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Computers and Mathematics with Applications 47 (2004) 1165–1176

An International Journal
**computers &
mathematics**
with applications

www.elsevier.com/locate/camwa

Application of the General Method of Lyapunov Functionals Construction for Difference Volterra Equations

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(Received December 2002; accepted October 2003)

Abstract—Lyapunov functionals are used usually for stability investigation of systems with aftereffect [1–3]. The general method of Lyapunov functionals construction which was proposed and developed in [4–19] is used here for stochastic second type Volterra difference equations. It is shown that using this method, there is a possibility to construct for a given equation, a sequence of extending stability regions. © 2004 Elsevier Ltd. All rights reserved.

Keywords—Difference equations, Method of Lyapunov functionals construction, Asymptotic stability.

1. STATEMENT OF THE PROBLEM

Let $\{\Omega, \sigma, \mathbf{P}\}$ be a basic probability space, $i \in Z = \{0, 1, \dots\}$ be a discrete time, $f_i \in \sigma$ be a sequence of σ -algebras, H be a space of sequences $x = \{x_i, i \in Z\}$ f_i -adapted random values $x_i \in R^n$ with norm

$$\|x\|^2 = \sup_{i \in Z} \mathbf{E} |x_i|^2.$$

Consider the stochastic difference equation in the form

$$x_{i+1} = \eta_{i+1} + F(i, x_0, \dots, x_i), \quad i \in Z, \quad x_0 = \eta_0, \quad (1.1)$$

and auxiliary nonstochastic difference equation

$$x_{i+1} = F(i, x_0, \dots, x_i), \quad i \in Z. \quad (1.2)$$

It is assumed that the functional F from equations (1.1) and (1.2) such that $F : Z * H \Rightarrow R^n$ and $F(i, \cdot)$ does not depend on x_j for $j > i$, $F(i, 0, \dots, 0) = 0$, $\eta \in H$.

DEFINITION 1.1. A sequence x_i from H is called:

- uniformly mean square bounded if $\|x\|^2 < \infty$;
- asymptotically mean square trivial if $\lim_{i \rightarrow \infty} \mathbf{E}|x_i|^2 = 0$;
- mean square integrable if $\sum_{i=0}^{\infty} \mathbf{E}|x_i|^2 < \infty$.

REMARK 1.1. It is easy to see that if the sequence x_i is mean square integrable, then it is uniformly mean square bounded and asymptotically mean square trivial.

DEFINITION 1.2. The trivial solution of equation (1.2) is called:

- stable if for any $\epsilon > 0$ there exist $\delta > 0$ such that $\sup_{i \in Z} |x_i| < \epsilon$ if $|x_0| < \delta$;
- asymptotically stable if it is stable and $\lim_{i \rightarrow \infty} x_i = 0$ for any x_0 .

THEOREM 1.1. Let there exists a nonnegative functional $V_i = V(i, x_0, \dots, x_i)$ and a sequence of nonnegative numbers γ_i such that

$$\mathbf{E}V(0, x_0) < \infty, \quad \sum_{i=0}^{\infty} \gamma_i < \infty, \tag{1.3}$$

$$\mathbf{E}\Delta V_i \leq -c\mathbf{E}|x_i|^2 + \gamma_i, \quad i \in Z, \quad c > 0. \tag{1.4}$$

Then, the solution of equation (1.1) is mean square integrable.

PROOF. From (1.4), it follows

$$\sum_{j=0}^i \mathbf{E}\Delta V_j = \mathbf{E}V(i+1, x_0, \dots, x_{i+1}) - \mathbf{E}V(0, x_0) \leq -c \sum_{j=0}^i \mathbf{E}|x_j|^2 + \sum_{j=0}^i \gamma_j.$$

From here, by virtue of (1.3), we obtain

$$c \sum_{j=0}^i \mathbf{E}|x_j|^2 \leq \mathbf{E}V(0, x_0) + \sum_{j=0}^{\infty} \gamma_j < \infty.$$

Therefore, the solution of equation (1.1) is mean square integrable. The theorem is proven.

COROLLARY 1.1. Let the sequence η_i be mean square integrable and there exists the nonnegative functional $V_i = V(i, x_0, \dots, x_i)$ such that

$$\begin{aligned} \mathbf{E}V(0, x_0) &< c_1 \mathbf{E}|x_0|^2, \\ \mathbf{E}\Delta V_i &\leq -c_2 \mathbf{E}|x_i|^2 + c_3 \mathbf{E}|\eta_{i+1}|^2, \quad i \in Z, \end{aligned} \tag{1.5}$$

$c_k > 0$, $k = 1, 2, 3$. Then, the solution of equation (1.1) is mean square integrable.

Similar to Theorem 1.1, one can prove the following.

THEOREM 1.2. Let there exist a nonnegative functional $V_i = V(i, x_0, \dots, x_i)$ such that for some $p > 0$

$$\begin{aligned} V(0, x_0) &< c_1 |x_0|^p, \\ \Delta V_i &\leq -c_2 |x_i|^p, \quad i \in Z, \end{aligned}$$

$c_k > 0$, $k = 1, 2$. Then, the trivial solution of equation (1.2) is asymptotically stable.

From Theorems 1.1 and 1.2, it follows that investigation of asymptotic behavior of solutions of difference equations type (1.1) and (1.2) can be reduced to construction of appropriate Lyapunov functionals. For this, it is possible to use the formal procedure of Lyapunov functionals construction which was described in [5]. Below, this procedure is demonstrated for linear Volterra equation with constant coefficients.

2. LINEAR VOLTERRA EQUATIONS WITH CONSTANT COEFFICIENTS

Consider the scalar equation

$$x_{i+1} = \eta_{i+1} + \sum_{j=0}^i a_j x_{i-j}, \quad i \in Z, \quad x_0 = \eta_0. \quad (2.1)$$

Here $a_i, i \in Z$, are known constants.

Let us apply the procedure of Lyapunov functionals construction to equation (2.1). Represent the right-hand side of this equation in the form

$$x_{i+1} = \eta_{i+1} + \sum_{j=0}^k a_j x_{i-j} + \sum_{j=k+1}^i a_j x_{i-j}, \quad k \geq 0, \quad (2.2)$$

and consider the auxiliary difference equation

$$y_{i+1} = \sum_{j=0}^k a_j y_{i-j}, \quad i \in Z. \quad (2.3)$$

Note that in (2.2) and (2.3), it is supposed $x_j = 0$ if $j < 0$.

Introduce into consideration the vector $y(i) = (y_{i-k}, \dots, y_i)'$ and represent the auxiliary equation (2.3) in the form

$$y(i+1) = Ay(i), \quad A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ a_k & a_{k-1} & a_{k-2} & \cdots & a_1 & a_0 \end{pmatrix}. \quad (2.4)$$

Consider now the matrix equation

$$A'DA - D = -U, \quad U = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \quad (2.5)$$

where D is a symmetric matrix of dimension $k+1$.

Let us suppose that solution D of matrix equation (2.5) is a positive semidefinite matrix with $d_{k+1, k+1} > 0$. In this case, the function $v_i = y'(i)Dy(i)$ is Lyapunov function for equation (2.4). Really,

$$\Delta v_i = y'(i+1)Dy(i+1) - y'(i)Dy(i) = y'(i)[A'DA - D]y(i) = -y'(i)Uy(i) = -y_i^2.$$

So, if matrix equation (2.5) has a positive semidefinite solution D , then the trivial solution of equation (2.4) (or (2.3)) is asymptotically stable [5].

Following the procedure of Lyapunov functionals construction, the main part V_{1i} of Lyapunov functional $V_i = V_{1i} + V_{2i}$ must be chosen in the form $V_{1i} = x'(i)Dx(i)$, where $x(i) = (x_{i-k}, \dots, x_i)'$.

Represent equation (2.1) in the form

$$x(i+1) = \eta(i+1) + Ax(i) + B(i), \quad (2.6)$$

where matrix A is defined by (2.4), $\eta(i) = (0, \dots, 0, \eta_i)$ and $B(i) = (0, \dots, 0, b_i)'$,

$$b_i = \sum_{j=k+1}^i a_j x_{i-j}. \tag{2.7}$$

Calculating ΔV_{1i} , by virtue of equation (2.6), we have

$$\begin{aligned} \Delta V_{1i} &= x'(i+1)Dx(i+1) - x'(i)Dx(i) \\ &= (\eta(i+1) + Ax(i) + B(i))'D(\eta(i+1) + Ax(i) + B(i)) - x'(i)Dx(i) \\ &= -x_i^2 + \eta'(i+1)D\eta(i+1) + B'(i)DB(i) \\ &\quad + 2\eta'(i+1)DB(i) + 2\eta'(i+1)DAx(i) + 2B'(i)DAx(i). \end{aligned} \tag{2.8}$$

Note that

$$\eta'(i+1)D\eta(i+1) = d_{k+1,k+1}\eta_{i+1}^2. \tag{2.9}$$

Put

$$\alpha_l = \sum_{j=l}^{\infty} |a_j|, \quad l = 0, 1, \dots \tag{2.10}$$

Then, using (2.7), we obtain

$$\begin{aligned} B'(i)DB(i) &= d_{k+1,k+1}b_i^2 = d_{k+1,k+1} \left(\sum_{j=k+1}^i a_j x_{i-j} \right)^2 \\ &\leq d_{k+1,k+1} \sum_{m=k+1}^i |a_m| \sum_{j=k+1}^i |a_j| x_{i-j}^2 \\ &\leq d_{k+1,k+1} \alpha_{k+1} \sum_{j=k+1}^i |a_j| x_{i-j}^2. \end{aligned} \tag{2.11}$$

Using (2.7) and $\lambda > 0$, we have

$$\begin{aligned} 2\eta'(i+1)DB(i) &= 2\eta_{i+1}d_{k+1,k+1}b_i = 2\eta_{i+1}d_{k+1,k+1} \sum_{j=k+1}^i a_j x_{i-j} \\ &\leq d_{k+1,k+1} \sum_{j=k+1}^i |a_j| (\lambda^{-1}\eta_{i+1}^2 + \lambda x_{i-j}^2) \\ &\leq d_{k+1,k+1} \left(\lambda^{-1}\alpha_{k+1}\eta_{i+1}^2 + \lambda \sum_{j=k+1}^i |a_j| x_{i-j}^2 \right). \end{aligned} \tag{2.12}$$

Since

$$D\eta(i+1) = \eta_{i+1} \begin{pmatrix} d_{1,k+1} \\ d_{2,k+1} \\ \dots \\ d_{k,k+1} \\ d_{k+1,k+1} \end{pmatrix}, \quad Ax(i) = \begin{pmatrix} x_{i-k+1} \\ x_{i-k+2} \\ \dots \\ x_i \\ \sum_{m=0}^k a_m x_{i-m} \end{pmatrix},$$

then

$$\begin{aligned} 2\eta'(i+1)DAx(i) &= 2\eta_{i+1} \left[\sum_{l=1}^k d_{l,k+1}x_{i-k+l} + d_{k+1,k+1} \sum_{m=0}^k a_m x_{i-m} \right] \\ &= 2\eta_{i+1} \left[\sum_{m=0}^{k-1} (a_m d_{k+1,k+1} + d_{k-m,k+1}) x_{i-m} + a_k d_{k+1,k+1} x_{i-k} \right] \\ &= 2\eta_{i+1} d_{k+1,k+1} \sum_{m=0}^k Q_{km} x_{i-m}, \end{aligned} \tag{2.13}$$

where

$$Q_{km} = a_m + \frac{d_{k-m,k+1}}{d_{k+1,k+1}}, \quad m = 0, \dots, k-1, \quad Q_{kk} = a_k. \tag{2.14}$$

So, putting

$$\beta_k = \sum_{m=0}^k |Q_{km}| = |a_k| + \sum_{m=0}^{k-1} \left| a_m + \frac{d_{k-m,k+1}}{d_{k+1,k+1}} \right| \tag{2.15}$$

and using $\lambda > 0$, we have

$$\begin{aligned} 2\eta'(i+1)DAx(i) &\leq d_{k+1,k+1} \sum_{m=0}^k |Q_{km}| (\lambda^{-1}\eta_{i+1}^2 + \lambda x_{i-m}^2) \\ &= d_{k+1,k+1} \left(\lambda^{-1}\beta_k \eta_{i+1}^2 + \lambda \sum_{m=0}^k |Q_{km}| x_{i-m}^2 \right). \end{aligned} \tag{2.16}$$

Using (2.13)–(2.15), (2.7), (2.10), similar to (2.16), we obtain

$$\begin{aligned} 2B'(i)DAx(i) &= 2b_i d_{k+1,k+1} \sum_{m=0}^k Q_{km} x_{i-m} \\ &= 2d_{k+1,k+1} \sum_{m=0}^k \sum_{j=k+1}^i Q_{km} a_j x_{i-m} x_{i-j} \\ &\leq d_{k+1,k+1} \sum_{m=0}^k \sum_{j=k+1}^i |Q_{km}| |a_j| (x_{i-m}^2 + x_{i-j}^2) \\ &\leq d_{k+1,k+1} \sum_{m=0}^k |Q_{km}| \left(\alpha_{k+1} x_{i-m}^2 + \sum_{j=k+1}^i |a_j| x_{i-j}^2 \right) \\ &= d_{k+1,k+1} \left(\alpha_{k+1} \sum_{m=0}^k |Q_{km}| x_{i-m}^2 + \beta_k \sum_{j=k+1}^i |a_j| x_{i-j}^2 \right). \end{aligned} \tag{2.17}$$

As a result of (2.8), by virtue of (2.9), (2.11), (2.12), (2.16), (2.17), it follows that

$$\Delta V_{1i} \leq -x_i^2 + d_{k+1,k+1} \left(\lambda^{-1} q_k \eta_{i+1}^2 + (\lambda + \alpha_{k+1}) \sum_{m=0}^k |Q_{km}| x_{i-m}^2 + q_k \sum_{j=k+1}^i |a_j| x_{i-j}^2 \right), \tag{2.18}$$

where

$$q_k = \lambda + \alpha_{k+1} + \beta_k. \tag{2.19}$$

Now put

$$R_{kj} = \begin{cases} (\lambda + \alpha_{k+1}) |Q_{kj}|, & 0 \leq j \leq k, \\ q_k |a_j|, & j > k. \end{cases} \tag{2.20}$$

Then, (2.18) can be written as follows

$$\Delta V_{1i} \leq -x_i^2 + d_{k+1,k+1} \left(\lambda^{-1} q_k \eta_{i+1}^2 + \sum_{j=0}^i R_{kj} x_{i-j}^2 \right). \tag{2.21}$$

Choose the functional V_{2i} in the form

$$V_{20} = 0, \quad V_{2i} = d_{k+1,k+1} \sum_{l=0}^{i-1} x_l^2 \sum_{j=i-l}^{\infty} R_{kj}, \quad i > 0.$$

Calculating ΔV_{2i} , we obtain

$$\begin{aligned} \Delta V_{2i} &= d_{k+1,k+1} \left(\sum_{l=0}^i x_l^2 \sum_{j=i+1-l}^{\infty} R_{kj} - \sum_{l=0}^{i-1} x_l^2 \sum_{j=i-l}^{\infty} R_{kj} \right) \\ &= d_{k+1,k+1} \left(x_i^2 \sum_{j=1}^{\infty} R_{kj} - \sum_{l=0}^{i-1} R_{k,i-l} x_l^2 \right) \\ &= d_{k+1,k+1} \left(x_i^2 \sum_{j=1}^{\infty} R_{kj} - \sum_{j=1}^i R_{kj} x_{i-j}^2 \right). \end{aligned} \tag{2.22}$$

From (2.20)–(2.22) for the functional $V_i = V_{1i} + V_{2i}$, it follows

$$\mathbf{E} \Delta V_i \leq - \left(1 - d_{k+1,k+1} \sum_{j=0}^{\infty} R_{kj} \right) \mathbf{E} x_i^2 + \lambda^{-1} d_{k+1,k+1} q_k \mathbf{E} \eta_{i+1}^2.$$

Therefore, if

$$\sum_{j=0}^{\infty} R_{kj} < \delta_k = d_{k+1,k+1}^{-1}, \tag{2.23}$$

then the functional V_i satisfies the conditions of Corollary 1.1, and therefore, the solution of equation (2.1) is mean square integrable.

Using (2.20), (2.10), (2.15), (2.19), transform the left-hand part of inequality (2.23) in the following way

$$\begin{aligned} \sum_{j=0}^{\infty} R_{kj} &= \sum_{j=0}^k R_{kj} + \sum_{j=k+1}^{\infty} R_{kj} \\ &= (\lambda + \alpha_{k+1}) \sum_{j=0}^k |Q_{kj}| + q_k \sum_{j=k+1}^{\infty} |a_j| \\ &= (\lambda + \alpha_{k+1}) \beta_k + (\lambda + \alpha_{k+1} + \beta_k) \alpha_{k+1} \\ &= \lambda(\beta_k + \alpha_{k+1}) + \alpha_{k+1}^2 + 2\alpha_{k+1}\beta_k. \end{aligned}$$

So, if

$$\alpha_{k+1}^2 + 2\alpha_{k+1}\beta_k < \delta_k, \tag{2.24}$$

then there exists sufficiently small $\lambda > 0$ such that

$$\lambda(\beta_k + \alpha_{k+1}) + \alpha_{k+1}^2 + 2\alpha_{k+1}\beta_k < \delta_k$$

and condition (2.23) holds. It means that if condition (2.24) holds, then the solution of equation (2.1) is mean square integrable.

Note that condition (2.24) can be represented also in the form

$$\alpha_{k+1} < \sqrt{\beta_k^2 + \delta_k} - \beta_k. \tag{2.25}$$

REMARK 2.1. Using the same Lyapunov functional as above, one can prove that inequality (2.25) is a sufficient condition for asymptotic stability of the trivial solution of equation

$$x_{i+1} = \sum_{j=0}^i a_j x_{i-j}, \quad i \in Z. \tag{2.26}$$

REMARK 2.2. Suppose that in equation (2.1) (or (2.26))

$$a_j = 0, \quad j \geq k + 1. \tag{2.27}$$

Then, $\alpha_{k+1} = 0$ and condition (2.25) holds. So, if condition (2.27) holds and matrix equation (2.5) has a positive semidefinite solution with $d_{k+1,k+1} > 0$, then the solution of equation (2.1) is mean square integrable (the trivial solution of equation (2.26) is asymptotically stable).

REMARK 2.3. Suppose that for each $k \geq 0$, matrix equation (2.5) has a positive semidefinite solution D_k with $d_{k+1,k+1} > 0$ and there exist the limits $D = \lim_{k \rightarrow \infty} D_k$, $\beta = \lim_{k \rightarrow \infty} \beta_k$, such that $D \geq 0$, $|D| < \infty$ ($|D|$ is Euclidean norm of matrix D), $\beta < \infty$. Then, the solution of equation (2.1) is mean square integrable (the trivial solution of equation (2.26) is asymptotically stable). Really, this statement follows from (2.25) since $\lim_{k \rightarrow \infty} \alpha_{k+1} = 0$.

3. PARTICULAR CASES

CASE 3.1. $k = 0$. Condition (2.25) takes the form

$$\alpha_1 < \sqrt{\beta_0^2 + \delta_0} - \beta_0. \tag{3.1}$$

From (2.15), it follows $\beta_0 = |a_0|$. Equation (2.5) gives the solution $d_{11} = (1 - a_0^2)^{-1}$, which is a positive one if $|a_0| < 1$. Condition (3.1) takes the form $\alpha_1 < 1 - |a_0|$ or

$$\alpha_0 < 1. \tag{3.2}$$

Under condition (3.2), the solution of equation (2.1) is mean square integrable.

REMARK 3.1. Note that by condition

$$a_j \geq 0, \quad j \geq 0, \tag{3.3}$$

inequality (3.2) is the necessary and sufficient condition for asymptotic stability of the trivial solution of equation (2.26). Really, suppose first that $x_0 > 0$. Then, from (3.3) and (2.26), it follows $x_i \geq 0$ for $i \geq 0$. Consider the Lyapunov functional

$$V_i = x_i + \sum_{l=0}^{i-1} x_l \sum_{j=i-l}^{\infty} a_j. \tag{3.4}$$

Calculating ΔV_i , we obtain

$$\begin{aligned} \Delta V_i &= x_{i+1} + \sum_{l=0}^i x_l \sum_{j=i+1-l}^{\infty} a_j - x_i - \sum_{l=0}^{i-1} x_l \sum_{j=i-l}^{\infty} a_j \\ &= a_0 x_i + \sum_{j=1}^i a_j x_{i-j} + x_i \sum_{j=1}^{\infty} a_j - x_i - \sum_{l=0}^{i-1} x_l a_{i-l} = (\alpha_0 - 1)x_i. \end{aligned} \tag{3.5}$$

So, if inequality (3.2) holds, then from Theorem 1.2, it follows that the trivial solution of equation (2.26) is asymptotically stable. In the contrary case, i.e., $\alpha_0 \geq 1$, from (3.5) it follows that the functional V_i is nondecreasing one, and therefore, x_i does not go to zero. If $x_0 < 0$, then $x_i \leq 0$ for $i \geq 0$ and in Lyapunov functional (3.4), it is necessary to change x_l on $|x_l|$, $l = 0, 1, \dots$

CASE 3.2. $k = 1$. Condition (2.25) takes the form

$$\alpha_2 < \sqrt{\beta_1^2 + \delta_1} - \beta_1. \tag{3.6}$$

Matrix equation (2.5) is equivalent to the system of the equations

$$\begin{aligned} a_1^2 d_{22} - d_{11} &= 0, \\ (a_1 - 1) d_{12} + a_0 a_1 d_{22} &= 0, \\ d_{11} + 2a_0 d_{12} + (a_0^2 - 1) d_{22} &= -1, \end{aligned}$$

with the solution

$$\begin{aligned} d_{11} &= a_1^2 d_{22}, & d_{12} &= \frac{a_0 a_1}{1 - a_1} d_{22}, \\ d_{22} &= \frac{1 - a_1}{(1 + a_1) \left[(1 - a_1)^2 - a_0^2 \right]}. \end{aligned} \tag{3.7}$$

Matrix D is a positive semidefinite one with $d_{22} > 0$ by conditions $|a_1| < 1$, $|a_0| < 1 - a_1$. Using (2.15) and (3.7), we have

$$\begin{aligned} \beta_1 &= |a_1| + \left| a_0 + \frac{d_{12}}{d_{22}} \right| = |a_1| + \left| a_0 + \frac{a_0 a_1}{1 - a_1} \right| = |a_1| + \frac{|a_0|}{1 - a_1}, \\ \delta_1 &= d_{22}^{-1} = 1 - a_1^2 - a_0^2 \frac{1 + a_1}{1 - a_1}. \end{aligned} \tag{3.8}$$

Substituting (3.8) into (3.6), we obtain

$$\alpha_2 < 1 - |a_0| \frac{1 - |a_1|}{1 - a_1} - |a_1|. \tag{3.9}$$

Under condition (3.9), the solution of equation (2.1) is mean square integrable.

It is easy to see that condition (3.9) is not worse than condition (3.2). In particular, if $a_1 \geq 0$, then condition (3.9) coincides with condition (3.2), if $a_1 < 0$, then condition (3.9) is better than condition (3.2).

CASE 3.3. $k = 2$. Condition (2.25) takes the form

$$\alpha_3 < \sqrt{\beta_2^2 + \delta_2} - \beta_2. \tag{3.10}$$

Matrix equation (2.5) is equivalent to the system of the equations

$$\begin{aligned} a_2^2 d_{33} - d_{11} &= 0, \\ a_2 d_{13} + a_1 a_2 d_{33} - d_{12} &= 0, \\ a_2 d_{23} + a_0 a_2 d_{33} - d_{13} &= 0, \\ d_{11} + 2a_1 d_{13} + a_1^2 d_{33} - d_{22} &= 0, \\ d_{12} + a_0 d_{13} + a_0 a_1 d_{33} + (a_1 - 1) d_{23} &= 0, \\ d_{22} + 2a_0 d_{23} + (a_0^2 - 1) d_{33} &= -1, \end{aligned} \tag{3.11}$$

with the solution

$$\begin{aligned} d_{11} &= a_2^2 d_{33}, \\ d_{12} &= \frac{a_2 (1 - a_1) (a_1 + a_0 a_2)}{1 - a_1 - a_2 (a_0 + a_2)} d_{33}, \\ d_{13} &= \frac{a_2 (a_0 + a_1 a_2)}{1 - a_1 - a_2 (a_0 + a_2)} d_{33}, \\ d_{22} &= \left[a_1^2 + a_2^2 + \frac{2a_1 a_2 (a_0 + a_1 a_2)}{1 - a_1 - a_2 (a_0 + a_2)} \right] d_{33}, \\ d_{23} &= \frac{(a_0 + a_2) (a_1 + a_0 a_2)}{1 - a_1 - a_2 (a_0 + a_2)} d_{33}, \\ d_{33} &= \left[1 - a_0^2 - a_1^2 - a_2^2 - 2 \frac{a_1 a_2 (a_0 + a_1 a_2) + a_0 (a_0 + a_2) (a_1 + a_0 a_2)}{1 - a_1 - a_2 (a_0 + a_2)} \right]^{-1}. \end{aligned} \tag{3.12}$$

Using (2.15) and (3.11), (3.12), we have

$$\begin{aligned}\beta_2 &= |a_2| + \left| a_0 + \frac{d_{23}}{d_{33}} \right| + \left| a_1 + \frac{d_{13}}{d_{33}} \right| = |a_2| + \frac{|d_{13}| + |d_{12}|}{|a_2|d_{33}} \\ &= |a_2| + \frac{|a_0 + a_1a_2| + |(1-a_1)(a_1 + a_0a_2)|}{|1 - a_1 - a_2(a_0 + a_2)|}, \quad \delta_2 = d_{33}^{-1}.\end{aligned}\quad (3.13)$$

If matrix D with the elements defined by (3.12) is a positive semidefinite one with $d_{33} > 0$, then under condition (3.10) and (3.13), the solution of equation (2.1) is mean square integrable.

EXAMPLE 3.1. Consider the scalar equation

$$x_{i+1} = \eta_{i+1} + ax_i + \sum_{j=1}^i b^j x_{i-j}, \quad x_0 = \eta_0. \quad (3.14)$$

Condition (3.2) gives

$$|a| + \frac{|b|}{1 - |b|} < 1, \quad |b| < 1. \quad (3.15)$$

From (3.9), it follows

$$\frac{b^2}{1 - |b|} < 1 - |a| \frac{1 - |b|}{1 - b} - |b|, \quad |b| < 1. \quad (3.16)$$

From (3.10), (3.12), (3.13), we have

$$\begin{aligned}\frac{|b|^3}{1 - |b|} &< \sqrt{\beta_2^2 + \delta_2} - \beta_2, \quad |b| < 1, \\ \beta_2 &= b^2 + \frac{|a + b^3| + (1-b)|b(1+ab)|}{|1 - b - b^2(a + b^2)|}, \\ \delta_2 &= 1 - a^2 - b^2 - b^4 - 2b \frac{b^2(a + b^3) + a(a + b^2)(1 + ab)}{1 - b - b^2(a + b^2)}.\end{aligned}\quad (3.17)$$

For $k = 3$, condition (2.25) of equation (3.14) takes the form

$$\begin{aligned}\frac{b^4}{1 - |b|} &< \sqrt{\beta_3^2 + \delta_3} - \beta_3, \quad |b| < 1, \\ \beta_3 &= |b^3| + \left| a + \frac{d_{34}}{d_{44}} \right| + \left| b + \frac{d_{24}}{d_{44}} \right| + \left| b^2 + \frac{d_{14}}{d_{44}} \right|, \quad \delta_3 = d_{44}^{-1},\end{aligned}\quad (3.18)$$

where

$$\begin{aligned}\frac{d_{14}}{d_{44}} &= b^3 [b^3 + b^5 - b^8 + a(1 - b^3 + b^4)] G^{-1}, \\ \frac{d_{24}}{d_{44}} &= b^2 [a^2b + b^2 + b^5 - b^6 - b^8 + a(1 + b^4 + b^6)] G^{-1}, \\ \frac{d_{34}}{d_{44}} &= b [b^2 + a^3b^2 + b^4 - b^7 + a^2(b + b^4) + a(1 + 2b^3 + b^5 - b^6 - b^8)] G^{-1}, \\ d_{44} &= G [1 - b - b^2 - a^4b^3 - 2b^4 + 2b^7 - 2b^8 + 2b^9 - b^{10} - b^{12} + b^{13} \\ &\quad - b^{14} + b^{17} - a^3(b^2 + b^5) - a^2(1 + b + 5b^4 - b^5 + b^6 - 2b^7 - b^9) \\ &\quad - ab^2(1 + 4b - b^2 + 5b^3 - b^4 + b^5 - 4b^6 + 4b^7 - b^{10} + b^{11})]^{-1}, \\ G &= 1 - b - ab^2 - (1 + a^2)b^3 - b^4 - ab^5 - b^6 + b^7 + b^9.\end{aligned}$$

Note that the solution of equation (2.5) for $k = 3$ was obtained by the program "MATHEMATICA".

In Figure 1, the regions of mean square integrability of the solution of equation (3.14) (and at the same time the regions of asymptotic stability of the trivial solution of equation (2.26)) are shown, given by conditions (3.15) (curve number 1), (3.16) (curve number 2), (3.17) (curve number 3), and (3.18) (curve number 4). One can see that for $b \geq 0$, the bound of the region of mean square integrability, given by condition (3.16), coincides with the bound of region, given by condition (3.15). For $a \geq 0$ and $b \geq 0$, all conditions (3.15)–(3.18) give the same region of mean square integrability, which is defined by inequality

$$a + \frac{b}{1-b} < 1, \quad b < 1. \tag{3.19}$$

From Remark 3.1, it follows that by conditions $a \geq 0$ and $b \geq 0$, inequality (3.19) is the necessary and sufficient condition for asymptotic stability of the trivial solution of equation

$$x_{i+1} = ax_i + \sum_{j=1}^i b^j x_{i-j}. \tag{3.20}$$

Note also that the region of mean square integrability Q_k , obtained for equation (3.14), (or the region of asymptotic stability of the trivial solution of equation (3.20)) expands if k increases, i.e., $Q_0 \subset Q_1 \subset Q_2 \subset Q_3$. So, to get a greater region of mean square integrability (or asymptotic stability), one can use the method of Lyapunov functionals construction for $k = 4, k = 5$, etc. However, it is easy to see that using condition (2.25) it is impossible to get a region of mean square integrability for equation (3.14) (or the region of asymptotic stability of the trivial solution of equation (3.20)) if $|b| \geq 1$. For comparison on Figure 1, the exact region of asymptotic stability of the trivial solution of equation (3.20) is shown (curve number 5) obtained in [13] by virtue of some another stability condition. In Figure 2, the solution of equation (3.20) with the initial condition $x_0 = 0.5$ is shown in the point M of the region of asymptotic stability (Figure 1) with coordinates $a = 0.63, b = -2.5$.

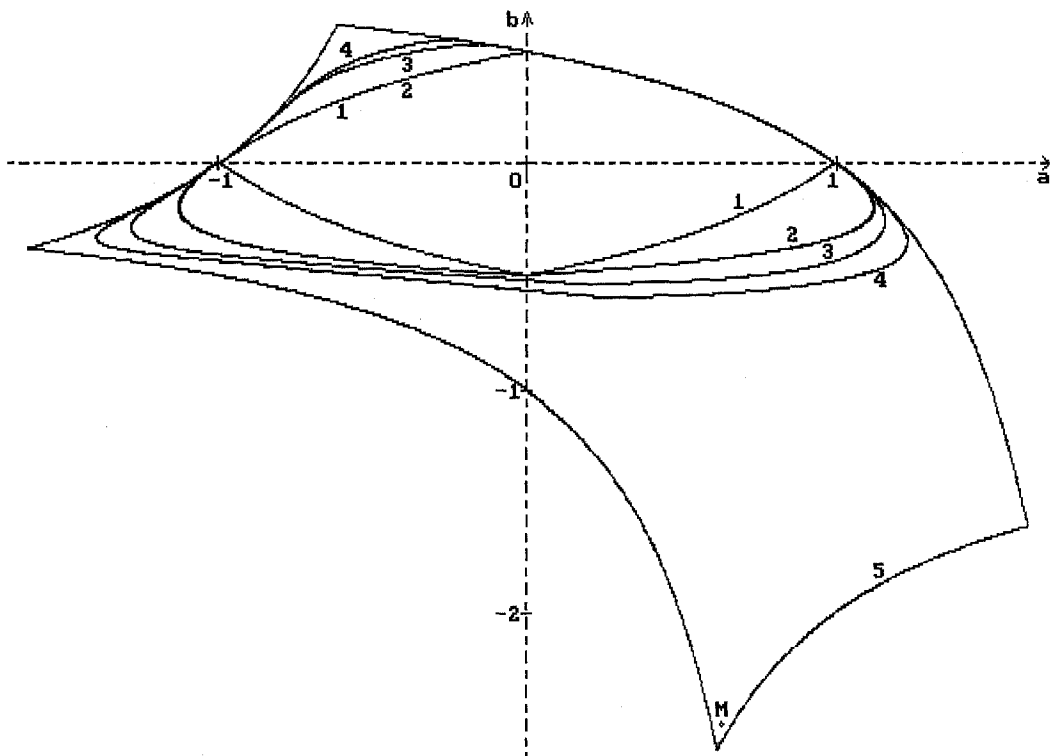


Figure 1.

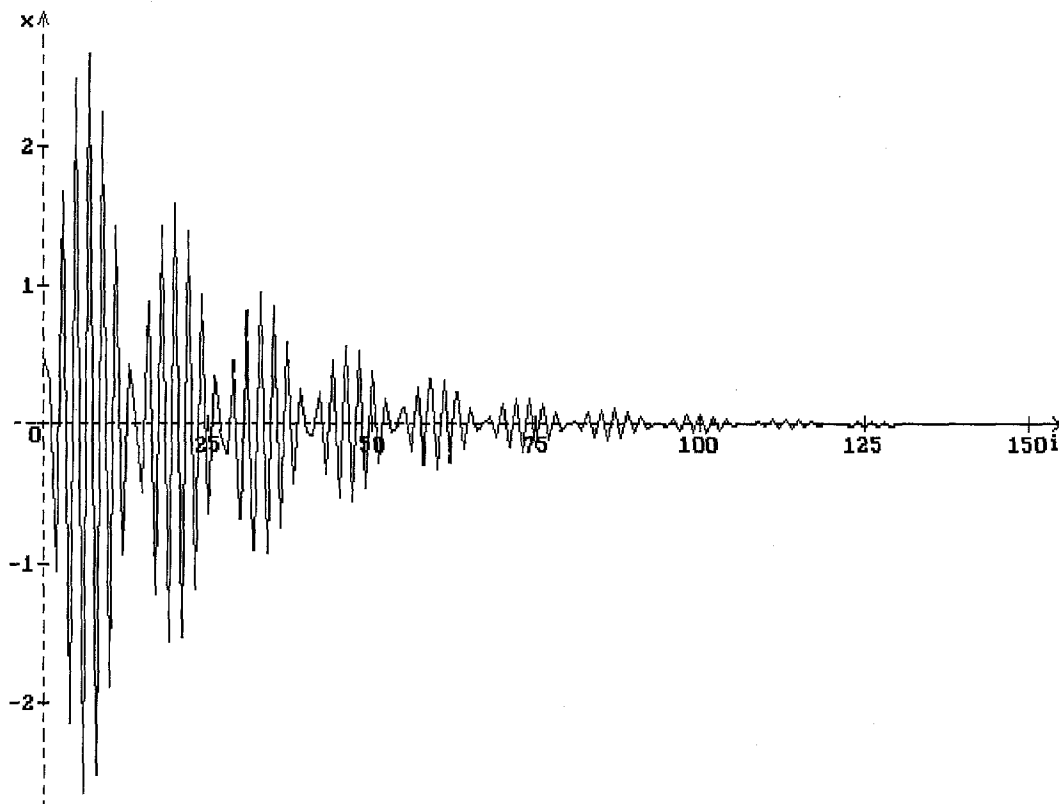


Figure 2.

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