(s) : t - \tau(t) \leqslant s \leqslant t$$
$$= \limsup_{t \to +\infty} x(t) \leqslant M < \infty,$$

which is a contradiction and thus $u(t) > 0$ for any $t \geqslant T_x^1$. Since $u(T_x^1) = x(T_x^1) - g(T_x^1, x(\cdot)) > 0$ and $x(t) > 0$, $T_x \leqslant t \leqslant T_x^1$, thus $u(t) > 0$ for any $t \geqslant T_x$, which completes the proof of the proposition.

Now we are ready to prove our main statements.

THEOREM 2.1. *Assume that* $-\infty \leqslant t_{-1} \leqslant t_0 < \infty$ *and* f, $g : [t_0, \infty) \times C([t_{-1}, \infty), R) \to R$ *are continuous Volterra functionals such that Assumption* (A_1) *holds and*

(A_2) *for any* $x, u \in C([t_{-1}, \infty), R)$ *and* $T \geqslant t_{-1}$, *the inequality* $x(t) \geqslant u(t)$ $(t \geqslant T)$ *implies the existence of a* $T_1 = T_1(T)$ *such that*

$$f(t, x(\cdot)) \geqslant f(t, u(\cdot)) \text{ and } g(t, x(\cdot)) \geqslant g(t, u(\cdot)), \qquad t \geqslant T_1. \tag{2.10}$$

If $x(t)$ is a solution of Eq. (2.1) on $[t_{-1}, \infty)$ and there exists a constant $T_x \geqslant t_0$ such that $x(t) > 0$ $(t \geqslant T_x)$, moreover (2.3) and

$$g(t, x(\cdot)) \geqslant 0, \qquad t \geqslant T_x, \tag{2.11}$$

hold, then the function $u(t)$ defined by (2.4) is positive on $[T_x, \infty)$ and there exists $\tilde{T}_x \geqslant T_x$ for which

$$\dot{u}(t) \leqslant -f(t, u(\cdot) + g_0(\cdot, u(\cdot))), \qquad t \geqslant \tilde{T}_x, \tag{2.12}$$

where $g_0(t, u(\cdot))$ is a continuous non-negative function on $[t_{-1}, \infty)$ such that $g_0(t, u(\cdot)) = g(t, u(\cdot))$ for all t large enough.

Proof. By virtue of Proposition 2.1, we have $u(t) > 0$ $(t \geqslant T_x)$, and (2.11) implies

$$x(t) = u(t) + g(t, x(\cdot)) \geqslant u(t), \qquad t \geqslant T_x.$$

From Assumption (A_2), we can define some $T(T_x) \geqslant T_x$ such that (2.10) holds and $g_0(t, u(\cdot)) = g(t, u(\cdot))$ $(t \geqslant T(T_x))$; therefore,

$$x(t) \geqslant u(t) + g_0(t, u(\cdot)), \qquad t \geqslant T(T_x).$$

Using also Assumption (A_2), we have that for a suitable $\tilde{T}_x$ depending on $T(T_x)$,

$$f(t, x(\cdot)) \geqslant f(u(\cdot) + g_0(\cdot, u(\cdot))), \qquad t \geqslant \tilde{T}_x.$$

Combining this inequality with (2.6), we obtain the required inequality (2.12), and the proof is complete.

THEOREM 2.2. *Assume that* $-\infty \leqslant t_{-1} \leqslant t_0 < \infty$ *and* f, $g: [t_0, \infty) \times C([t_{-1}, \infty), R) \to R$ *are continuous Volterra functionals such that Assumptions* (A_1) *and* (A_2) *hold, moreover*

(A_3) *there exist* $a > 0$ *and* $b \geqslant c > 0$ *such that*

$$g(t, x(\cdot)) \geqslant a \min\{x(s): t-b \leqslant s \leqslant t-c\}, \tag{2.13}$$

for any $(t, x) \in [t_0, \infty) \times C([t_{-1}, \infty), R)$.

If on these conditions, $x(t)$ is a solution of Eq. (2.1) on $[t_{-1}, \infty)$ which is positive on some interval $[T_x, \infty)$, then the function

$$\hat{u}_n(t) = \begin{cases} u(t) + \sum\limits_{i=1}^{n} a^i u(t-ic), & t \geqslant T_x + (n+1)\, b, \\ u_n(T_x + (n+1)\, b), & t < T_x + (n+1)\, b, \end{cases} \tag{2.14}$$

is continuous and positive on $[t_{-1}, \infty)$ for any fixed $n \geqslant 1$, moreover for some $\tilde{T}_x(n) \geqslant T_x$,

$$\dot{u}(t) \leqslant -f(t, u_n(\cdot)), \qquad t \geqslant \tilde{T}_x(N). \tag{2.15}$$

Proof. By virtue of Proposition 2.1, we have $u(t) > 0$ and $\dot{u}(t) < 0$ $(t \geqslant T_x)$, and from (2.4), it follows that

$$\begin{aligned} x(t) &= u(t) + g(t, x(\cdot)) \\ &\geqslant u(t) + a \min\{x(s): t-b \leqslant s \leqslant t-c\}, \qquad t \geqslant t_0, \end{aligned} \tag{2.16}$$

i.e.,

$$x(t) \geqslant u(t) + au(t-c), \qquad t \geqslant T_x + b. \tag{2.17}$$

We shall show that for any $k \geqslant 1$ and for all $t \geqslant T_x + (k+1)\, b$,

$$x(t) \geqslant u_k(t), \tag{2.18}$$

where

$$u_k(t) = \begin{cases} u(t) + \sum\limits_{i=1}^{k} a^i u(t-ic), & t \geqslant T_x + (k+1)\, b, \\ u_k(T_x + (k+1)\, b), & t_{-1} \leqslant t < T_x + (k+1)\, b. \end{cases}$$

Indeed, (2.17) implies that $x(t) \geqslant u_1(t)$, $t \geqslant T_x + b$. So, by virtue of the mathematical induction it is enough to show that if $x(t) \geqslant u_k(t)$, $t \geqslant T_x + (k+1)\, b$, for an arbitrary fixed $k \geqslant 1$ then $x(t) \geqslant u_{k+1}(t)$, $t \geqslant T_x + (k+2)\, b$. Let us assume that (2.18) holds for all $t \geqslant T_x + (k+1)\, b$. Then (2.16) implies

$$x(t) \geqslant u(t) + a \min\{u_k(s) : t-b \leqslant s \leqslant t-c\}, \qquad t \geqslant T_x + (k+1)\, b.$$

Since $\dot{u}(t) \leqslant 0$, $t \geqslant T_x$, we have $\dot{u}_k(t) \leqslant 0$, $t \geqslant T_x + (k+1)\, b$. Thus

$$x(t) \geqslant u(t) + a u_k(t-c) = u(t) + \sum_{i=1}^{k+1} a^i u(t-ic) = u_{k+1}(t),$$

for any $t \geqslant T_x + (k+2)\, b$; i.e., (2.18) is valid for any $k \geqslant 1$. Particularly, if $k = n$, then $x, u_n \in C([t_{-1}, \infty), R)$ are such that the inequality $x(t) \geqslant u_n(t)$ holds for any $t \geqslant T = T_x + (n+1)\, b$. By virtue of Assumption (A_2), there exists a $\tilde{T}_x(n) \geqslant T$ such that

$$f(t, x(\cdot)) \geqslant f(t, u_n(\cdot)), \qquad t \geqslant \tilde{T}_x(n),$$

and thus (2.5) implies

$$\dot{u}(t) = -f(t, x(\cdot)) \leqslant -f(t, u_n(\cdot)), \qquad t \geqslant \tilde{T}_x(n).$$

The proof of the theorem is complete.

3. Some Special Equations with Finite Delays

Let us consider the neutral delay differential equation

$$\frac{d}{dt}\left[x(t) - \sum_{j=1}^{M} a_j(t)\, x(t-\delta_j(t))\right] = -\sum_{i=1}^{N} b_i(t)\, x(t-r_i(t)), \quad (3.1)$$

where

(P_1) a_j, δ_j ($1 \leqslant j \leqslant M$), and b_i, r_i ($1 \leqslant i \leqslant N$) are given continuous non-negative functions on $[t_0, \infty)$ such that

$$\sum_{j=1}^{M} a_j(t) \leqslant 1, \qquad \sup\left\{\sum_{i=1}^{N} b_i(s) : s \geqslant t\right\} > 0, \quad (3.2)$$

for all t large enough and for some $\Delta > 0$,

$$\delta_j(t) \leqslant \Delta \text{ and } r_i(t) \leqslant \Delta, \qquad t \geqslant t_0,\ 1 \leqslant j \leqslant M,\ 1 \leqslant i \leqslant N.$$

By virtue of Theorem 2.1, we can prove the following.

THEOREM 3.1. *If the property* (P_1) *is satisfied and*

$$\dot{u}(t) = -\sum_{i=1}^{N} b_i(t)\Bigg[u(t-r_i(t)) + \sum_{j=1}^{M} a_j(t-r_i(t))\,u(t-r_i(t)-\delta_j(t-r_i(t)))\Bigg] \tag{3.3}$$

has only oscillatory solutions on $[T-2\Delta, \infty)$ *for all* $T \geqslant t_0 + 2\Delta$ *large enough, then Eq.* (3.1) *is oscillatory on* $[t_0 - \Delta, \infty)$.

Proof. Let us define the functions f and g by the formulas

$$f(t, x(\cdot)) = \sum_{i=1}^{N} b_i(t)\,x(t-r_i(t))$$

and

$$g(t, x(\cdot)) = \sum_{j=1}^{M} a_j(t)\,x(t-\delta_j(t)),$$

for any $t, x \in [t_0, \infty) \times C([t_{-1}, \infty), R)$, where $t_{-1} = t_0 - \Delta$.

Then f and g are continuous Volterra functionals and Eqs. (2.1) and (3.1) are equivalent.

Define $\tau(t) = \Delta$ $(t \geqslant t_0)$, then $t - \tau(t) \to +\infty$ $(t \to +\infty)$, and for any eventually positive function $x \in C([t_{-1}, \infty), R)$, there exists a $t_x \geqslant t_0$ such that $x(s) \geqslant 0$ $(s \geqslant t_x - \Delta)$ and

$$\begin{aligned} g(t, x(\cdot)) &= \sum_{j=1}^{M} a_j(t)\,x(t-\delta_j(t)) \\ &\leqslant \max\{x(s) : t-\Delta \leqslant s \leqslant t\},\ t \geqslant t_x, \end{aligned}$$

and thus the Assumption (A_1) is satisfied. From the positivity of the functions a_j $(1 \leqslant j \leqslant M)$ and b_i $(1 \leqslant i \leqslant N)$, it follows that f and g satisfy Assumption (A_2).

Now let us assume that Eq. (3.1) is not oscillatory on $[t_0 - \Delta, \infty)$; i.e., Eq. (3.1) has an eventually positive or negative solution $x(t)$ on $[t_0 - \Delta, \infty)$. Without loss of generality, we can assume that $x(t)$ is eventually positive, since if $x(t)$ is a solution of Eq. (3.1) then $-x(t)$ is also a solution.

Now, we are able to apply Theorem 2.1. From this theorem, we have that the function $u(t)$ defined by (2.4) is eventually positive and there exists a continuous function $g_0(t, u(\cdot))$ on $[t_{-1}, \infty)$ such that $g_0(t, u(\cdot)) = g(t, u(\cdot))$ and inequality (2.12) is satisfied for all t large enough. But for any t large enough

$$
\begin{aligned}
&f(t, u(\cdot)+g_0(\cdot, u(\cdot)))\\
&\quad=\sum_{i=1}^{N} b_i(t)[u(t-r_i(t))+g_0(t-r_i(t), u(\cdot))]\\
&\quad=\sum_{i=1}^{N} b_i(t)\left[u(t-r_i(t))+\sum_{j=1}^{M} a_j(t-r_i(t))\,u(t-r_i(t)-\delta_j(t-r_i(t))\right];
\end{aligned}
$$

i.e., (2.12) implies

$$
\dot{u}(t)\leqslant -\sum_{i=1}^{N} b_i(t)\left[u(t-r_i(t))+\sum_{j=1}^{M} a_j(t-r_i(t))\,u(t-r_i(t)-\delta_j(t-r_i(t)))\right]
$$

for a suitable interval $T\leqslant t<\infty$, where $T\geqslant t_0$ is defined such that $u(t)>0$ for all $t\geqslant T-2\Delta$. But in that case, by a comparison theorem (see [3, p. 224]), we have that Eq. (3.3) has a positive solution on $[T-2\Delta, \infty)$, which is a contradiction. Thus Eq. (3.1) is oscillatory and the proof of the theorem is complete.

COROLLARY 3.1. *If property* (P_1) *is satisfied and*

$$
\liminf_{t\to+\infty} \sum_{i=1}^{n} b_i(t)\left[r_i(t)+\sum_{j=1}^{n} a_j(t-r_i(t))(r_i(t)+\delta_j(t-r_i(t))\right]>1/e \tag{3.4}
$$

then Eq. (3.1) *is oscillatory on* $[t_0-\Delta, \infty)$.

Proof. From [10, Theorem 1.1], we have that if (3.4) is satisfied then Eq. (3.3) is oscillatory on $[t_0-\Delta, \infty)$. Therefore by virtue of our Theorem 3.1, Eq. (3.1) is also oscillatory on $[t_0-\Delta, \infty)$. The proof of the corollary is complete.

From the last corollary, we get a generalized version of Theorem 3 of Ladas and Sficas [12]:

COROLLARY 3.2. *If* a_j, δ_j $(1\leqslant j\leqslant N)$ *and* b_i, r_i $(1\leqslant i\leqslant N)$ *are given positive constants such that* $\sum_{j=1}^{N} a_j\leqslant 1$ *and*

$$
\left(\sum_{i=1}^{N} b_i r_i\right)\left(1+\sum_{j=1}^{M} a_j\right)+\left(\sum_{i=1}^{N} b_i\right)\left(\sum_{j=1}^{M} a_j d_j\right)>1/e,
$$

then

$$
\frac{d}{dt}\left[x(t)-\sum_{j=1}^{M} a_j x(t-\delta_j)\right]=-\sum_{i=1}^{N} b_i x(t-r_i) \tag{3.5}
$$

is oscillatory on $[t_0-\Delta, \infty)$, *where* $\Delta=\max\{\delta_j, r_i\colon 1\leqslant j\leqslant M, 1\leqslant i\leqslant N\}$.

COROLLARY 3.3. *Assume that* $a \in [0, 1]$, $\delta > 0$ *are given constants and* b, $r: [t_0, \infty) \to R$ *are continuous functions*, $0 \leqslant r(t) \leqslant \Delta < \infty$ $(t \geqslant t_0)$, $\limsup_{t \to +\infty} b(t) < \infty$. *If*

$$\liminf_{t \to +\infty} b(t)[r(t) + 1] > 0, \qquad \textit{whenever } a = 1, \tag{3.6}$$

and

$$\liminf_{t \to +\infty} b(t) \left[r(t) \frac{1}{1-a} + \frac{a\delta}{(1-a)^2} \right] > 1/e, \qquad \textit{whenever } a \in [0, 1), \tag{3.7}$$

then

$$\frac{d}{dt}[x(t) - ax(t-\delta)] = -b(t)\, x(t - r(t)) \tag{3.8}$$

is oscillatory on $[t_0 - \Delta_0, \infty)$, *where* $\Delta_0 = \max\{\delta, \Delta\}$.

Proof. Let us define the functions f and g by the formulas

$$f(t, x(\cdot)) = b(t)\, x(t - r(t))$$

and

$$g(t, x(\cdot)) = ax(t - \delta),$$

for any $t, x \in [t_0, \infty) \times C([t_{-1}, \infty), R)$, where $t_{-1} = t_0 - \Delta_0$.

Then one can see that f and g satisfy Assumptions (A_1)–(A_3), since

$$g(t, x(\cdot)) \geqslant a \min\{x(s): t - b \leqslant s \leqslant t - c\}$$

for any $(t, x) \in [t_0, \infty) \times C([t_{-1}, \infty), R)$, whenever $b = c = \delta$.

If we now assume that Eq. (3.8) is not oscillatory, then Eq. (3.8) has an eventually positive solution $x(t)$ on $[t_0 - \Delta, \infty)$. Define a constant $T_x \geqslant t_0 - \Delta$ such that $x(t) > 0$, $t \geqslant T_x$. Then in view of Theorem 2.2, we have that the function $u_n(t)$ defined by (2.14) is positive on $[t_0 - \Delta, \infty)$ and

$$\dot{u}(t) \leqslant -f(t, u_n(\cdot)) = -b(t)\, u_n(t - r(t)),$$

for any t large enough.

But if t is large enough then from (2.14), it follows that

$$u_n(t - r(t)) = u(t - r(t)) + \sum_{i=1}^{n} a^i u(t - r(t) - i\delta),$$

since $\delta = c$. Thus for any fixed $n \geqslant 1$, there exists a real number $T_n \geqslant t_0$ such that $u(t) > 0$ $(t \geqslant T_n - \Delta)$ and

$$\dot{u}(t) \leqslant -b(t)\, u(t - r(t)) - b(t) \sum_{i=1}^{n} a^i u(t - r(t) - i\delta), \qquad t \geqslant T_n.$$

In view of a comparison result (see [3, p. 224], we have that

$$\dot{v}(t) = -b(t)\, v(t-r(t)) - b(t) \sum_{i=1}^{n} a^i v(t-r(t)-i\delta) \tag{3.8$_n$}$$

has a positive solution on $[T_n - \Delta, \infty)$, for any fixed $n \geqslant 1$. From (3.6) and (3.7), it follows that there exists an integer $n_0 \geqslant 1$ such that the real number $B(a)$ defined by

$$B(a) = \liminf_{t \to +\infty} \left\{ b(t)\, r(t) + b(t) \sum_{i=1}^{n_0} a^i (r(t) + i\delta) \right\}$$

is larger than $1/e$, since

$$\lim_{n \to +\infty} \sum_{i=0}^{n} a^i = \begin{cases} +\infty, & \text{if} \quad a = 1, \\ \dfrac{1}{1-a}, & \text{if} \quad 0 \leqslant a < 1, \end{cases}$$

and

$$\lim_{n \to +\infty} \sum_{i=1}^{n} i a^i = \begin{cases} +\infty, & \text{if} \quad a = 1, \\ \dfrac{a}{(1-a)^2}, & \text{if} \quad 0 \leqslant a < 1, \end{cases}$$

is larger than $1/e$. But by virtue of Theorem 1.1 in [10], from the inequality

$$B(a) > 1/e,$$

it follows that Eq. $(3.8)_n$ has no positive solution, whenever $n = n_0$, which is a contradiction. Thus Eq. (3.8) is oscillatory and the proof of the corollary is complete.

Remark 3.1. Corollary 3.3 contains Theorems 2, 3, and 4 of Ladas and Sficas [12] when $b(t) = b_0$ and $r(t) = r_0$ are some constants in (3.8).

Now let us consider the differential equation

$$\dot{x}(t) = -px(t-\tau) + qx(t-\sigma), \tag{3.9}$$

where p, q, τ, and σ are positive constants.

Recently, this equation has been investigated by Arino, Ladas, and Sficas [1] and they obtained that if $\sigma \leqslant \tau$, $q \leqslant p$, $q(\tau - \sigma) \leqslant 1$, and $(p-q)\,\tau e > 1$ then every solution of (3.9) oscillates.

By the application of our Theorem 2.1 from the previous section, we have the following theorem which is a significant extension of the above-mentioned result in [1].

THEOREM 3.2. *If* p, q, τ, *and* σ *are positive constants such that* $0 \leqslant q(\tau-\sigma) \leqslant 1$ *and*

$$(p-q)[\tau+q(\tau-\sigma)^2] > 1/e, \tag{3.10}$$

then Eq. (3.9) *is oscillatory.*

Proof. Equation (3.9) is equivalent to

$$\frac{d}{dt}\left[x(t)-q\int_{t-\tau}^{t-\sigma} x(s)\,ds\right] = -(p-q)\,x(t-\tau). \tag{3.11}$$

Let us define the functions f and g by the formulas

$$f\,t, x(\cdot)) = (p-q)\,x(t-\tau)$$

and

$$g(t, x(\cdot)) = q\int_{t-\tau}^{t-\sigma} x(s)\,ds,$$

for any $(t, x) \in [t_0, \infty) \times C([t_{-1}, \infty))$, where $t_{-1} = t_0 - \tau$.

Then f and g satisfy Assumptions (A_1) and (A_2), moreover Eq. (3.11) is a special case of Eq. (2.1).

Now let us assume that (3.10) holds but Eq. (3.9) is not oscillatory. In that case Eq. (3.9) has an eventually positive solution $x(t)$ on $[t_{-1}, \infty)$, and $x(t)$ is also a solution of Eq. (3.11).

By virtue of Theorem 2.1, we have that there exists a $T_x \geqslant t_0 + \tau$ such that the function

$$u(t) = x(t) - q\int_{t-\tau}^{t-\sigma} x(s)\,ds$$

is positive on $[T_x - \tau, \infty)$ and

$$\dot{u}(t) \leqslant -(p-q)\,u(t-\tau) - (p-q)\,q\int_{t-2\tau}^{t-\tau-\sigma} u(s)\,ds, \qquad t \geqslant T_x. \tag{3.12}$$

Since $u(t)$ is continuously differentiable and positive on $T_x - \tau$, the function

$$\alpha(t) = -\frac{\dot{u}(t)}{u(t)}$$

is also continuous and

$$u(t) = u(T_x - \tau)\exp\left(-\int_{T_x-\tau}^{t} \alpha(s)\,ds\right), \qquad t \geqslant T_x - \tau.$$

From (3.12), it follows that

$$\alpha(t) \geqslant (p-q) \exp\left(\int_{t-\tau}^{t} \alpha(s)\, ds\right)$$
$$+ (p-q)\, q \int_{t-2\tau}^{t-\tau-\sigma} \exp\left(\left(\int_{s}^{t} \alpha(u)\, du\right) ds\right), \qquad t \geqslant T_x, \quad (3.13)$$

and thus

$$\alpha(t) \geqslant (p-q)\, e \int_{t-\tau}^{t} \alpha(s)\, ds$$
$$+ (p-q)\, qe \int_{t-2\tau}^{t-\tau-\sigma} \left(\int_{s}^{t} \alpha(u)\, du\right) ds, \quad (3.14)$$

since $\exp(x) \geqslant ex$, $x \geqslant 0$. From [8, Lemma 2.1] and (3.13), we have

$$0 < m = \liminf_{t \to +\infty} \alpha(t) < \infty.$$

Since

$$\liminf_{t \to +\infty} \int_{t-2\tau}^{t-\tau-\sigma} \left(\int_{s}^{t} \alpha(u)\, du\right) ds = m \int_{t-2\tau}^{t-\tau-\sigma} (t-s)\, ds \geqslant (\tau-\sigma)^2,$$

thus (3.14) yields

$$m \geqslant (p-q)\, e\tau m + (p-q)\, qem(\tau-\sigma)^2,$$

i.e.,

$$1/e \geqslant (p-q)[\tau + q(\tau-\sigma)^2].$$

But the latest inequality contradicts our assumption (3.10); therefore, Eq. (3.9) is oscillatory. The proof of the theorem is complete.

Remark 3.2. Theorem 3.2 can be extended to the equations with several delays generalizing [1, Theorem 3].

4. An Integro-differential Equation with Infinite Delay

Let us consider the integro-differential equation of the neutral type

$$\frac{d}{dt}\left[x(t) - \int_{t-\tau(t)}^{t} x(s)\, d_s p_0(t, s)\right]$$
$$= -\int_{t-1}^{t} x(s)\, d_s q_0(t, s), \qquad t \geqslant t_0, \quad (4.1)$$

where $-\infty < t_{-1} \leqslant t_0 < \infty$ and

(H_1) *$\tau: [t_0, \infty) \to R_+$ is a continuous function such that $t-\tau(t)$ is monotone non-decreasing and $t-\tau(t) \geqslant t_{-1}$ $(t \geqslant t_0)$, moreover $t-\tau(t) \to +\infty$ $(t \to +\infty)$;*

(H_2) *for any fixed $t \geqslant t_0$, the function $p_0(t, s)$ is monotone non-decreasing on $t-\tau(t) \leqslant s \leqslant t$ and for some $T_0 \geqslant t_0$,*

$$p_0(t, t) - p_0(t, t-\tau(t)) \leqslant 1, \qquad t \geqslant T_0, \tag{4.2}$$

moreover for any $x \in C([t_{-1}, \infty), R)$, the function

$$g(t, x(\cdot)) = \int_{t-\tau(t)}^{t} x(s)\, d_s p_0(t, s) \tag{4.3}$$

is continuous on $[t_0, \infty)$;

(H_3) *$q_0: [t_0, \infty) \times [t_{-1}, \infty) \to R$ is a given function such that for any fixed $t \in [t_0, \infty)$, $q_0(t, s)$ is a monotone non-decreasing function in $s \in [t_{-1}, \infty)$ and for any $x \in C([t_{-1}, \infty), R)$, the function*

$$f(t, x(\cdot)) = \int_{t_{-1}}^{t} x(s)\, d_s q_0(t, s) \tag{4.4}$$

is continuous on $[t_0, \infty)$.

THEOREM 4.1. *Assume* (H_1)–(H_3) *hold and there exist two continuous functions T_0, T: $[t_0, \infty) \to R$ such that $T_0(t) \leqslant T(t) \leqslant t$ $(t \geqslant t_0)$,* $\lim_{t\to+\infty} T_0(t) = +\infty$, $\limsup_{t\to+\infty} (t - T(t)) > 0$, *and*

$$\liminf_{t\to+\infty} [q_0(t, T(t)) - q_0(t, T_0(t))] > 0. \tag{4.5}$$

If

$$m_0 = \inf\left\{\frac{1}{\mu} \liminf_{t\to+\infty} \int_{T_0(t)}^{T(t)} \exp(\mu(t-s))\, d_s q_0(t, s) : \mu > 0\right\} > 1, \tag{4.6}$$

then Eq. (4.1) *has no positive negative solution on* $[t_{-1}, \infty)$.

Proof. We will show that the existence of a positive negative solution $x(t)$ of (4.1) leads to a contradiction. Without loss of generality, we may assume that $x(t)$ is a positive solution, since $-x(t)$ is also a solution of Eq. (4.1).

From (4.2), it follows that

$$g(t, x(\cdot)) = \int_{t-\tau(t)}^{t} x(s)\, d_s p_0(t, s) \leqslant [p_0(t, t) - p_0(t, t-\tau(t))]$$
$$\max\{x(s)\colon t-\tau(t) \leqslant s \leqslant t\}$$
$$\leqslant \max\{x(s)\colon t-\tau(t) \leqslant s \leqslant t\},$$

and thus Assumption (A_1) of Proposition 2.1 is satisfied. Since $q_0(t, s)$ is monotone non-decreasing in its independent variable s, we have

$$f(t, x(\cdot)) = \int_{t-1}^{t} x(s)\, d_s q_0(t, s)$$
$$\geqslant \int_{T_0(t)}^{T(t)} x(s)\, d_s q_0(t, s)$$
$$\geqslant [q_0(t, T(t)) - q_0(t, T_0(t))] \min\{x(s)\colon T_0(t)$$
$$\leqslant s \leqslant T(t)\} > 0,$$

for any $t \geqslant t_0$, which means that all conditions of Proposition 2.1 are satisfied.

Thus for the positive solution $x \in C([t_{-1}, \infty), R)$, by Proposition 2.1, we have that the function

$$u(t) = \begin{cases} x(t) - \int_{t-\tau(t)}^{t} x(s)\, d_s p_0(t, s), & t \geqslant t_0, \\ \min\left\{x(t), x(t_0) - \int_{t_0}^{t_0} - \tau(t_0)\, x(s)\, d_s p_0(t, s), \right. & t_{-1} \leqslant t < t_0 \end{cases} \tag{4.7}$$

is continuous and

$$\dot{u}(t) \leqslant -\int_{t-1}^{t} x(s)\, d_s q_0(t, s), \qquad t \geqslant t_0. \tag{4.8}$$

But, (4.7) implies that

$$u(t) \geqslant x(t), \qquad t_{-1} \leqslant t < \infty,$$

and thus (4.8) yields

$$\dot{u}(t) \leqslant -\int_{t-1}^{t} u(s)\, d_s q_0(t, s), \qquad t \geqslant t_0, \tag{4.9}$$

since $q_0(t, s)$ is monotone non-decreasing in $s \in [t_{-1}, \infty)$, for all fixed $t \geqslant t_0$.

Since for any $t_0 \leqslant t_1 \leqslant t_2$ and $t \geqslant t_0$,

$$\int_{t_1}^{t_2} u(s)\, d_s q_0(t, s) \geqslant 0,$$

from (4.9) it follows that

$$\begin{aligned}
\dot{u}(t) &\leqslant -\int_{t-1}^{t_0} u(s)\, d_s q_0(t, s) \\
&\leqslant -\int_{t_0}^{T(t)} u(s)\, d_s q_0(t, s) - \int_{T(t)}^{t} u(s)\, d_s q_0(t, s) \\
&\leqslant -\int_{t_0}^{T(t)} u(s)\, d_s q_0(t, s), \qquad t \geqslant t_0. \qquad (4.10)
\end{aligned}$$

The function $u(t)$ is continuously differentiable on $[t_0, \infty)$, therefore we may write it in the form

$$x(t) = x(t_0) \exp\left(-\int_{t_0}^{t} \alpha(s)\, ds\right), \qquad t \geqslant t_0,$$

where the continuous, non-negative function α is defined by

$$\alpha(t) = \frac{\dot{x}(t)}{x(t)}, \qquad t \geqslant t_0.$$

Again, with the aid of inequality (4.10), we obtain

$$\alpha(t) \geqslant \int_{t_0}^{T} \exp\left(\int_{\alpha}^{t} \alpha(u)\, du\right) d_s q_0(t, s), \qquad t \geqslant t_0. \qquad (4.11)$$

Thus, (4.5) and (4.11) imply

$$\begin{aligned}
\alpha(t) &\geqslant [q_0(t, T(t)) - q_0(t, t_0)] \exp\left(\int_{T(t)}^{t} \alpha(u)\, du\right) \\
&\geqslant m \exp\left(\int_{t-r}^{t} \alpha(u)\, du\right), \qquad (4.12)
\end{aligned}$$

for any t large enough, where m and r are some constants such that

$$0 < m < \liminf_{t \to +\infty} \{q_0(t, T(t)) - q_0(t, t_0)\}$$

and

$$0 < r < \limsup_{t \to +\infty} (t - T(t)).$$

In that case, by virtue of [8, Lemma 2.1], we have

$$\liminf_{t \to +\infty} \int_{t-r}^{t} \alpha(u)\, du < \infty,$$

i.e.,

$$\beta = \liminf_{t \to +\infty} \alpha(t) < \infty.$$

From (4.12), we obtain $\beta \geqslant m > 0$, therefore for any $\varepsilon \in (0, \beta)$, there exists a $T(\varepsilon) > t_0$ such that

$$\alpha(t) \geqslant \beta - \varepsilon, \qquad t \geqslant T(\varepsilon).$$

Using this inequality and (4.11), we have

$$\begin{aligned} \alpha(t) &\geqslant \int_{T_0(t)}^{T(t)} \exp\left(\int_s^t \alpha(u)\, du\right) d_s q_0(t, s) \\ &\geqslant \int_{T_0(t)}^{T(t)} \exp((\beta - \varepsilon)(t - s))\, d_s q_0(t, s) \end{aligned}$$

for any large enough t, since $T_0(t) \to +\infty$, as $t \to +\infty$. But in that case

$$\beta = \liminf_{t \to +\infty} \alpha(t) \geqslant \liminf_{t \to +\infty} \int_{T_0(t)}^{T(t)} \exp((\beta - \varepsilon)(t - s))\, d_s q_0(t, s),$$

for any fixed $\varepsilon \in (0, \beta)$.

From (4.6) and the last inequality, it follows that

$$\frac{\beta}{\beta - \varepsilon} \geqslant \frac{1}{\beta - \varepsilon} \liminf_{t \to +\infty} \int_{T_0(t)}^{T(t)} \exp((\beta - \varepsilon)(t - s))\, d_s q_0(t, s) \geqslant m_0 > 1,$$

for any $\varepsilon \in (0, \beta)$. But this inequality leads to a contradiction as $\varepsilon \to 0$. Therefore Eq. (4.1) has no positive/negative solution on $[t_{-1}, \infty)$ and the proof is complete.

COROLLARY 4.1. *Assume that* (H_1)–(H_2) *hold, moreover* $p_i\colon [t_0, \infty) \to R_+$ *and* $\tau_i\colon [t_0, \infty) \to R_+$ $(1 \leqslant i \leqslant n)$ *are continuous functions such that*

$$\inf_{t \geqslant t_0} \{t - \tau_i(t)\} \geqslant t_{-1}, \qquad \lim_{t \to +\infty} [t - \tau_i(t)] = +\infty \qquad (1 \leqslant i \leqslant n) \tag{4.13}$$

and

$$\liminf_{t \to +\infty} \sum_{i=1}^{n} p_i(t) > 0, \qquad \limsup_{t \to +\infty} [\min\{\tau_i(t)\colon 1 \leqslant i \leqslant n\}] > 0. \tag{4.14}$$

If

$$m_1 = \inf\left\{\frac{1}{\mu}\liminf_{r\to+\infty}\sum_{i=1}^{n} p_i(t)\exp(\tau_i(t)) : \mu > 0\right\} > 1, \tag{4.15}$$

then

$$\frac{d}{dt}\left[x(t) - \int_{t-\tau(t)}^{t} x(s)\, d_s\, p_0(t, s)\right]$$

$$= -\sum_{i=1}^{n} p_i(t)\, x(t-\tau_i(t)), \qquad t \geqslant t_0, \tag{4.16}$$

has no positive negative solution on $[t_{-1}, \infty)$.

Proof. Set

$$T_0(t) = \min\{t - \tau_i(t) : 1 \leqslant i \leqslant n\}, \qquad T(t) = \max\{t - \tau_i(t) : 1 \leqslant i \leqslant n\},$$

and

$$q_0(t, s) = \sum_{i=1}^{n} p_i(t)\, e(s - (t - \tau_i(t))), \qquad t \geqslant t_0,\ T_0(t) \leqslant s \leqslant T(t),$$

where $e(s) = 0$, $(s \leqslant 0)$, and $e(s) = 1$ $(s > 0)$. Then $\lim_{t\to+\infty} T_0(t) = +\infty$ and $\limsup_{t\to+\infty}(t - T(t)) > 0$, moreover (H_3) and (4.5) are satisfied.

On the other hand Eq. (4.16) is a special case of Eq. (4.1) and $m_0 = m_1 > 1$ where m_0 and m_1 are defined by (4.6) and (4.15), respectively. Thus Theorem 4.1 implies that Eq. (4.16) is oscillatory on $[t_{-1}, \infty)$ and the proof is complete.

Remark 4.1. Consider the delay differential equation

$$\dot{x}(t) = -\sum_{i=1}^{n} p_i(t)\, x(t - \tau_i(t)), \qquad t \geqslant 0, \tag{4.17}$$

where $p_i, \tau_i : R_+ \to R_+$ $(1 \leqslant i \leqslant n)$ are some given continuous functions. Hunt and Yorke [10] conjectured that all solutions of (4.17) oscillate if p_i and τ_i $(1 \leqslant i \leqslant n)$ are bounded on R_+ and

$$\inf_{\lambda, t \in (0, \infty)} \frac{1}{\lambda} \sum_{i=1}^{n} p_i(t)\exp(\lambda\tau_i(t)) > 1. \tag{4.18}$$

The conjecture was proved in some particular cases in a recent paper [9] and now Corollary 4.1 shows that the conjecture is valid for some neutral delay differential equations, too.

COROLLARY 4.2. *Assume* $0<p<1$ *and* $\tau>0$ *are constants and* $k: R_+ \to R_+$ *is a continuous function.*

Then

$$\frac{d}{dt}[x(t)-px(t-\tau)] = -\int_0^t k(t-s)\,x(s)\,ds, \qquad t \geqslant 0, \tag{4.19}$$

has no positive negative solution on $[-\tau, \infty)$ *if*

$$e\int_0^{\infty} k(s)\,s\,ds > 1. \tag{4.20}$$

Proof. Let $\tau(t)=\tau$ $(t \geqslant 0)$, $t_{-1}=-\tau$, and

$$p_0(t,s) = pe(s-[t-\tau]),$$

where e is the unit step function, i.e., $e(u)=0$ for $u \leqslant 0$ and $e(u)=1$ for $u>0$. Then the assumptions ((H_1) and (H_2)) about τ and $p_0(t,s)$ are satisfied. Set

$$q_0(t,s) = \begin{cases} -\displaystyle\int_0^t k(u)\,du, & t>0,\ -\tau \leqslant s < 0, \\ -\displaystyle\int_0^{t-s} k(u)\,du, & 0 \leqslant s \leqslant t, \end{cases}$$

and observe that in this case Eq. (4.1) reduces to (4.19).

Since (4.20) holds, we can define a constant $\varepsilon>0$ such that

$$\int_\varepsilon^{\infty} k(s)\,ds > 0 \qquad \text{and} \qquad e\int_\varepsilon^{\infty} sk(s)\,ds > 1. \tag{4.21}$$

Set $T_0(t)=t/2$ and $T(t)=\max\{T_0(t), t-\varepsilon\}$, for any $t \geqslant 0$.

Then, taking into account (4.21), we have

$$\begin{aligned} m_0 &= \inf\left\{\frac{1}{\mu}\liminf_{t\to+\infty}\int_{T_0(t)}^{T(t)} \exp(\mu(t-s))\,ds\,q_0(t,s) : \mu>0\right\} \\ &= \inf\left\{\frac{1}{\mu}\liminf_{t\to+\infty}\int_{t/2}^{t-\varepsilon} \exp(\mu(t-s))\,k(t-s)\,ds\right\} \\ &\geqslant e\liminf_{t\to+\infty}\int_{t/2}^{t-\varepsilon} (t-s)\,k(t-s)\,ds \\ &= e\liminf_{t\to+\infty}\int_\varepsilon^{t/2} sk(s)\,ds > 1, \end{aligned}$$

where we used the well-known inequality $\exp(x) \geqslant ex$ $(x \geqslant 0)$. On the other hand

$$\liminf_{t \to +\infty} [q_0(t, T(t)) - q_0(t, T_0(t))] = \liminf_{t \to +\infty} \int_{\varepsilon}^{t/2} k(s)\, ds > 0,$$

that is, all conditions of Theorem 4.1 are satisfied and the proof of the corollary is complete.

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