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Generalized Bi-quasi-variational Inequalities

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Let E, F be Hausdorff topological vector spaces over the field Φ (which is either the real field or the complex field), let $\langle\ ,\ \rangle: F\times E\to\Phi$ be a bilinear functional, and let X be a non-empty subset of E. Given a multi-valued map $S: X\to 2^X$ and two multi-valued maps $M, T: X\to 2^F$, the *generalized bi-quasi-variational inequality* (GBQVI) problem is to find a point $\hat{y}\in X$ such that $\hat{y}\in S(\hat{y})$ and $\inf_{w\in T(\hat{y})}\operatorname{Re}\langle f-w, \hat{y}-x\rangle\leqslant 0$ for all $x\in S(\hat{y})$ and for all $f\in M(\hat{y})$. In this paper two general existence theorems on solutions of GBQVIs are obtained which simultaneously unify, sharpen, and extend existence theorems for multi-valued versions of Hartman–Stampacchia variational inequalities proved by Browder and by Shih and Tan, variational inequalities due to Browder, existence theorems for generalized quasi-variational inequalities achieved by Shih and Tan, theorems for monotone operators obtained by Debrunner and Flor, Fan, and Browder, and the Fan–Glicksberg fixed-point theorem. © 1989 Academic Press, Inc.

1. INTRODUCTION

Throughout this paper, Φ denotes either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. For a non-empty set Y, 2^Y will denote the family of all *non-empty* subsets of Y. Let E, F be vector spaces over Φ, $\langle\ ,\ \rangle: F\times E\to\Phi$ be a bilinear functional, and X be a non-empty subset of E. If $S: X\to 2^X$ and

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$M, T: X \to 2^F$, the *generalized bi-quasi-variational inequality* (GBQVI) problem for the triple (S, M, T) is to find $\hat{y} \in X$ satisfying the properties

(i) $\hat{y} \in S(\hat{y})$ and

(ii) $\inf_{w \in T(\hat{y})} \mathrm{Re}\langle f - w, \hat{y} - x \rangle \leqslant 0$ for all $x \in S(\hat{y})$ and for all $f \in M(\hat{y})$.

In case T is single-valued, a generalized bi-quasi-variational inequality problem will be called a *bi-quasi-variational inequality* problem.

We remark that the generalized bi-quasi variational inequality problem includes the following generally known variational-type inequality problems:

Suppose E is a topological vector space, $F = E'$, the vector space of all continuous linear functionals on E, and $\langle\ ,\ \rangle$ is the usual duality pairing between E' and E. Then:

(i) If $T \equiv 0$, a generalized bi-quasi-variational inequality problem for $(S, M, 0)$ becomes a generalized quasi-variational inequality problem which was studied by Chan and Pang [9] in the finite-dimensional case and by Shih and Tan [24] in the infinite-dimensional case.

(ii) If $T \equiv 0$ and M is single-valued, a generalized bi-quasi-variational inequality problem for $(S, M, 0)$ becomes a quasi-variational inequality problem which was introduced by Bensoussan and Lions in 1973 in connection with impulse control (cf. [4]; see also Aubin [1] and Baiocchi and Capelo [3]).

(iii) If $S(x) \equiv X$, $M \equiv 0$, and T is single-valued, a generalized bi-quasi-variational inequality problem becomes a variational inequality problem which was introduced by Stampacchia [25] (see also Kinderlehrer and Stampacchia [18] and Lions and Stampacchia [20]).

(iv) If $S(x) \equiv X$ and $M \equiv 0$, a generalized bi-quasi-variational inequality problem becomes a generalized variational inequality problem which was studied by Browder [7], Yen [27], and Shih and Tan [23], etc.

In this paper we prove two general existence theorems on solutions of the generalized bi-quasi-variational inequality problem, which contain the fixed point theorem of Fan [11] and Glicksberg [15] and which simultaneously unify, sharpen, and extend existence theorems of seemingly diverse variational-type inequalities including (a) the existence theorems of variational inequalities due to Hartman and Stampacchia [17] and Browder [6, 7], (b) the existence theorems of generalized variational inequalities due to Browder [7, Theorem 6] and Shih and Tan [23, Theorem 11], (c) the existence theorems of generalized quasi-variational inequalities due to Shih and Tan [24, Theorems 1–4], and (d) the theorem on monotone operators due to Browder [7, Theorem 8] which in turn sharpens results due to Debrunner and Flor [10] and Fan [13, Theorem 12].

2. Two Main Theorems

The following three types of multi-valued maps will be needed in the discussion of our results.

Let X, Y be topological spaces and let $g: X \to 2^Y$. Then g is said to be *upper semi-continuous* on X [2, p. 109] if for every $x_0 \in X$ and for each open set G in Y with $g(x_0) \subset G$ there exists an open neighborhood $N(x_0)$ of x_0 such that $g(x) \subset G$ for all $x \in N(x_0)$. g is said to be *lower semi-continuous* on X [2, p. 109] if for every $x_0 \in X$ and for each open set G in Y with $g(x_0) \cap G \neq \emptyset$ there exists an open neighborhood $N(x_0)$ of x_0 such that $g(x) \cap G \neq \emptyset$ for all $x \in N(x_0)$. g is said to be *continuous* on X if g is both upper semi-continuous and lower semi-continuous on X.

Let E be a topological vector space over Φ, F be a vector space over Φ and $\langle\ ,\ \rangle: F \times E \to \Phi$ be a bilinear functional. For each $x_0 \in E$, for each non-empty subset A of E, and for $\varepsilon > 0$, let

$$W(x_0; \varepsilon) = \{y \in F: |\langle y, x_0 \rangle| < \varepsilon\},$$

$$U(A; \varepsilon) := \{y \in F: \sup_{x \in A} |\langle y, x \rangle| < \varepsilon\}.$$

Let $\sigma\langle F, E\rangle$ be the topology on F generated by the family $\{W(x; \varepsilon): x \in E$ and $\varepsilon > 0\}$ as a subbase for the neighborhood system at 0 and let $\delta\langle F, E\rangle$ be the topology on F generated by the family $\{U(A; \varepsilon)$: A is a non-empty compact subset of E and $\varepsilon > 0\}$ as a base for the neighborhood system at 0. We note then that F, when equipped with the topology $\sigma\langle F, E\rangle$ or the topology $\delta\langle F, E\rangle$, becomes a locally convex topological vector space but not necessarily Hausdorff. Furthermore, for a net $\{y_\alpha\}_{\alpha \in \Gamma}$ in F and for $y \in F$, (i) $y_\alpha \to y$ in $\sigma\langle F, E\rangle$ if and only if $\langle y_\alpha, x\rangle \to \langle y, x\rangle$ for each $x \in E$ and (ii) $y_\alpha \to y$ in $\delta\langle F, E\rangle$ if and only if $\langle y_\alpha, x\rangle \to \langle y, x\rangle$ uniformly for $x \in A$ for each non-empty compact subset A of E. Now if X is a non-empty subset of E, then a map $T: X \to 2^F$ is called *monotone* (with respect to the bilinear functional $\langle\ ,\ \rangle$) if for any $x, y \in X$, $u \in T(x)$, $w \in T(y)$, $\mathrm{Re}\langle w - u, y - x\rangle \geqslant 0$. Note that when $F = E'$, the vector space of all continuous linear functionals on E, and $\langle\ ,\ \rangle$ is the usual pairing between E' and E, the monotonicity notion coincides with the usual definition (see, e.g., Browder [8, p. 79]). Note also that $T: X \to 2^F$ is monotone if and only if its graph $G(T) = \{(x, y): y \in T(x)\}$ is a monotone subset of $X \times F$; i.e., for all (x_1, y_1), $(x_2, y_2) \in G(T)$, $\mathrm{Re}\langle y_2 - y_1, x_2 - x_1\rangle \geqslant 0$.

We now state two main theorems.

THEOREM 1. *Let E be a locally convex Hausdorff topological vector space over Φ, X be a non-empty compact convex subset of E, and F be a topological vector space over Φ. Let $\langle\ ,\ \rangle: F\times E \to \Phi$ be a bilinear functional which is continuous on compact subsets of $F\times X$. Suppose that*

(a) *$S: X\to 2^X$ is an upper semi-continuous map such that each $S(x)$ is closed convex.*

(b) *$M: X\to 2^F$ is monotone (with respect to $\langle\ ,\ \rangle$).*

(c) *$T: X\to 2^F$ is an upper semi-continuous map such that each $T(x)$ is compact.*

(d) *The set*

$$\Sigma := \{y\in X\colon \sup_{x\in S(y)}\ \sup_{f\in M(x)}\ \inf_{w\in T(y)} \operatorname{Re}\langle f-w, y-x\rangle > 0\}$$

is open in X.

Then there exists a point $\hat{y}\in X$ such that

(i) *$\hat{y}\in S(\hat{y})$ and*

(ii) *$\inf_{w\in T(\hat{y})} \operatorname{Re}\langle f-w, \hat{y}-x\rangle \leqslant 0$ for all $x\in S(\hat{y})$ and for all $f\in M(x)$.*

If, in addition, M is lower semi-continuous along the line segments in X to the topology $\sigma\langle F, E\rangle$ on F, then

(iii) *$\inf_{w\in T(\hat{y})} \operatorname{Re}\langle f-w, \hat{y}-x\rangle \leqslant 0$ for all $x\in S(\hat{y})$ and for all $f\in M(\hat{y})$.*

Moreover, if $S(x)=X$ for all $x\in X$, E is not required to be locally convex and if $T\equiv 0$, the continuity assumption on $\langle\ ,\ \rangle$ can be weakened to: for each $f\in F$, $x\mapsto\langle f, x\rangle$ is continuous on X.

THEOREM 2. *Let E be a locally convex Hausdorff topological vector space over Φ, X be a non-empty compact convex subset of E, and F be a vector space over Φ. Let $\langle\ ,\ \rangle: F\times E\to\Phi$ be a bilinear functional such that for each $f\in F$, $x\mapsto\langle f, x\rangle$ is continuous on X. Equip F with the topology $\delta\langle F, E\rangle$. Suppose that*

(a) *$S: X\to 2^X$ is a continuous map such that each $S(x)$ is closed convex.*

(b) *$M: X\to 2^F$ is monotone (with respect to $\langle\ ,\ \rangle$) and lower semi-continuous.*

(c) *$T: X \to 2^F$ is an upper semi-continuous map such that each $T(x)$ is compact.*

Then there exists a point $\hat{y} \in X$ such that

(i) *$\hat{y} \in S(\hat{y})$ and*

(ii) $\inf_{w \in T(\hat{y})} \operatorname{Re}\langle f - w, \hat{y} - x\rangle \leqslant 0$ *for all* $x \in S(\hat{y})$ *and for all* $f \in M(\hat{y})$.

3. Four Lemmas

The following lemmas will be needed in our proofs.

Lemma 1. *Let X be a non-empty subset of a Hausdorff topological vector space E and let $S: X \to 2^E$ be upper semi-continuous such that each $S(x)$ is bounded. Then for each continuous linear functional p on E, the map $f_p: X \to \mathbb{R}$ defined by $f_p(y) := \sup_{x \in S(y)} \operatorname{Re}\langle p, x\rangle$ is upper semi-continuous; i.e., for each $\lambda \in \mathbb{R}$, the set $\{y \in X: f_p(y) = \sup_{x \in S(y)} \operatorname{Re}\langle p, x\rangle < \lambda\}$ is open in X.*

Proof. We refer to Lemma 1 in Shih and Tan [24]. ∎

Lemma 2. *Let E, F be two topological vector spaces over Φ, and let $\langle\ ,\ \rangle: F \times E \to \Phi$ be a bilinear functional. Let X be a non-empty compact set in E and let $T: X \to 2^F$ be an upper semi-continuous map such that each $T(x)$ is compact. Let $\psi: X \times X \to \mathbb{R}$ be defined by*

$$\psi(x, y) := \inf_{w \in T(y)} \operatorname{Re}\langle w, y - x\rangle.$$

If $\langle\ ,\ \rangle$ is continuous on the (compact) subset $(\bigcup_{x \in X} T(x)) \times X$ of $F \times E$, then for each fixed $x \in X$, the map $y \mapsto \psi(x, y)$ is lower semi-continuous on X; i.e., for each $\lambda \in \mathbb{R}$, the set $\{y \in X: \psi(x, y) \leqslant \lambda\}$ is closed in X.

Proof. Let $x \in X$ be fixed and $\lambda \in \mathbb{R}$ be given. Let $A_\lambda := \{y \in X: \psi(x, y) \leqslant \lambda\}$. Suppose $\{y_\alpha\}_{\alpha \in \Gamma}$ is a net in A_λ and $y_\alpha \to y_0 \in X$. Since for each $\alpha \in \Gamma$, $w \mapsto \operatorname{Re}\langle w, y_\alpha - x\rangle$ is continuous on $T(y_\alpha)$ and $T(y_\alpha)$ is compact, for each $\alpha \in \Gamma$, we can choose $w_\alpha \in T(y_\alpha)$ such that

$$\operatorname{Re}\langle w_\alpha, y_\alpha - x\rangle = \inf_{w \in T(y_\alpha)} \operatorname{Re}\langle w, y_\alpha - x\rangle = \psi(x, y_\alpha) \leqslant \lambda.$$

Since T is upper semi-continuous and each $T(y)$ is compact, $\bigcup_{y \in X} T(y)$ is compact; thus $\{w_\alpha\}_{\alpha \in \Gamma}$ has a subnet $\{w_{\alpha'}\}_{\alpha' \in \Gamma'}$ which converges to some

$w_0 \in \bigcup_{y \in X} T(y)$. As T is upper semi-continuous, $w_0 \in T(y_0)$. Because $\langle\ ,\ \rangle$ is continuous on the compact set $(\bigcup_{y \in X} T(y)) \times X$, we have

$$\begin{aligned}\lambda &\geqslant \lim_{\alpha'} \operatorname{Re}\langle w_{\alpha'}, y_{\alpha'} - x\rangle \\ &= \operatorname{Re}\langle w_0, y_0 - x\rangle \\ &\geqslant \inf_{w \in T(y_0)} \operatorname{Re}\langle w, y_0 - x\rangle = \psi(x, y_0).\end{aligned}$$

Thus $y_0 \in A_\lambda$ and hence A_λ is closed in X. Therefore $y \mapsto \psi(x, y)$ is lower semi-continuous on X. ∎

The statement of Lemma 3 below may be found in Takahashi [26, Lemma 3]; however, we shall provide a complete proof (of which only Case 1 was proved by Takahashi in [26]).

LEMMA 3. *Let X, Y be topological spaces, let $f: X \to \mathbb{R}$ be non-negative and continuous, and let $g: Y \to \mathbb{R}$ be lower semi-continuous. Then the map $F: X \times Y \to \mathbb{R}$ defined by $F(x, y) := f(x)\, g(y)$ for all $(x, y) \in X \times Y$ is lower semi-continuous.*

Proof. Let $\alpha \in \mathbb{R}$. We need to show only that the set

$$H := \{(x, y) \in X \times Y: F(x, y) > \alpha\}$$

is open in $X \times Y$. Let $(x_0, y_0) \in H$; then $F(x_0, y_0) = f(x_0)\, g(y_0) > \alpha$.

Case 1. Suppose $f(x_0) \neq 0$ so that $f(x_0) > 0$. Since $f(x_0)\, g(y_0) > \alpha$, $g(y_0) > \alpha/f(x_0)$. Choose $\beta \neq 0$ such that $g(y_0) > \beta > \alpha/f(x_0)$. Let

$$N_{x_0} := \{x \in X: f(x)\, \beta > \alpha\} = \begin{cases} \{x \in X: f(x) > \alpha/\beta\}, & \text{if } \beta > 0, \\ \{x \in X: f(x) < \alpha/\beta\}, & \text{if } \beta < 0, \end{cases}$$

which is an open neighborhood of x_0 in X since f is continuous. Let

$$N_{y_0} := \{y \in Y: g(y) > \beta\},$$

which an open neighborhood of y_0 as g is lower semi-continuous. Thus for all $(x, y) \in N_{x_0} \times N_{y_0}$, $f(x)\, \beta > \alpha$ and $g(y) > \beta$, it follows that

(1) if $f(x) = 0$, then $f(x)\, g(y) = 0 > \alpha$.

(2) if $f(x) \neq 0$, then $f(x)\, g(y) > f(x)\, \beta > \alpha$.

Case 2. Suppose $f(x_0) = 0$ and $g(y_0) \geqslant 0$. But then $\alpha < 0$. Let $\varepsilon := \sqrt{-\alpha/2}$. As f is continuous at x_0 and $f \geqslant 0$, there exists an open neighborhood N_{x_0} of x_0 such that $0 \leqslant f(x) < \varepsilon$ for all $x \in N_{x_0}$. As g is lower semi-continuous at y_0, there exists an open neighborhood N_{y_0} of y_0 such that $g(y) > g(y_0) - \varepsilon$ for all $y \in N_{y_0}$. It follows that for all $(x, y) \in N_{x_0} \times N_{y_0}$, $0 \leqslant f(x) < \varepsilon$ and $g(y) - g(y_0) + \varepsilon > 0$; thus $f(x)[g(y) - g(y_0) + \varepsilon] \geqslant 0$ and hence

$$\begin{aligned} f(x)\, g(y) &\geqslant f(x)\, g(y_0) - \varepsilon f(x) \\ &\geqslant -\varepsilon f(x) \geqslant -\varepsilon^2 \\ &= \alpha/4 > \alpha. \end{aligned}$$

Case 3. Suppose $f(x_0) = 0$ and $g(y_0) < 0$. Set

$$\varepsilon := \alpha/(3g(y_0)), \qquad \varepsilon' := -1/(2g(y_0)).$$

Then ε, $\varepsilon' > 0$. By continuity of f at x_0, there exists an open neighborhood N_{x_0} of x_0 such that $0 \leqslant f(x) < \varepsilon$ for all $x \in N_{x_0}$. Also, by lower semi-continuity of g at y_0, there exists an open neighborhood N_{y_0} of y_0 such that $g(y) > g(y_0) - \varepsilon'$ for all $y \in N_{y_0}$. Thus, for all $(x, y) \in N_{x_0} \times N_{y_0}$, $0 \leqslant f(x) < \varepsilon$ and $g(y) > g(y_0) - \varepsilon' = 3/(2g(y_0))$, so that

$$\begin{aligned} f(x)\, g(y) &\geqslant 3f(x)\, g(y_0)/2 \\ &> 3\varepsilon g(y_0)/2 = \alpha/2 > \alpha. \end{aligned}$$

In any case, H is open in $X \times Y$. This completes the proof. ∎

LEMMA 4. *Let E be a topological vector space over Φ, F be a vector space over Φ, and let $\langle\ ,\ \rangle: F \times E \to \Phi$ be a bilinear functional. Let X be a non-empty subset of E and let $M: X \to 2^F$ be lower semi-continous along line segments in X to the topology $\sigma\langle F, E\rangle$ on F. Suppose $S: X \to 2^X$ and $T: X \to 2^F$. If there exists $\hat{y} \in X$ such that*

(i) *$\hat{y} \in S(\hat{y})$, $S(\hat{y})$ is convex and*

(ii) *$\inf_{w \in T(\hat{y})} \operatorname{Re}\langle f - w, \hat{y} - x\rangle \leqslant 0$ for all $x \in S(\hat{y})$ and for all $f \in M(x)$,*

then

$$\inf_{w \in T(\hat{y})} \operatorname{Re}\langle f - w, \hat{y} - x\rangle \leqslant 0 \qquad \textit{for all} \quad x \in S(\hat{y}) \textit{ and for all } f \in M(\hat{y}).$$

Proof. Fix $x \in S(\hat{y})$. For each $t \in [0, 1]$, let $x_t := tx + (1 - t)\,\hat{y}$; then by (i), $x_t \in S(\hat{y})$. By (ii),

$$\inf_{w \in T(\hat{y})} \operatorname{Re}\langle f - w, \hat{y} - x_t\rangle \leqslant 0 \qquad \text{for all } t \in [0, 1] \text{ and for all } f \in M(x_t),$$

so that

$$t \cdot \inf_{w \in T(\hat{y})} \operatorname{Re}\langle f - w, \hat{y} - x \rangle \leqslant 0 \qquad \text{for all} \quad t \in [0, 1] \text{ and for all } f \in M(x_t).$$

Hence

(*) $\inf_{w \in T(\hat{y})} \operatorname{Re}\langle f - w, \hat{y} - x \rangle \leqslant 0$ for all $t \in (0, 1]$ and for all $f \in M(x_t)$.

Fix $f_0 \in M(\hat{y})$. Let $\varepsilon > 0$ be given and let

$$U_{f_0} := \{ f \in F: |\langle f - f_0, \hat{y} - x \rangle| < \varepsilon \};$$

then U_{f_0} is a $\sigma\langle F, E \rangle$ neighborhood of f_0. As $U_{f_0} \cap M(\hat{y}) \neq \varnothing$ and M is lower semi-continuous on $L := \{x_t: t \in [0, 1]\}$, there exists an open neighborhood N of $\hat{y}$ such that $U_{f_0} \cap M(y) \neq \varnothing$ for all $y \in N$. But then there exists $\delta \in (0, 1)$ such that $x_t \in N$ for all $t \in (0, \delta)$. Choosing any $t \in (0, \delta)$ and $f_t \in U_{f_0} \cap M(x_t)$, we have $|\langle f_0 - f_t, \hat{y} - x \rangle| < \varepsilon$. Thus

$$\begin{aligned} &\inf_{w \in T(\hat{y})} \operatorname{Re}\langle f_0 - w, \hat{y} - x \rangle \\ &\quad = \operatorname{Re}\langle f_0 - f_t, \hat{y} - x \rangle + \inf_{w \in T(\hat{y})} \operatorname{Re}\langle f_t - w, \hat{y} - x \rangle \\ &\quad \leqslant \varepsilon + \inf_{w \in T(\hat{y})} \operatorname{Re}\langle f_t - w, \hat{y} - x \rangle \leqslant \varepsilon \qquad \text{(by (*))}. \end{aligned}$$

As $\varepsilon > 0$ is arbitrary, $\inf_{w \in T(y)} \operatorname{Re}\langle f_0 - w, \hat{y} - x \rangle \leqslant 0$. Since $x \in S(\hat{y})$ and $f_0 \in M(\hat{y})$ are arbitrary,

$$\inf_{w \in T(\hat{y})} \operatorname{Re}\langle f - w, \hat{y} - x \rangle \leqslant 0 \qquad \text{for all} \quad x \in S(\hat{y}) \text{ and for all } f \in M(\hat{y}). \quad \blacksquare$$

When $F = E'$ and $\langle \ , \ \rangle$ is the usual pairing between E' and E, $S(x) \equiv X$, $T \equiv 0$, and M is single-valued, Lemma 4 reduces to that of Minty [21].

4. Proof of Theorem 1

The proof of Theorem 1 may either use Fan's infinite-dimensional generalization [12, Lemma 1] of the classical Knaster–Kuratowski–Mazurkiewicz theorem or the following Yen's generalization [27] of Fan's minimax inequality [14]:

THEOREM A. *Let X be a non-empty compact convex subset of a Hausdorff topological vector space and let φ, ψ be two real-valued functions on $X \times X$. Suppose that*

(a) *$\varphi(x, y) \leqslant \psi(x, y)$ for all $(x, y) \in X \times X$ and $\psi(x, x) \leqslant 0$ for all $x \in X$.*

(b) *For each fixed $x \in X$, $\varphi(x, y)$ is a lower semi-continuous function of y on X.*

(c) *For each fixed $y \in X$, $\psi(x, y)$ is a quasi-concave function of x on X.*

Then there exists $\hat{y} \in X$ such that $\varphi(x, \hat{y}) \leqslant 0$ for all $x \in X$.

Here, a real-valued function ψ on a convex set X is said to be *quasi-concave* if for each $\lambda \in \mathbb{R}$, the set $\{x \in X: \psi(x) > \lambda\}$ is convex.

We now proceed to prove Theorem 1. Suppose that the assertions of Theorem 1 were false. Then for each $y \in X$, either $y \notin S(y)$ or there exist $x \in S(y)$ and $f \in M(x)$ such that $\inf_{w \in T(y)} \operatorname{Re}\langle f - w, y - x\rangle > 0$; that is, either $y \notin S(y)$ or $y \in \Sigma$. If $y \notin S(y)$, then by a separation theorem for convex sets in locally convex Hausdorff topological vector spaces, there exists $p \in E'$ such that

$$\operatorname{Re}\langle p, y\rangle - \sup_{x \in S(y)} \operatorname{Re}\langle p, x\rangle > 0.$$

For each $p \in E'$, let

$$V(p) := \{y \in X: \operatorname{Re}\langle p, y\rangle - \sup_{x \in S(y)} \operatorname{Re}\langle p, x\rangle > 0\}.$$

Then $X = \Sigma \cup \bigcup_{p \in E'} V(p)$. As each $V(p)$ is open in X by Lemma 1 and Σ is open in X by hypothesis, by compactness of X, there exist $p_1, p_2, \ldots, p_n \in E'$ such that $X = \Sigma \cup \bigcup_{i=1}^{n} V(p_i)$. For simplicitly of notations, let $V_0 := \Sigma$ and $V_i = V(p_i)$ for $i = 1, 2, \ldots, n$. Let $\{\beta_0, \beta_1, \ldots, \beta_n\}$ be a continuous partition of unity on X subordinate to the covering $\{V_0, V_1, \ldots, V_n\}$. Then $\beta_0, \beta_1, \ldots, \beta_n$ are continuous non-negative real-valued functions on X such that β_i vanishes on $X \backslash V_i$ for each $i = 0, 1, \ldots, n$ and $\sum_{i=0}^{n} \beta_i(x) = 1$ for all $x \in X$. Define $\varphi, \psi: X \times X \to \mathbb{R}$ by

$$\varphi(x, y) := \beta_0(y) \sup_{f \in M(x)} \inf_{w \in T(y)} \operatorname{Re}\langle f - w, y - x\rangle + \sum_{i=1}^{n} \beta_i(y) \operatorname{Re}\langle p_i, y - x\rangle,$$

$$\psi(x, y) := \beta_0(y) \inf_{g \in M(y)} \inf_{w \in T(y)} \operatorname{Re}\langle g - w, y - x\rangle + \sum_{i=1}^{n} \beta_i(y) \operatorname{Re}\langle p_i, y - x\rangle.$$

Then we have:

(1) $\psi(x, x) = 0$ for all $x \in X$.

(2) Since M is monotone, for any $x, y \in X$, we have $\operatorname{Re}\langle f - g, x - y\rangle \geqslant 0$ for all $f \in M(x)$ and for all $g \in M(y)$. It follows that

$\text{Re}\langle f-w, y-x\rangle \leqslant \text{Re}\langle g-w, y-x\rangle$ for all $f\in M(x)$, for all $g\in M(y)$, and for all $w\in T(y)$, and hence

$$\sup_{f\in M(x)} \inf_{w\in T(y)} \text{Re}\langle f-w, y-x\rangle \leqslant \inf_{g\in M(y)} \inf_{w\in T(y)} \text{Re}\langle g-w, y-x\rangle.$$

Therefore $\varphi(x, y)\leqslant \psi(x, y)$ for all $x, y\in X$.

(3) For each fixed $x\in X$ and for each fixed $f\in M(x)$, the map $T_f: X\to 2^F$ defined by

$$T_f(y) := f - T(y) \qquad \text{for each } y\in X$$

is an upper semi-continuous map such that each $T_f(y)$ is a compact subset of F. Thus by Lemma 2, the map

$$y\mapsto \inf_{w\in T(y)} \text{Re}\langle f-w, y-x\rangle = \inf_{w\in T_f(y)} \text{Re}\langle w, y-x\rangle$$

is lower semi-continuous on X. By Lemma 3, the map

$$y\mapsto \beta_0(y) \sup_{f\in M(x)} \inf_{w\in T(y)} \text{Re}\langle f-w, y-x\rangle$$

is lower semi-continuous on X. Therefore for each fixed $x\in X$, the map $y\mapsto \varphi(x, y)$ is lower semi-continuous on X.

(4) Clearly, for each fixed $y\in X$, the map $x\mapsto \varphi(x, y)$ is quasi-concave.

Thus by Theorem A, there exists $\hat{y}\in X$ such that $\varphi(x, \hat{y})\leqslant 0$ for all $x\in X$, i.e.,

(**) $\beta_0(\hat{y})\ \sup_{f\in M(x)} \inf_{w\in T(\hat{y})} \text{Re}\langle f-w, \hat{y}-x\rangle + \sum_{i=1}^n \beta_i(\hat{y})\ \text{Re}\langle p_i, \hat{y}-x\rangle \leqslant 0$ for all $x\in X$.

Choose $\hat{x}\in S(\hat{y})$ such that

$$\sup_{f\in M(\hat{x})} \inf_{w\in T(\hat{y})} \text{Re}\langle f-w, \hat{y}-\hat{x}\rangle > 0 \qquad \text{whenever } \beta_0(\hat{y})>0;$$

it follows that

$$\beta_0(\hat{y}) \sup_{f\in M(\hat{x})} \inf_{w\in T(\hat{y})} \text{Re}\langle f-w, \hat{y}-\hat{x}\rangle > 0 \qquad \text{whenever } \beta_0(\hat{y})>0.$$

If $i\in\{1, ..., n\}$ is such that $\beta_i(\hat{y})>0$, then $\hat{y}\in V(p_i)$ and hence

$$\text{Re}\langle p_i, \hat{y}\rangle > \sup_{x\in S(\hat{y})} \text{Re}\langle p_i, x\rangle \geqslant \text{Re}\langle p_i, \hat{x}\rangle$$

so that

$$\operatorname{Re}\langle p_i, \hat{y}-\hat{x}\rangle > 0.$$

Note then that

$$\beta_i(\hat{y}) \operatorname{Re}\langle p_i, \hat{y}-\hat{x}\rangle > 0 \qquad \text{whenever } \beta_i(\hat{y}) > 0 \qquad \text{for} \quad i=1, ..., n.$$

Since $\beta_i(\hat{y}) > 0$ for at least one $i \in \{0, 1, ..., n\}$, it follows that

$$\beta_0(\hat{y}) \sup_{f \in M(\hat{x})} \inf_{w \in T(\hat{y})} \operatorname{Re}\langle f-w, \hat{y}-\hat{x}\rangle + \sum_{i=1}^{n} \beta_i(\hat{y}) \operatorname{Re}\langle p_i, \hat{y}-\hat{x}\rangle > 0,$$

which contradicts (**). This proves parts (i) and (ii) of Theorem 1. Now part (iii) of Theorem 1 follows from parts (i) and (ii) and Lemma 4.

Next we note from the above proof that E is required to be locally convex when and only when the separation theorem is applied to the case $y \notin S(y)$. Thus if $S\colon X \to 2^X$ is the constant map $S(x) = X$ for all $x \in X$, E is not required to be locally convex.

Finally, if $T \equiv 0$, in order to show that for each $x \in X$, $y \mapsto \varphi(x, y)$ is lower semi-continuous, Lemma 2 is no longer needed and the weaker continuity assumption on $\langle\ ,\ \rangle$ that for each $f \in F$, $x \mapsto \langle f, x\rangle$ is continuous on X is sufficient. This completes the proof. ∎

5. Proof of Theorem 2

As $\langle\ ,\ \rangle\colon F \times E \to \Phi$ is a bilinear functional such that for each $f \in F$, $x \mapsto \langle f, x\rangle$ is continuous on X and as F is equipped with the topology $\delta\langle F, E\rangle$, it is easy to see that $\langle\ ,\ \rangle$ is continuous on compact subsets of $F \times X$. Thus by Theorem 1, it suffices to show that the set

$$\Sigma := \{y \in X\colon \sup_{x \in S(y)} \sup_{f \in M(x)} \inf_{w \in T(y)} \operatorname{Re}\langle f-w, y-x\rangle > 0\}$$

is open in X. Indeed, let $y_0 \in \sum$; then there exist $x_0 \in S(y_0)$ and $f_0 \in M(x_0)$ such that

$$\alpha := \inf_{w \in T(y_0)} \operatorname{Re}\langle f_0 - w, y_0 - x_0\rangle > 0.$$

Let

$$W := \{w \in F\colon \sup_{z_1, z_2 \in X} |\langle w, z_1 - z_2\rangle| < \alpha/6\}.$$

Then W is an open neighborhood of 0 in F so that

$$U_0 := T(y_0) + W$$

is an open neighborhood of $T(y_0)$. Since T is upper semi-continuous at y_0, there exists an open neighborhood N_1 of y_0 such that if $y \in N_1$ then $T(y) \subset U_0$. Let

$$Z := \{f \in F: \sup_{z_1, z_2 \in X} |\langle f - f_0, z_1 - z_2 \rangle| < \alpha/6\}.$$

Then Z is an open neighborhood of f_0. As M is lower semi-continuous at x_0 and $Z \cap M(x_0) \neq \emptyset$, there exists an open neighborhood V_1 of x_0 in X such that $M(x) \cap Z \neq \emptyset$ for all $x \in V_1$. As the map

$$x \mapsto \inf_{w \in T(y_0)} \operatorname{Re}\langle f_0 - w, x_0 - x \rangle$$

is continuous at x_0, there exists an open neighborhood V_2 of x_0 in X such that

$$\left| \inf_{w \in T(y_0)} \operatorname{Re}\langle f_0 - w, x_0 - x \rangle \right| < \alpha/6 \qquad \text{for all} \quad x \in V_2.$$

Let $V_0 := V_1 \cap V_2$; then V_0 is an open neighborhood of x_0 in X. Since $V_0 \cap S(y_0) \neq \emptyset$ and S is lower semi-continuous at y_0, there exists an open neighborhood N_2 of y_0 such that $S(y) \cap V_0 \neq \emptyset$ for all $y \in N_2$. Since the map

$$y \mapsto \inf_{w \in T(y_0)} \operatorname{Re}\langle f_0 - w, y - y_0 \rangle$$

in continuous at y_0, there exists an open neighborhood N_3 of y_0 such that

$$\left| \inf_{w \in T(y_0)} \operatorname{Re}\langle f_0 - w, y - y_0 \rangle \right| < \alpha/6 \qquad \text{for all} \quad y \in N_3.$$

Let $N_0 := N_1 \cap N_2 \cap N_3$. Then N_0 is an open neighborhood of y_0 such that for each $y_1 \in N_0$, we have

(i) $T(y_1) \subset U_0 = T(y_0) + W$ as $y_1 \in N_1$;

(ii) $S(y_1) \cap V_0 \neq \emptyset$ as $y_1 \in N_2$; choose any $x_1 \in S(y_1) \cap V_0$;

(iii) $|\inf_{w \in T(y_0)} \operatorname{Re}\langle f_0 - w, y_1 - y_0 \rangle| < \alpha/6$ as $y_1 \in N_3$;

(iv) $M(x_1) \cap Z \neq \emptyset$ as $x_1 \in V_1$; choose any $f_1 \in M(x_1) \cap Z$,

so that

$$\sup_{z_1, z_2 \in X} |\langle f_1 - f_0, z_1 - z_2 \rangle| < \alpha/6;$$

(v) $|\inf_{w \in T(y_0)} \text{Re}\langle f_0 - w, x_0 - x_1 \rangle| < \alpha/6$ as $x_1 \in V_2$;

it follows that

$$\begin{aligned} &\inf_{w \in T(y_1)} \text{Re}\langle f_1 - w, y_1 - x_1 \rangle \\ &\quad \geqslant \text{Re}\langle f_1 - f_0, y_1 - x_1 \rangle + \inf_{w \in T(y_1)} \text{Re}\langle f_0 - w, y_1 - x_1 \rangle \\ &\quad \geqslant -\alpha/6 + \inf_{w \in T(y_0) + W} \text{Re}\langle f_0 - w, y_1 - x_1 \rangle \qquad \text{(by (i) and (iv))} \\ &\quad \geqslant -\alpha/6 + \inf_{w \in T(y_0)} \text{Re}\langle f_0 - w, y_1 - x_1 \rangle + \inf_{w \in W} \text{Re}\langle w, y_1 - x_1 \rangle \\ &\quad \geqslant -\alpha/6 + \inf_{w \in T(y_0)} \text{Re}\langle f_0 - w, y_1 - y_0 \rangle + \inf_{w \in T(y_0)} \text{Re}\langle f_0 - w, y_0 - x_0 \rangle \\ &\quad + \inf_{w \in T(y_0)} \text{Re}\langle f_0 - w, x_0 - x_1 \rangle - \alpha/6 \\ &\quad \geqslant -\alpha/6 - \alpha/6 + \alpha - \alpha/6 - \alpha/6 \qquad \text{(by (iii) and (v))} \\ &\quad = \alpha/3 > 0; \end{aligned}$$

therefore

$$\sup_{x \in S(y_1)} \sup_{f \in M(x)} \inf_{w \in T(y_1)} \text{Re}\langle f - w, y_1 - x \rangle > 0$$

as $x_1 \in S(y_1)$ and $f_1 \in M(x_1)$. This shows that $y_1 \in \Sigma$ for all $y_1 \in N_0$, so that Σ is open in X. This proves the theorem. ▌

6. Consequences of Theorem 1

In this section we shall give a variety of applications of Theorem 1.

THEOREM 3 (Fan [11] and Glicksberg [15]). *Let E be a locally convex Hausdorff topological vector space and X be a non-empty compact convex subset of E. Let $S\colon X \to 2^X$ be upper semi-continuous such that each $S(x)$ is closed convex. Then there exists a point $\hat{x} \in X$ such that $\hat{x} \in S(\hat{x})$.*

Proof. Apply Theorem 1 where $F = E'$, the vector space of all continuous linear functionals on E, $\langle\ ,\ \rangle$ is the usual pairing between E'

and E (or simply, $\langle\ ,\ \rangle \equiv 0$), F is equipped with the topology, $\delta\langle F, E\rangle$, and $M \equiv 0$ and $T \equiv 0$; the assertion follows. ∎

THEOREM 4. *Let E be a Hausdorff topological vector space (not necessarily locally convex) over Φ, X be a non-empty compact convex subset of E, and F be a topological vector space over Φ. Let $\langle\ ,\ \rangle: F \times E \to \Phi$ be a bilinear functional which is continuous on compact subsets of $F \times X$. Suppose that*

(a) *$M: X \to 2^F$ is monotone (with respect to $\langle\ ,\ \rangle$).*

(b) *$T: X \to 2^F$ is an upper semi-continuous map such that each $T(x)$ is compact.*

Then there exists a point $\hat{y} \in X$ such that

(i) $\inf_{w \in T(\hat{y})} \operatorname{Re}\langle f - w, \hat{y} - x\rangle \leqslant 0$ *for all $x \in X$ and for all $f \in M(x)$.*

If, in addition, M is lower semi-continuous along the line segments in X to the topology $\sigma\langle F, E\rangle$ on F, then

(ii) $\inf_{w \in T(\hat{y})} \operatorname{Re}\langle f - w, \hat{y} - x\rangle \leqslant 0$ *for all $x \in X$ and for all $f \in M(\hat{y})$.*

Proof. Let

$$\Sigma' := \{y \in X: \sup_{x \in X} \sup_{f \in M(x)} \inf_{w \in T(y)} \operatorname{Re}\langle f - w, y - x\rangle > 0\}.$$

If $\Sigma' = X$, then by applying Theorem 1 with $S(x) = X$ for all $x \in X$, we obtain a contradiction. Thus $\Sigma' \neq X$ which proves (i). Now (ii) follows from (i) and Lemma 4. ∎

Note that even when T is single-valued, part (i) of Theorem 4 sharpens Theorem 8 in Browder [7], (where $\langle\ ,\ \rangle$ is assumed to be continuous on $F \times E$) which in turn sharpens Theorem 12 in Fan [13] (where $\langle\ ,\ \rangle$ is assumed to be continuous on $F \times E$ and F is required to be locally convex and quasi-complete) and the theorem in Debrunner and Flor [10] (where $\langle\ ,\ \rangle$ is assumed to be continuous on $F \times E$ and E is required to be locally convex). We remark here that Debrunner and Flor's theorem is one of the fundamental theorems in the theory of monotone operators. Note also that Debrunner and Flor's theorem extends earlier results of Minty [22] and Grünbaum [16] (see also [5]).

It is of interest to compare Theorem 4 with the following:

THEOREM 5. *Let E, F be Hausdorff topological vector spaces over Φ, of which F is locally convex. Let $\langle\ ,\ \rangle: F \times E \to \mathbb{R}$ be a bilinear functional which is continuous on compact subsets of $F \times E$. Let X and Y be non-empty compact convex subsets of E and F, respectively. Let $M: X \to 2^Y$ be* *monotone (with respect to* $\langle\ ,\ \rangle$*) and* $T: X \to 2^Y$ *be upper semi-continuous. Then*

(i) *there exist* $\hat{y} \in X$ *and* $\hat{w} \in T(\hat{y})$ *such that* $\langle f - \hat{w}, \hat{y} - x \rangle \leqslant 0$ *for all* $x \in X$ *and for all* $f \in M(x)$.

(ii) *If, in addition, M is lower semi-continuous along the line segments in X to the topology* $\sigma\langle F, E \rangle$ *on F, then* $\langle f - \hat{w}, \hat{y} - x \rangle \leqslant 0$ *for all* $x \in X$ *and for all* $f \in M(\hat{y})$.

Proof. Part (i) is a theorem of Browder [7, Theorem 10]. Part (ii) follows from part (i) and Lemma 4. ▮

THEOREM 6. *Let E be a locally convex Hausdorff topological vector space over* Φ, *X be a non-empty compact convex subset of E, and F be a topological vector space over* Φ. *Let* $\langle\ ,\ \rangle: F \times X \to \Phi$ *be a bilinear functional such that for each* $f \in F$, $x \mapsto \langle f, x \rangle$ *is continuous on X. Suppose that*

(a) $S: X \to 2^X$ *is an upper semi-continuous map such that each* $S(x)$ *is closed convex;*

(b) $M: X \to 2^F$ *is monotone (with respect to* $\langle\ ,\ \rangle$*);*

(c) *the set*

$$\Sigma := \{ y \in X: \sup_{x \in S(y)} \sup_{f \in M(x)} \operatorname{Re}\langle f, y - x \rangle > 0 \}$$

is open in X.

Then there exists a point $\hat{y} \in X$ *such that*

(i) $\hat{y} \in S(\hat{y})$ *and*

(ii) $\sup_{f \in M(x)} \operatorname{Re}\langle f, \hat{y} - x \rangle \leqslant 0$ *for all* $x \in S(\hat{y})$.

If, in addition, M is lower semi-continuous along the line segments in X to the topology $\sigma\langle F, E \rangle$ *on F, then*

(iii) $\sup_{f \in M(\hat{y})} \operatorname{Re}\langle f, \hat{y} - x \rangle \leqslant 0$ *for all* $x \in S(\hat{y})$.

If $S(x) = X$ *for all* $x \in X$, *E is not required to be locally convex.*

Proof. Apply Theorem 1 with $T \equiv 0$; the conclusion follows. ▮

When $F = E'$ and $\langle\ ,\ \rangle$ is the usual pairing between E' and E, Theorem 6 was first proved by Shih and Tan [24, Theorem 1].

THEOREM 7. *Let E be a Hausdorff topological vector space over* Φ, *X be a non-empty compact convex subset of E, and F be a topological vector space over* Φ. *Let* $\langle\ ,\ \rangle: F \times E \to \Phi$ *be a bilinear functional such that for each* $f \in F$, $x \mapsto \langle f, x \rangle$ *is continuous on X. If* $M: X \to 2^F$ *is monotone (with respect* *to* $\langle\ ,\ \rangle$) *and is lower semi-continuous along the line segments in* X *to the topology* $\sigma\langle F, E\rangle$ *on* F, *then there exists a point* $\hat{y}\in X$ *such that*

$$\sup_{f\in M(\hat{y})} \operatorname{Re}\langle f, \hat{y}-x\rangle \leqslant 0 \qquad \textit{for all } x\in X.$$

Proof. Let

$$\Sigma' := \{y\in X: \sup_{x\in X}\ \sup_{f\in M(x)} \operatorname{Re}\langle f, y-x\rangle > 0\}.$$

If $\Sigma' = X$, then by applying Theorem 6 with $S(x)\equiv X$, we obtain a contradiction. Thus $\Sigma' \neq X$ and hence there exists $y\in X$ such that

$$\sup_{f\in M(x)} \operatorname{Re}\langle f, \hat{y}-x\rangle \leqslant 0 \qquad \text{for all} \quad x\in X.$$

By Lemma 4 with $T\equiv 0$ and $S(x)\equiv X$, we conclude that

$$\sup_{f\in M(\hat{y})} \operatorname{Re}\langle f, \hat{y}-x\rangle \leqslant 0 \qquad \text{for all} \quad x\in X. \quad \blacksquare$$

We remark that Theorem 7 is an improvement of a multi-valued version of the Hartman–Stampacchia variational inequality [17] given in Shih and Tan [23, Theorem 11].

THEOREM 8. *Let* E *be a locally convex Hausdorff topological vector space over* Φ, X *be a non-empty compact convex subset of* E, *and* F *be a topological vector space over* Φ. *Let* $\langle\ ,\ \rangle: F\times E\to\Phi$ *be a bilinear functional which is continuous on compact subsets of* $F\times X$. *Suppose that*

(a) $S: X\to 2^X$ *is an upper semi-continuous map such that each* $S(x)$ *is closed convex.*

(b) $T: X\to 2^F$ *is an upper semi-continuous map such that each* $T(x)$ *is compact convex.*

(c) *The set*

$$\Sigma := \{y\in X: \sup_{x\in S(y)}\ \inf_{w\in T(y)} \operatorname{Re}\langle w, y-x\rangle > 0\}$$

is open in X.

Then there exists a point $\hat{y}\in X$ *such that*

(i) $\hat{y}\in S(y)$ *and*

(ii) *there exists a point* $\hat{z}\in T(\hat{y})$ *with* $\operatorname{Re}\langle\hat{z}, \hat{y}-x\rangle \leqslant 0$ *for all* $x\in S(\hat{y})$.

If $S(x) = X$ *for all* $x\in X$, E *is not required to be locally convex.*

Proof. Apply Theorem 1 with $M \equiv 0$ and T being replaced by $-T$; there exists $\hat{y} \in X$ such that

(*) $\hat{y} \in S(\hat{y})$ and

(**) $\inf_{w \in T(\hat{y})} \mathrm{Re}\langle w, \hat{y} - x\rangle \leqslant 0$ for all $x \in S(\hat{y})$.

Now define $g\colon S(\hat{y}) \times T(\hat{y}) \to \mathbb{R}$ by

$$g(x, z) := \mathrm{Re}\langle z, \hat{y} - x\rangle.$$

As $S(\hat{y})$ is convex, $T(\hat{y})$ is compact convex, and for each fixed $x \in S(\hat{y})$, $z \mapsto g(x, z)$ is continuous and affine and for each fixed $z \in T(\hat{y})$, $x \mapsto g(x, z)$ is affine, by Kneser's minimax theorem [19] we have

$$\min_{z \in T(\hat{y})} \max_{x \in S(\hat{y})} g(x, z) = \max_{x \in S(\hat{y})} \min_{z \in T(\hat{y})} g(x, z).$$

Thus

$$\min_{z \in T(\hat{y})} \max_{x \in S(\hat{y})} \mathrm{Re}\langle z, \hat{y} - x\rangle \leqslant 0 \qquad \text{by } (**).$$

Since $T(\hat{y})$ is compact, there exists $\hat{z} \in T(\hat{y})$ such that

$$\mathrm{Re}\langle \hat{z}, \hat{y} - x\rangle \leqslant 0 \qquad \text{for all} \quad x \in S(\hat{y}). \quad \blacksquare$$

Theorem 8 is an improvement of Theorem 3 in our paper [24].

THEOREM 9. *Let E be a Hausdorff topological vector space over Φ, X be a non-empty compact convex subset of E, and F be a topological vector space over Φ. Let $\langle\ ,\ \rangle\colon F \times E \to \Phi$ be a bilinear functional which is continuous on compact subsets of $F \times X$. Suppose that $T\colon X \to 2^F$ is an upper semi-continuous map such that each $T(x)$ is compact convex. Then there exist a point $\hat{y} \in X$ and a point $\hat{z} \in T(\hat{y})$ such that*

$$\mathrm{Re}\langle \hat{z}, \hat{y} - x\rangle \leqslant 0 \qquad \textit{for all} \quad x \in X.$$

Proof. Let

$$\Sigma' := \{y \in X\colon \sup_{x \in X} \inf_{w \in T(y)} \mathrm{Re}\langle w, y - x\rangle > 0\}.$$

If $\Sigma' = X$, then by applying Theorem 1 with $M \equiv 0$, T being replaced by $-T$, and with $S(x) \equiv X$, we obtain a contradiction. Thus $\Sigma' \neq X$ and hence there exists $\hat{y} \in X$ such that

$$\sup_{x \in X} \inf_{w \in T(\hat{y})} \mathrm{Re}\langle w, \hat{y} - x\rangle \leqslant 0.$$

By another application of Kneser's minimax theorem as seen in the proof of Theorem 8 (with $S(\hat{y}) = X$), the conclusion follows. ∎

Theorem 9 is an improvement of Theorem 6 in [7] (and hence also Theorem 2 in [7]). Thus Theorem 1 is a unified account of Browder's theorems (Theorems 2 and 6 in [7]) and Debrunner and Flor's theorem [10] which are seemingly unrelated results.

7. Consequences of Theorem 2

THEOREM 10. *Let E be a locally convex Hausdorff topological vector space over Φ, X be a non-empty compact convex subset of E, and F be a vector space over Φ. Let $\langle\ ,\ \rangle: F \times E \to \Phi$ be a bilinear functional such that for each $f \in F$, $x \mapsto \langle f, x \rangle$ is continuous on X. Equip F with the topology $\delta\langle F, E\rangle$. Suppose that*

(a) *$S: X \to 2^X$ is a continuous map such that each $S(x)$ is closed convex.*

(b) *$M: X \to 2^F$ is monotone (with respect to $\langle\ ,\ \rangle$) and lower semi-continuous.*

Then there exists $\hat{y} \in X$ such that

(i) *$\hat{y} \in S(\hat{y})$ and*

(ii) $\operatorname{Re}\langle f, \hat{y} - x \rangle \leqslant 0$ *for all $x \in S(\hat{y})$ and for all $f \in M(\hat{y})$.*

Proof. Applying Theorem 2 with $T \equiv 0$, the conclusion follows. ∎

THEOREM 11. *Let E be a locally convex Hausdorff topological vector space over Φ, X be a non-empty compact convex subset of E, and F be a vector space over Φ. Let $\langle\ ,\ \rangle: F \times E \to \Phi$ be a bilinear functional such that for each $f \in F$, $x \mapsto \langle f, x \rangle$ is continuous on X. Equip F with the topology $\delta\langle F, E\rangle$. Suppose that*

(a) *$S: X \to 2^X$ is a continuous map such that each $S(x)$ is closed convex.*

(b) *$T: X \to 2^F$ is an upper semi-continuous map such that each $T(x)$ is compact convex.*

Then there exists a point $\hat{y} \in X$ such that

(i) *$\hat{y} \in S(\hat{y})$ and*

(ii) *there exists a point $\hat{z} \in T(\hat{y})$ with* $\operatorname{Re}\langle \hat{z}, \hat{y} - x \rangle \leqslant 0$ *for all $x \in S(\hat{y})$.*

Proof. Applying Theorem 2 with $M \equiv 0$ and T being replaced by $-T$, there exists $\hat{y} \in X$ such that

(*) $\hat{y} \in S(\hat{y})$ *and*

(**) $\inf_{w \in T(\hat{y})} \operatorname{Re}\langle w, \hat{y} - x\rangle \leqslant 0$ for all $x \in S(\hat{y})$.

By another application of Kneser's minimax theorem as seen in the proof of Theorem 8, the conclusion follows. ∎

Theorem 10 improves Theorem 2 in our earlier paper [24] where $F = E'$ and $\langle\ ,\ \rangle$ is the usual pairing between E' and E; Theorem 11 generalizes Theorem 4 in our earlier paper [24] where E is a normed space, $F = E'$ is equipped with normed topology, and $\langle\ ,\ \rangle$ is the usual pairing between E' and E.

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