A Generalization of Lie's "Counting" Theorem for Second-Order Ordinary Differential Equations*⁺

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I. INTRODUCTION

The results presented in this paper are based on a generalization [1] of the Lie-Ovsjannikov theory [2]. Fundamental to this generalization is the recognition that the sets of algebraically independent covariance transformation labels employed in the Lie-Ovsjannikov theory are not in general maximal sets. In particular, in the case of partial differential equations, derivatives higher than those in the original system of partial differential equations may appear as covariance transformation labels, and, in practice, an infinite number of these derivatives may be necessary. Examples of the latter possibility have been demonstrated elsewhere [3]; in particular, it was necessary there to include an infinite number of labels in order to recover the known dynamical algebras of various quantum mechanical systems. In that work, since the emphasis was on covariance algebras, a mathematical formulation different from that which is the mathematical basis of the Lie-Ovsjannikov theory was employed. Because of this and in order to make the context of our results clearer, there is included in Section II a summary of the generalized Lie-Ovsjannikov covariance formulas for an arbitrary system (S) of partial differential equations which incorporate a maximal set of transformation labels involving higher derivatives. Further, by way of illustration, we have found a simple example of a covariance generator in this generalized Lie-Ovsjannikov theory which involves a derivative higher than any in the original partial differential equation.

The application of this generalization to an nth order ordinary differential equation yields that the maximal set of algebraically independent labels for a transformation is \( \{x, y, y', \ldots, y^{(n-1)}\} \). The corresponding formulas are also

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presented in Section II. The possibility of using higher derivatives in the transformation laws for ordinary differential equations is well known for second-order ordinary differential equations in the Hamiltonian Formulation of Classical Mechanics [4] and it appears in Ovsjannikov's [5] important generalization of Lie's theory. In the latter case, it is introduced as a device to pass from an nth order ordinary differential equation to an equivalent set of first order quasilinear equations in order to avoid consideration of high extensions in Lie's theory [6]. But, as we have previously pointed out [7], the implications of using the maximal set fundamentally alters the theory of the hierarchy of extensions as originally proposed by Lie, both for systems of partial differential equations and of ordinary differential equations.

The bulk of this paper, Sections III-VI, deals with a derivation of a generalization of Lie's "Counting" Theorem [8] for the maximal possible number of generators leaving a second-order ordinary differential equation covariant and the role and utility of these additional transformation labels.

II. A Generalization of the Lie–Ovsjannikov Covariance Formulas. Summary

The most general covariance transformation law for a system (S) of partial differential equations involving m unknown functions \( y = (y^1, ..., y^m) \) of the n independent variables \( x = (x^1, ..., x^n) \) can be labeled by a maximal set (M) of algebraically independent quantities consisting of \( x, y \) and various partial derivatives of the unknown functions with respect to the independent variables. The system (S)

\[
\Omega^{(r)}(x, y, y^x, y^{x^2}, ..., y^{x^n}) = 0 \quad (r = 1, ..., N) \tag{II-1}
\]

and all equations which are derivable from it by differentiation provide a set of algebraic relations which we shall denote by (A) between \( x, y \), and the partial derivatives of the \( y^x \) with respect to the \( x^r \). A maximal set (M) consists of any set of quantities \( x, y, y^x, y^{x^2}, \text{etc.} \), and/or their signs which themselves are algebraically independent with respect to the algebraic relations (A), but are sufficient to uniquely determine all the remaining derivatives through the relations established by (A). This maximal set, in general, includes an infinite number of derivatives.

A generalized Lie–Ovsjannikov covariance transformation law which

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1 The formulas presented in this section are summarized from lecture notes prepared by one of the authors (RLA) for a course in the Group-Theoretic Properties of Differential Equations conducted jointly with Carl E. Wulfman in the Spring of 1972.
utilizes the fact that $(M)$ provides a maximal set of labels for each one-
parameter covariance group is of the form\(^a\)

\[
\begin{align*}
  x^i &= \Phi^i(x, y, y^a, \ldots; a) \quad (i = 1, \ldots, n), \\
  y^\alpha &= \Psi^\alpha(x, y, y_x^\alpha, \ldots; a) \quad (\alpha = 1, \ldots, m),
\end{align*}
\]

(II-2)

where \(\{x, y, y^a_x, \ldots\} = (M)\), \(a\) is a group parameter, and a prime indicates
a transformed quantity. The finite transformation properties for all partial
derivatives follow from (II-2) in the usual way e.g.

\[
y^\alpha_{x^i} = \lim_{\Delta x^i \to 0} \frac{y^\alpha(x^1, \ldots, x^i + \Delta x^i, \ldots, x^n) - y^\alpha(x^1, \ldots, x^i, \ldots, x^n)}{\Delta x^i} \bigg|_{(A)}
\]

\[
= \left[ \sum_j \left( \partial_{y^j} \Psi^\alpha \right) x^j_{x^i} + \sum_{\beta} \sum_j \left( \partial_{y^j} \Psi^\alpha \right) y^\beta_{x^j} y^\alpha_{x^i} 
\right. \\
+ \left. \sum_{\beta} \sum_k \sum_j \left( \partial_{y^j} \Psi^\alpha \right) y^\beta_{x^j} x^k_{x^i} + \cdots \bigg|_{(A)}
\]

\[
= \Psi^\alpha_{x^i}(x, y, y_x^\alpha, \ldots; a).
\]

The notation \(\big|_{(A)}\) means "evaluated at" or subject to the algebraic relations
established by (A). We have adopted the notation in this paper that the usual
partial differentiation in which all but one of the members of the set of inde-
dependent variables \(\{x^i\}_{i=1}^n\) is varied, is denoted by a subscript indicating which
variable is varied, e.g., \(y^\alpha_{x^i}\), \(y^\alpha_{x^j x^k}\), etc. Similarly, if the differentiation is with
respect to a member of the transformed set of independent variables \(\{x'^i\}_{i=1}^n\),
we use the same notation e.g. \(y'^\alpha_{x'^i}\), \(y'^\alpha_{x'^j x'^k}\), etc. But partial differentiation
in which all but one of the members of the set \((M)\) is varied, is denoted by a
round \(d\) (\(\partial\)) with a subscript indicating which variable is varied e.g. \(\partial_{y^j} \Psi^\alpha\),
\(\partial_{y^j} \Phi^i\), \(\partial_{y^j} \Psi^\alpha\), etc. Similarly, if the transformed set \(M' = \{x', y', y'^\alpha, \ldots\}\) is
involved, we replace the unprimed subscript with a primed subscript on the
round \(d\) (\(\partial\)).

It can be shown by extending the Lie–Ovsjannikov arguments, that the
necessary and sufficient condition that (II-2) correspond to a one-parameter
covariance group of (II-1) is that

\[
U^{\Omega(r)} \bigg|_{(A)} = 0 \quad (r = 1, \ldots, N),
\]

(II-3)

\(^a\) As emphasized by R. Raczka and S. Woronowicz in private communications, this
is still not the most general form, but one could take \(\Phi\) and \(\Psi\) to be functionals or
quasidifferential operators. We note that such forms are utilized in the Weyl Calculus;
see e.g. K. E. Cahill and R. J. Glauber, Phys. Rev. 177, 1857 (1969), and references
therein.
where the generator $U$ is given by

$$U = \sum_{t} \xi^t \partial_{x^t} + \sum_{a} \eta^{a} \partial_{y^{a}} + \sum_{t} \sum_{a} \eta^{t \alpha} \partial_{y^{t \alpha}}, \ldots$$ \hspace{2cm} (II-4)

and

$$\xi^t(x, y, y^a, \ldots) = \lim_{\Delta a \to 0} \frac{\Phi(x, y, y^a_2, \ldots; \Delta a) - \Phi(x, y, y^a, \ldots; 0)}{\Delta a};$$

$$\eta^a(x, y, y^a, \ldots) = \lim_{\Delta a \to 0} \frac{\Psi^a(x, y, y^a_2, \ldots; \Delta a) - \Psi^a(x, y, y^a, \ldots; 0)}{\Delta a};$$

$$\eta^{a} \frac{y^a_1 \ldots y^a_2 \ldots y^a \ldots y^a_n}{j_1 j_1 j_1 + 1 \ldots j_n} = \lim_{\Delta a \to 0} \frac{\Psi^a(x, y, y^a_2, \ldots; \Delta a) - \Psi^a(x, y, y^a, \ldots; 0)}{\Delta a};$$

$$\eta^a \frac{y^a_1 \ldots y^a_2 \ldots y^a \ldots y^a_n}{j_1 j_1 j_1 + 1 \ldots j_n} = \lim_{\Delta a \to 0} \frac{\Psi^a(x, y, y^a_2, \ldots; \Delta a) - \Psi^a(x, y, y^a, \ldots; 0)}{\Delta a};$$

$$\frac{D}{Dx^t} = \partial_{x^t} + \sum_{a} \xi^a \partial_{y^a} + \sum_{a} \sum_{k} \eta^a \partial_{y^a} + \ldots.$$

A simple example of a covariance generator which involves a derivative higher that any occurring in the original partial differential equation is given by

$$U = \partial_{x^1} + y_{x^1 x^2} \partial_{y^2} + y_{x^1 x^2 x^1} \partial_{y^1} + y_{x^1 x^2 x^3} \partial_{y^3} + \ldots$$ \hspace{2cm} (II-4')

for the partial differential equation

$$y_{x^1} + y_{x^2} = 0.$$ \hspace{2cm} (II-1')

The maximal set (M) reduces in the case of an nth order ordinary differential equation

$$y^{(n)} = F(x, \dot{y}, \ldots, y^{(n-1)})$$ \hspace{2cm} (II-5)

to the set \{x, y, \dot{y}, \ldots, y^{(n-1)}\} where the overdot indicates differentiation with respect to $x$ and we have dropped the tensor notation.
A generalized point transformation law which utilizes the fact that \( \{x, y, \dot{y}, \ldots, y^{(n-1)}\} \) is the maximal set of algebraically independent quantities for each one-parameter covariance group is of the form

\[
\begin{align*}
    x' &= \Phi(x, y, \dot{y}, \ldots, y^{(n-1)}; a), \\
y' &= \Psi(x, y, \dot{y}, \ldots, y^{(n-1)}; a),
\end{align*}
\] (II-6)

where, as before, \( a \) is a group parameter and the primes indicate transformed quantities.

The necessary and sufficient condition that (II-6) corresponds to a one-parameter covariance group of (II-5) is then given by

\[
U[y^{(n)} = F(x, y, \dot{y}, \ldots, y^{(n-1)})] \big|_{y^{(n)}=F(x,y,y,...,y^{(n-1)}, y^{(n-1)})} = 0,
\] (II-7)

where the generator \( U \) is given by

\[
U = \xi \partial_x + \eta^x \partial_y + \eta^y \partial_\dot{y} + \cdots + \eta^{y^{(n-1)}} \partial_{y^{(n-1)}} + \eta^{y^{(n)}} \partial_{y^{(n)}},
\] (II-8)

and

\[
\begin{align*}
    \xi &= \frac{\partial \Phi}{\partial a} \bigg|_{a=0}, \\
    \eta^x &= \frac{\partial \Psi}{\partial a} \bigg|_{a=0} \\
    \eta^{y^{(n)}} &= \frac{d}{dx} \eta^{y^{(k-1)}} - y^{(k)} \frac{d \xi}{dx} \quad (\text{for } k = 1, \ldots, n).
\end{align*}
\] (II-8')

Further, the finite transformations can be expressed in terms of the generator \( U \) and the group parameter \( a \) as follows.

\[
\begin{pmatrix}
x' \\
y' \\
\vdots \\
y^{(n-1)}' \\
y^{(n)}'
\end{pmatrix} = e^{aU}
\begin{pmatrix}
x \\
y \\
\vdots \\
y^{(n-1)} \\
y^{(n)}
\end{pmatrix}.
\] (II-9)

In particular, Eq. (II-7) takes the following form for \( n = 2 \).

\[
\begin{align*}
(\eta^{y^0} - 2\xi_x)F - \xi F_x - \eta^y F_y &- \eta^y F_\dot{y} + \eta^y_{xx} \\
&+ [2\eta^y_{xy} - \xi_{xx} - 3\xi_y F - (\eta^y - \xi_x) F_\dot{y}] \dot{y} \\
&+ (\eta^y_{yy} - 2\xi_y + \xi_y F_y) \dot{y}^2 - \xi_{yy} \dot{y}^3 \\
&+ 2\eta^y_{xy} F + \eta^y_{yy} F^2 + \eta^y_y F_x - 2\xi_\dot{y} F^2 \\
&+ [2\eta^y_{yy} F + \eta^y_y F_v - 2\xi_y F_x - \xi_\dot{y} F_x] \dot{y} \\
&+ [-2\xi_y F - \xi_\dot{y} F_v] \dot{y}^2 &\equiv 0.
\end{align*}
\] (II-10)
III. Counting Theorem

It follows directly from Eq. (II–10) that if a general dependence on \( \dot{y} \) is assumed in \( \xi \) and \( \eta^y \) then, for a given \( F \), Eq. (II–10) involves two arbitrary functions, namely \( \xi \) and \( \eta^y \), and their derivatives. Hence, for each specification of, say, \( \eta^y \), we obtain a partial differential equation for \( \xi \) which in general yields an infinity of possible \( \xi \)'s. We recover in this manner the result shown by Ovsjannikov [9] that there exists an infinite number of one-parameter covariance transformations, not only for first-order ordinary differential equations, but for any order. Actually Ovsjannikov showed that any system of first-order ordinary differential equations admits a so-called “infinite” group and, as is well-known and he so stated, any \( n \)th order ordinary differential equation is equivalent to such a system.

In view of this conclusion, it would be of interest to illustrate one important implication of this with an example where generators do not exist under the assumption of the original Lie-type point transformation, but generators exist when derivatives are included in the transformation law. Examples of the former exist e.g. \( y = my + \dot{y}y \), but we have not been successful in producing an example with both features.

Because there exists in general an infinite number of one-parameter transformations, counting theorems analogous to those of Lie can only be obtained if an explicit \( \dot{y} \) dependence is assumed for \( \xi \) and \( \eta^y \). Here we derive such a new “counting” theorem by expanding the \( \dot{y} \) dependence of the various generators in terms of a subset of a particular linearly independent set of functions, namely simple powers of \( \dot{y} \).

We shall restrict ourselves to second-order ordinary differential equations,

\[
y = F(x, y, \dot{y}), \tag{III–1}
\]

which are integrable “in the large”, i.e., such that one and only one integral curve of Eq. (III–1) passes through any two points in the \((x, y)\) plane. As emphasized by Gel’fand and Fomin [10], the conditions for the existence and uniqueness of such integral curves do not simply reduce to the usual existence theorems for differential equations, and they cite Bernstein’s theorem [11] for one set of sufficient conditions. But Bernstein’s theorem is not sufficient to cover all known cases of integrability “in the large”, e.g. even the case \( y = \text{const} \). Hence, at this point we cannot to our satisfaction further characterize integrability “in the large”.

**Theorem.** A second-order ordinary differential equation of the form

\[
\dot{y} = F(x, y, \dot{y}), \tag{III–2}
\]
which is uniquely integrable "in the large" (defined above), is covariant under the action of at most \( r \) linearly independent generators of the form

\[
U_h = \xi_k \partial_x + \eta_k \partial_y + \eta_k \delta \partial_x + \eta_k \gamma \partial_y,
\]  

(III–3)

where \( \xi_k \) and \( \eta_k \) are restricted to be of the form

\[
\xi_k = \sum_{t=0}^{n} h_{tk}(x, y) (y)^t,
\]

(III–4)

\[
\eta_k = \sum_{t=0}^{n} l_{tk}(x, y) (y)^t,
\]

(III–5)

subject to the condition that precisely one function for each nontrivial pair \( l_{ik} \), \( h_{(i-1)k} \), \( i = 1, 2, \ldots \) and all \( k \) is specified a priori\(^3\) and \( r \) is given by

\[
y = \begin{cases} 
\frac{1}{2} (m + n + 2) (n + 4) & \text{for } n \geq m, \ n \geq 0, \ m \geq 0, \\
\frac{1}{2} (m + n + 2) (m + 3) & \text{for } n < m, \ n \geq 0, \ m > 0.
\end{cases}
\]

(III–6)

Proof. The argument is by contradiction and is patterned closely after that of Lie.

Fix \( n, m, \) and assume that the theorem is valid for \( r + 1 \) linearly independent generators: \( U_1, U_2, \ldots, U_{r+1} \) where \( r \) is determined by (III–6). Equation (III–2) is therefore also covariant under the action of a nontrivial linear combination of these generators

\[
U = \sum_{j=1}^{r+1} a_j U_j.
\]

(III–7)

Select a set of \( s = r/(m + n + 2) \) distinct points \( P_i = (x_i, y_i) \) such that each pair of points from this set determines a unique integral curve of (III–2) and no subset of three of these points lies on the same integral curve. We shall now fix the ratio of \( r \) of these constants, say, \( a_1, \ldots, a_r \) to \( a_{r+1} \) by requiring that the action of (III–7) leaves the \( s \) points \( P_1, \ldots, P_s \) invariant. The fact that, by assumption, an infinite number of integral curves pass through each point translates into the requirement that the points are invariant for arbitrary \( y \) which implies that the \( a \)'s simultaneously satisfy the following system of \( r \) equations.

\[
\begin{align*}
& a_1 h_{0,1}(x_j, y_j) + \cdots + a_{r+1} h_{0,r+1}(x_j, y_j) = 0, \\
& \vdots \\
& a_1 h_{n,1}(x_j, y_j) + \cdots + a_{r+1} h_{n,r+1}(x_j, y_j) = 0, \\
& a_1 l_{0,1}(x_j, y_j) + \cdots + a_{r+1} l_{0,r+1}(x_j, y_j) = 0, \\
& \vdots \\
& a_1 l_{m,1}(x_j, y_j) + \cdots + a_{r+1} l_{m,r+1}(x_j, y_j) = 0.
\end{align*}
\]

(III–8)

(III–9)

\(^3\) If this requirement is dropped, nondenumerably infinite classes of generators involving arbitrary functions would appear because it is only the difference \( l_{ik} - h_{(i-1)k} \) which is uniquely determined by Eq. (II–10).
for \( j = 1, \ldots, s \). There must exist at least one set of \( a \) points such that (III-8) and (III-9) yield a finite nontrivial set of ratios for some \( r \) of the \( a \)'s to the remaining one because of the assumed linear independence of the set \( \{ U_1, \ldots, U_{r+1} \} \).

Since the set of points \( \{ P_1, \ldots, P_s \} \) and the differential equation (III-2) are assumed to be invariant and covariant, respectively, under the action of the generator (III-7) where the \( a \)'s are constrained by Eqs. (III-8) and (III-9), then the unique integral curve of (III-2) passing through each pair of points is also invariant under the action of this generator. Hence, through each point there pass at least \( (s - 1) \) invariant integral curves. This implies that \( \eta^j = 0 \) when evaluated at each of these points and every \( \dot{y} \) corresponding to an invariant integral curve through that point. In particular, introducing \( \dot{y}_a \) for the slope of the integral curve passing through points \( q \) and \( j \) evaluated at \( q (q \neq j) \), and substituting (III-4) and (III-5) into the expression (II-8') for \( \eta^j \), it is required for each point \( P_1, \ldots, P_s \) (represented by \( q \) and all \( (s - 1) \) curves (represented by \( j \)) passing through each point that

\[
\sum_{k=1}^{r+1} a_k \eta_k \eta^j(qj) = \sum_{k=1}^{r+1} a_k \frac{d\eta_k \eta^j(qj)}{dx} - \dot{y} \sum_{k=1}^{r+1} a_k \frac{d\xi_k(qj)}{dx}
\]

\[
= \frac{d}{dx} \left[ \sum_{k=1}^{r+1} a_k \sum_{i=0}^{m} l_{ik}(q) (\dot{y}_a)^i \right] - \dot{y}_a \frac{d}{dx} \left[ \sum_{k=1}^{r+1} a_k \sum_{i=0}^{n} h_{ik}(q) (\dot{y}_a)^i \right]
\]

\[
= \sum_{k=1}^{r+1} a_k \left\{ \sum_{i=0}^{m} \left( \frac{\partial l_{ik}(q)}{\partial x} (\dot{y}_a)^i + \frac{\partial l_{ik}(q)}{\partial y} (\dot{y}_a)^{i+1} + il_{ik}(q) F(x, y, \dot{y}) (\dot{y}_a)^{i+1} \right) + \sum_{i=0}^{n} \left( \frac{\partial h_{ik}(q)}{\partial x} (\dot{y}_a)^{i+1} + \frac{\partial h_{ik}(q)}{\partial y} (\dot{y}_a)^{i+2} + ih_{ik}(q) F(x, y, \dot{y}) (\dot{y}_a)^{i+2} \right) \right\} = 0.
\]

Note that Eqs. (III-8) and (III-9) imply that the third and sixth terms of this last equation are zero. Since these are the only terms which depend on the form of (III-2), we have obtained the result that the highest power of \( \dot{y} \) in \( \eta^j \) is independent of the form of (III-2). It can be easily verified from the preceding equation that the highest power \( (s - 2) \) of \( \dot{y} \) is given by

(i) \( (n + 2) \) for \( n > m \) \((n \geq 0, m > 0)\),

(ii) \( (m + 1) \) for \( n < m \) \((n \geq 0, m > 0)\).
Let \( b_1, b_2, \ldots, b_{s+1} \) represent the coefficients of \((\dot{y}_q)^0, (\dot{y}_q)^1, \ldots, (\dot{y}_q)^{s-2}\), respectively. Then \( \eta^2 = 0 \) gives for each \( q \) and the \((s - 1)\)'s

\[
\begin{align*}
&b_{q(s-1)}(\dot{y}_{q1})^{(s-2)} + \cdots + b_{q1} \dot{y}_{q1} + b_{q0} = 0, \\
&b_{q(s-1)}(\dot{y}_{q(s-1)})^{(s-2)} + \cdots + b_{q1} \dot{y}_{q(s-1)} + b_{q0} = 0.
\end{align*}
\]

(III-10)

The determinant of the coefficients

\[
\begin{vmatrix}
\dot{y}_{q1}^{(s-2)} & \cdots & \dot{y}_{q1} & 1 \\
\vdots & \ddots & \vdots & \ddots \\
\dot{y}_{q(s-1)}^{(s-2)} & \cdots & \dot{y}_{q(s-1)} & 1
\end{vmatrix}
\]

is a Vandermonde determinant [13] and therefore

\[
A = \prod_{s-1 < k \leq s} (\dot{y}_{qk} - \dot{y}_{qk}).
\]

(III-11)

Now because \((s - 1)\) distinctly different integral curves were selected to pass through each point we have the result that \( A \neq 0 \), which implies that all the \( b \)'s are trivial, i.e.,

\[
b_{q1} = b_{q2} = \cdots = b_{q(s-1)} = 0 \quad q = 1, \ldots, s,
\]

which means that \( \eta^2 = 0 \) for all integral curves through each of the \( s \) points \( P_1, \ldots, P_s \), which means that all these integral curves are invariant under the action of (III-7) where the \( a \)'s are subject to (III-8) and (III-9).

Because the minimum \( s \) possible is four, corresponding to Lie's original case \( m = n = 0 \), if \( P_{s+1} \) is any \((s + 1)\)th point, it will lie on at least two distinct integral curves each of which passes through at least one of the \( s \) points. But we have just shown that these curves are invariant, hence the point \( P_{s+1} \) is invariant. Since \( P_{s+1} \) is arbitrary, we have the result that all points of the plane are invariant under the action of (III-7) subject to (III-8) and (III-9) which means that it must be identically zero for nontrivial \( a \)'s, i.e., the set \( \{U_1, \ldots, U_{s+1}\} \) is linearly dependent, which contradicts our original assumption.

This argument does not apply to any linear combination of \( r \) or less generators of the assumed form. This is because the highest power \((s - 2)\) of \( \dot{y} \) which depends on the set \( \{m, n\} \) and the properties of the Vandermonde determinant uniquely fix the number \( s \) of invariant points for which the argument is valid. Once this number is fixed, the number of generators which can be assumed ostensibly to be linearly independent must be such that all but one of the constants in their linear combination must be determined by requiring the invariance of the \( s \) points.
IV. EXAMPLES OF THE REALIZATION OF THE MAXIMAL NUMBER OF GENERATORS

The results of the search for generators of the form discussed in Section III are tabulated here for two examples from classical mechanics, namely, the one-dimensional free particle $y = 0$, and the simple harmonic oscillator $y = -ky$ ($k > 0$). In addition to the eight generators allowed for by Lie, we have found seven additional generators in which $\xi$ and $\eta y$ are linear in $\dot{y}$ and thus realize the maximal number allowed for this form. In the case of $y = 0$, the first eight generators were obtained by Lie, and are known to generate the projective group of the real line [14], whereas in the case of $y = -ky$ ($k > 0$), even the first eight generators were not previously reported.4

In particular, if we take $m = 0$ and $n = 1$, we obtain for $\dot{y} = 0$

\[
\begin{align*}
U_1 & = y\partial x - y^2\partial y, \\
U_2 & = x\partial x - y\partial y, \\
U_3 & = x\partial y + \partial y, \\
U_4 & = y\partial y + y\partial y, \\
U_5 & = \partial x, \\
U_6 & = \partial y, \\
U_7 & = x^2\partial x + xy\partial y + (y - xy) \partial y, \\
U_8 & = xy\partial x + y^2\partial y + (y\dot{y} - xy) \partial y, \\
U_9 & = (2x^2\dot{y} - xy) \partial x + y^2\partial y + (3y\dot{y} - 3xy^2) \partial y, \\
U_{10} & = \dot{y}\partial x, \\
U_{11} & = y\dot{y}\partial x - y^3\partial y, \\
U_{12} & = xy\partial x - y^2\partial y, \\
U_{13} & = (-y^2 + xy\dot{y}) \partial x + (y\dot{y}^2 - xy^3) \partial y, \\
U_{14} & = (-2xy^2 + x^2y\dot{y}) \partial x - y^3\partial y + (-y^2\dot{y} + 2xy\dot{y}^2 - x^2y^3) \partial y, \\
U_{15} & = \left(x^2y - \frac{x^3\dot{y}}{2}\right) \partial x + \frac{xy^2}{2} \partial y + \left(\frac{y^2}{2} - xy\dot{y} + \frac{x^2\dot{y}^2}{2}\right) \partial y.
\end{align*}
\]

4 These eight generators were derived by one of the authors (R.L.A.) and are contained in the lecture notes for the course on Group Theoretic Properties of Differential Equations previously cited.
Similarly, we obtain for $\dot{y} = -ky$, taking $m = 0$ and $n = 1$,

\begin{align*}
U_1 &= \sin k^{1/2}x \dot{y} + k^{1/2} \cos k^{1/2}x \ddot{y} - k \sin k^{1/2}x \dot{y}, \\
U_2 &= \cos k^{1/2}x \dot{y} - k^{1/2} \sin k^{1/2}x \ddot{y} - k \cos k^{1/2}x \dot{y}, \\
U_3 &= \partial x, \\
U_4 &= y \dot{y} + \dot{y} \dot{y} - ky \dot{y}, \\
U_5 &= \sin 2k^{1/2}x \dot{y} + k^{1/2} \cos 2k^{1/2}x \ddot{y} \\
&\quad + (-\dot{y} k^{1/2} \cos 2k^{1/2}x - 2ky \sin 2k^{1/2}x) \dot{y} - k^{3/2}y \cos 2k^{1/2}x \ddot{y}, \\
U_6 &= \cos 2k^{1/2}x \dot{y} - k^{1/2}y \sin 2k^{1/2}x \ddot{y} \\
&\quad + (\dot{y} k^{1/2} \sin 2k^{1/2}x - 2ky \cos 2k^{1/2}x) \dot{y} + k^{3/2}y \sin 2k^{1/2}x \ddot{y}, \\
U_7 &= y \cos k^{1/2}x \dot{y} - k^{1/2}y^2 \sin k^{1/2}x \ddot{y} \\
&\quad + (-ky^2 \cos k^{1/2}x - y^2 \cos k^{1/2}x - k^{1/2}y \sin k^{1/2}x) \dot{y} \\
&\quad + k^{3/2}y^2 \sin k^{1/2}x \ddot{y}, \\
U_8 &= y \sin k^{1/2}x \dot{y} + k^{1/2}y^2 \cos k^{1/2}x \ddot{y} \\
&\quad + (-ky^2 \cos k^{1/2}x - y^2 \sin k^{1/2}x + k^{1/2}y \cos k^{1/2}x) \dot{y} \\
&\quad - k^{3/2}y^2 \cos k^{1/2}x \ddot{y}, \\
U_9 &= (2k^{1/2}y^2 \sin 2k^{1/2}x + y \ddot{y} \cos 2k^{1/2}x) \dot{y} + ky^3 \cos 2k^{1/2}x \ddot{y} \\
&\quad + (-2k^{3/2}y^3 \sin 2k^{1/2}x - 2ky^2 \sin 2k^{1/2}x - y^3 \cos 2k^{1/2}x) \dot{y} \\
&\quad - k^{2}y^3 \sin 2k^{1/2}x \ddot{y}, \\
U_{10} &= (-2k^{1/2}y^2 \cos 2k^{1/2}x + y \ddot{y} \sin 2k^{1/2}x) \dot{y} + ky^3 \sin 2k^{1/2}x \ddot{y} \\
&\quad + (2k^{3/2}y^3 \cos 2k^{1/2}x + 2ky^2 \sin 2k^{1/2}x - y^3 \sin 2k^{1/2}x) \dot{y} \\
&\quad - k^{2}y^3 \sin 2k^{1/2}x \ddot{y}, \\
U_{11} &= y \cos k^{1/2}x \dot{y} - ky^2 \cos k^{1/2}x \ddot{y} \\
&\quad + (-ky^2 \cos k^{1/2}x + k^{3/2}y^2 \sin k^{1/2}x + k^{1/2}y \sin k^{1/2}x) \dot{y} \\
&\quad + k^{2}y^2 \cos k^{1/2}x \ddot{y}, \\
U_{12} &= y \sin k^{1/2}x \dot{y} - ky^2 \sin k^{1/2}x \ddot{y} \\
&\quad + (-ky^2 \sin k^{1/2}x - k^{3/2}y^2 \cos k^{1/2}x - k^{1/2}y \cos k^{1/2}x) \dot{y} \\
&\quad + k^{2}y^2 \sin k^{1/2}x \ddot{y}, \\
U_{13} &= (2k^{1/2}y \sin 3k^{1/2}x + y \ddot{y} \cos 3k^{1/2}x) \dot{y} + ky^2 \cos 3k^{1/2}x \ddot{y} \\
&\quad + (-3k^{3/2}y^2 \sin 3k^{1/2}x - 3ky^2 \cos 3k^{1/2}x + k^{1/2}y \sin 3k^{1/2}x) \dot{y} \\
&\quad - k^{2}y^2 \cos 3k^{1/2}x \ddot{y} \\
U_{14} &= (-2k^{1/2}y \cos 3k^{1/2}x + y \ddot{y} \sin 3k^{1/2}x) \dot{y} + ky^2 \sin 3k^{1/2}x \ddot{y} \\
&\quad + (3k^{3/2}y^2 \cos 3k^{1/2}x - 3ky^2 \sin 3k^{1/2}x - k^{1/2}y \cos 3k^{1/2}x) \dot{y} \\
&\quad - k^{2}y^2 \sin 3k^{1/2}x \ddot{y}, \\
U_{15} &= y \ddot{y} \dot{x} - ky^3 \dot{y} + (-2ky^2 \dot{y} - y^3) \dot{y} + k^{2}y^3 \ddot{y}.
\end{align*}

These operators are members of the first extension in the generalized theory.
Additional generators involving higher powers of \( \dot{y} \) can be obtained through the operation of commutation. But it can be shown that even if these sets are augmented by the additional generators produced by commutation, neither of these sets close under a finite number of commutations.

V. A Classification of the Generators

Addressing ourselves to the question of understanding the role and utility of these additional transformation labels, we are led to divide any set of generators (whether they close or not and/or involve derivative labels or not) into two mutually exclusive classes; those that generate nontrivial solutions directly by integration for an initial value problem, and those that do not. The former further divide themselves into two further subclasses; those that generate the general solution of the original ordinary differential equation, and those that do not. Each of the latter can then also be used in principle to generate potentially new solutions through its action on solutions that it cannot generate directly by integration, but which are known by other means, e.g. by integrating one of the other generators. Those that do not directly generate a solution still yield a distinctly new solution when operating on any known solution.\(^5\) What is interesting and extremely useful is that the generators which involve the additional derivative labels are also of both types, and we shall illustrate this by tabulating the solutions generated by the generators involving the additional labels for two simple second-order ordinary differential equations.

The criterion that a generator directly yields by integration a solution is that the initial conditions for the original solution remain a set of initial conditions for the transformed solution, i.e., the original differential equation must not only be covariant under the action of the generator in question, but in addition the initial conditions must be invariant under its action. This has been extensively discussed by Bluman and Cole [5] for partial differential equations in the Lie-Ovsjannikov theory and reduces here to the requirement that, in addition to satisfying the original differential equation, \( y \) also satisfy

\[
\xi \dot{y} = \eta^y. \tag{V-1}
\]

We note that in Lie's original treatment \( \xi \) and \( \eta^y \) were functions only of \( x \) and \( y \) and this was directly a quasilinear first-order ordinary differential equation, but in our generalization this is not the case because \( \xi \) and \( \eta^y \) in general involve \( \dot{y} \).\(^6\)

\(^5\) This characterization is also valid and useful for partial differential equations.

\(^6\) The result analogous to (V-1) in the case of a system of partial differential equations is \( \sum \xi^a y^a \dot{x} = \eta^y \) which is no longer in general directly a set of quasilinear first-order
In Table I, we have tabulated the solutions generated by the seven generators in which $\xi$ and $\eta^\nu$ involve $\dot{y}$ for the example $\dot{y} = 0$. For this case, every one of these generators yields some subclass of solutions. In Table II, we have similarly tabulated the results for the case $y = -ky$. In this case, only three of those generators yield nontrivial solutions. In the tables, $C_1$, $C_2$, $A_1$, and $A_2$ are arbitrary constants.

**TABLE I**

<table>
<thead>
<tr>
<th>Generators</th>
<th>Solutions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U_9$</td>
<td>$y = C_1 x$</td>
</tr>
<tr>
<td>$U_{10}$</td>
<td>$y = C_2$</td>
</tr>
<tr>
<td>$U_{11}$</td>
<td>$y = C_2$</td>
</tr>
<tr>
<td>$U_{12}$</td>
<td>$y = C_2$</td>
</tr>
<tr>
<td>$U_{13}$</td>
<td>$y = C_2$, $y = C_1 x$</td>
</tr>
<tr>
<td>$U_{14}$</td>
<td>$y = C_1 x$</td>
</tr>
<tr>
<td>$U_{15}$</td>
<td>$y = C_1 x$</td>
</tr>
</tbody>
</table>

**TABLE II**

<table>
<thead>
<tr>
<th>Generators</th>
<th>Solutions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U_9$</td>
<td>$y = 0$</td>
</tr>
<tr>
<td>$U_{10}$</td>
<td>$y = 0$</td>
</tr>
<tr>
<td>$U_{11}$</td>
<td>$y = A_1 e^{+ (-k)^{1/2}}$, $A_2 e^{-(-k)^{1/2}}$</td>
</tr>
<tr>
<td>$U_{12}$</td>
<td>$y = A_1 e^{+ (-k)^{1/2}}$, $A_2 e^{-(-k)^{1/2}}$</td>
</tr>
<tr>
<td>$U_{13}$</td>
<td>$y = 0$</td>
</tr>
<tr>
<td>$U_{14}$</td>
<td>$y = 0$</td>
</tr>
<tr>
<td>$U_{15}$</td>
<td>$y = A_1 e^{+ (k)^{1/2}}$, $A_2 e^{-(k)^{1/2}}$</td>
</tr>
</tbody>
</table>

**VI. INTEGRATION**

In this generalization, the problem of the integration of generators in order to obtain the corresponding finite covariance transformations for an $n$th-

---

...partial differential equations. For the example given in the Eqs. (II-1') and (II-4'), the analog of (V-1) is $y_{\alpha_2 \beta_2} = u_{\alpha_2}$ and the common solutions of this equation and (II-1') are given by $y(x^1, x^3) = C_1 \exp(x^3 - x^1) + C_2$ where $C_1$ and $C_2$ are arbitrary constants.
order ordinary differential equation is equivalent to the problem of the integration of the simultaneous system of ordinary differential equations

\[
\frac{da}{1} = \frac{dx'}{\xi(x', y', y', \ldots, y'(n-1))} = \frac{dy'}{\eta'(x', y', y', \ldots, y'(n-1))} = \ldots
\]

\[
\frac{dy'^{(n-1)}}{\eta'^{(n-1)}(x', y', y', \ldots, y'(n-1))}
\]

where \(x', y', y', \ldots, y'(n-1)\) are to be regarded as independent functions of the group parameter \(a\) and a set of initial conditions \(x, y, y', \ldots, y'(n-1)\). The operation \(d(\cdot)/dx'\) represents the rate of change of \((\cdot)\) with respect to \(x'\) along a path curve starting at a point specified by the maximal set

\[
(M) = \{x, y, y', \ldots, y'(n-1)\}
\]

and evaluated at a point specified by (II-6) e.g.

\[
\frac{d\dot{y}}{dx} = \lim_{da \to 0} \frac{\Psi(x, y, \dot{y}, \ldots, y'^{(n-1)}; a + \Delta a) - \Psi(x, y, \dot{y}, \ldots, y'^{(n-1)}; a)}{\Phi(x, y, \dot{y}, \ldots, y'^{(n-1)}; a + \Delta a) - \Phi(x, y, \dot{y}, \ldots, y'^{(n-1)}; a)}
\]

The solutions are subject to the consistency relations

\[
y'^{(k)} = \frac{dy'^{(k-1)}}{dx'} = \left(\frac{d/dx}{}\right)\frac{y'^{(k-1)}(x, y(x), \dot{y}, \ldots, y'^{(n-1)}; a)}{x'(x, y(x), \dot{y}, \ldots, y'^{(n-1)}; a)}
\]

for arbitrary but fixed \(a\).

The essential difference between (VI-1) and the corresponding simultaneous set in Lie's original theory is that in the latter \(\xi = \xi(x, y)\) and the \(\eta^{(k)}\)'s are functions of at most \(x, y', \ldots, y'^{(k)}\) \((k = 0, 1, \ldots, n - 1)\), whereas in (VI-1) \(\xi\) and the \(\eta^{(k)}\)'s are in general functions of the maximal set \(\{x, y, \dot{y}, \ldots, y'^{(n-1)}\}\).

By way of illustrating the appearance of the additional transformation labels, we have tabulated in Table III the finite covariance transformations for four of the readily integrable generators in which \(\xi\) and \(\eta\) involve \(\dot{y}\) for the example \(\dot{y} = 0\).

<table>
<thead>
<tr>
<th>Generators</th>
<th>Finite transformations (continuously connected to the identity transformation)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(U_{10})</td>
<td>(x' = x + \gamma a) (y' = y) (\dot{y}' = \dot{y})</td>
</tr>
<tr>
<td>(U_{11})</td>
<td>(x' = x + (y/\dot{y})[1 + 2a\dot{y}^{3}/2 - 1]) (y' = y) (\dot{y}' = [\dot{y}/(1 + 2a\dot{y})]^{1/2})</td>
</tr>
<tr>
<td>(U_{12})</td>
<td>(x' = x(1 + a\dot{y})) (y' = y) (\dot{y}' = \dot{y}/(1 + a\dot{y}))</td>
</tr>
<tr>
<td>(U_{13})</td>
<td>(x' = x - (y/\dot{y})[1 - (1 - 2y\dot{y}a + 2\dot{y}^{2}xa)^{1/2}]) (y' = y) (\dot{y}' = [\dot{y}/(1 - 2y\dot{y}a + 2\dot{y}^{2}xa)]^{1/2})</td>
</tr>
</tbody>
</table>
ACKNOWLEDGMENTS

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5. Ref. [2d]
6. Ref. [2d, p 45].
7. Ref. [1].
8. Ref. [2c, p. 294].
9. Ref. [2d].
10. Ref. [4b, p. 16]
12. Ref. [2c, p. 294].
14. Ref. [2a–c].