Multiplicity results for second-order $m$-point boundary value problem

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Abstract

By using fixed-point theorems, some new results for multiplicity of positive solutions for a class of second-order $m$-point boundary value problem are obtained. The associated Green’s function of the problem is also given.

Keywords: Second-order $m$-point boundary value problem; Positive solution; Cone; Fixed point

1. Introduction

Recently, the existence and multiplicity of positive solutions for nonlinear ordinary differential equations and difference equations have been studied extensively. To identify a few, we refer the reader to [1–6] and references therein. However, many works about positive solutions were done under the assumption that the first-order derivative is not involved explicitly in the nonlinear term. On the other hand, there are very few works considering the multiplicity of positive solutions with dependence on the first-order derivative.

Recently Bai [7] has obtained sufficient conditions for existence of at least three positive solutions for the equation

$$x''(t) + a(t)f(t, x, x') = 0$$

(1.1)
subject to one of the following two pairs of boundary conditions:

\[ x(0) = x(1) = 0 \quad \text{or} \quad x(0) = x'(1) = 0. \] (1.2)

In the article the authors depended on a fixed-point theorem due to Avery and Peterson [8], which can be regarded as an extension of the Leggett and Williams [9] fixed-point theorem in cone.

Bai [10] gave a new fixed-point theorem in cone which also can be regarded as a generalization of Leggett–Williams fixed-point theorem and applied the results to deal with problem

\[ x''(t) + f(t, x, x') = 0, \] (1.3)
\[ x(0) = x(1) = 0. \] (1.4)

Motivated by all the works above, we concentrate in getting three positive solutions for second-order \( m \)-point boundary value problem

\[ x''(t) + f(t, x, x') = 0, \quad 0 < t < 1, \] (1.5)
\[ x'(0) = \sum_{i=1}^{m-2} \alpha_i x' (\xi_i), \quad x(1) = \sum_{i=1}^{m-2} \beta_i x (\xi_i), \] (1.6)

where \( 0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1 \). In this article it is assumed that:

\begin{itemize}
  \item \((C_1)\) \( f \in C([0, 1] \times [0, +\infty) \times R, [0, +\infty)) \);
  \item \((C_2)\) \( \alpha_i \geq 0, \beta_i \geq 0, i = 1, 2, \ldots, m - 2 \) satisfying \( 0 < \sum_{i=1}^{m-2} \alpha_i < 1 \) and \( 0 < \sum_{i=1}^{m-2} \beta_i < 1 \).
\end{itemize}

Remark 1.1. We extend the main results of [10] and other results about positive solutions of second-order boundary value problems. Further, we give the Green’s function of this problem first, which simplifies the discussion greatly.

2. Background

For the convenience of the reader, we present here the necessary definitions from cone theory in Banach spaces. These definitions can be found in the recent literature.

Definition 2.1. Let \( E \) be a real Banach space over \( R \). A nonempty convex closed set \( P \subset E \) is said to be a cone provided that

\begin{itemize}
  \item (i) \( au \in P \), for all \( u \in P, a \geq 0 \);
  \item (ii) \( u, -u \in P \) implies \( u = 0 \).
\end{itemize}

Note that every one cone \( P \subset E \) induces an ordering in \( E \) given by \( x \leq y \) if \( y - x \in P \).

Definition 2.2. An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

Definition 2.3. The map \( \alpha \) is said to be a nonnegative continuous convex functional on cone \( P \) of a real Banach space \( E \) provided that \( \alpha : P \to [0, +\infty) \) is continuous and

\[ \alpha(tx + (1-t)y) \leq t\alpha(x) + (1-t)\alpha(y), \quad \text{for all} \ x, y \in P \ \text{and} \ t \in [0, 1]. \]
Definition 2.4. The map $\beta$ is said to be a nonnegative continuous concave functional on cone $P$ of a real Banach space $E$ provided that $\beta : P \to [0, +\infty)$ is continuous and

$$
\beta(tx + (1-t)y) \geq t\beta(x) + (1-t)\beta(y), \quad \text{for all } x, y \in P \text{ and } t \in [0, 1].
$$

3. Preliminaries

In this section we give some lemmas which are useful in the proof of our main results. Firstly we introduce a fixed-point theorem due to Avery and Peterson [8].

Let $\gamma, \theta$ be nonnegative continuous convex functionals on $P$, $\alpha$ be a nonnegative continuous concave functional on $P$ and $\psi$ be a nonnegative continuous functional on $P$. Then for positive numbers $a, b, c$ and $d$, we define the following convex sets:

$$
P(\gamma, d) = \{ x \in P \mid \gamma(x) < d \},
$$

$$
P(\gamma, \alpha, b, d) = \{ x \in P \mid b \leq \alpha(x), \gamma(x) \leq d \},
$$

$$
P(\gamma, \theta, \alpha, b, c, d) = \{ x \in P \mid b \leq \alpha(x), \theta(x) \leq c, \gamma(x) \leq d \}
$$

and a closed set

$$
R(\gamma, \psi, a, d) = \{ x \in P \mid a \leq \psi(x), \gamma(x) \leq d \}.
$$

Lemma 1. Let $P$ be a cone in a real Banach space $E$. Let $\gamma, \theta$ be nonnegative continuous convex functionals on $P$, $\alpha$ be a nonnegative continuous concave functional on $P$ and $\psi$ be a nonnegative continuous functional on $P$ satisfying:

$$
\psi(\lambda x) \leq \lambda \psi(x), \quad \text{for } 0 \leq \lambda \leq 1,
$$

such that for some positive numbers $l$ and $d$,

$$
\alpha(x) \leq \psi(x), \quad \|x\| \leq l\gamma(x),
$$

for all $x \in \overline{P(\gamma, d)}$. Suppose $T : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$ is completely continuous and there exist positive numbers $a, b, c$ with $a < b$ such that

$(S_1)$ $\{ x \in P(\gamma, \theta, \alpha, b, c, d) \mid \alpha(x) > b \} \neq \emptyset$ and $\alpha(Tx) > b$ for $x \in P(\gamma, \theta, \alpha, b, c, d)$;

$(S_2)$ $\alpha(Tx) > b$ for $x \in P(\gamma, \alpha, b, d)$ with $\theta(Tx) > c$;

$(S_3)$ $0 \notin R(\gamma, \psi, a, d)$ and $\psi(Tx) < a$ for $x \in R(\gamma, \psi, a, d)$ with $\psi(x) = a$.

Then $T$ has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma, d)}$ such that

$$
\gamma(x_i) \leq d, \quad i = 1, 2, 3; \quad b < \alpha(x_1); \quad a < \psi(x_2), \quad \alpha(x_2) < b; \quad \psi(x_3) < a.
$$

Then we give another fixed-point theorem which Bai established in [10].

Suppose $\gamma, \theta : P \to [0, +\infty)$ are two nonnegative continuous convex functionals satisfying

$$
\|x\| \leq k \max \{ \gamma(x), \theta(x) \}, \quad \text{for } x \in P,
$$

where $k$ is a positive constant and

$$
\Omega = \{ x \in P \mid \gamma(x) < L, \theta(x) < r \} \neq \emptyset, \quad \text{for } r > 0, \ L > 0.
$$
Let \( r > a > 0, \ L > 0 \) be given, \( \gamma, \theta : P \to [0, +\infty) \) be two nonnegative continuous convex functionals satisfying (3.4), (3.5) and \( \alpha \) be a nonnegative continuous concave functional on the cone \( P \). Define the bounded convex sets

\[
P(\gamma, L; \theta) = \{ x \in P \mid \gamma(x) < L, \ \theta(x) < r \},
\]

\[
\overline{P}(\gamma, L; \theta) = \{ x \in P \mid \gamma(x) \leq L, \ \theta(x) \leq r \},
\]

\[
P(\gamma, L; \theta, r; \alpha, a) = \{ x \in P \mid \gamma(x) < L, \ \theta(x) < r, \ \alpha(x) > a \},
\]

\[
\overline{P}(\gamma, L; \theta, r; \alpha, a) = \{ x \in P \mid \gamma(x) \leq L, \ \theta(x) \leq r, \ \alpha(x) \geq a \}.
\]

**Lemma 2.** Let \( E \) be a Banach space, \( P \subset E \) be a cone and \( r_2 \geq c > b > r_1 > 0, \ L_2 \geq L_1 > 0 \) be given. Assume that \( \gamma, \theta \) are nonnegative continuous convex functionals on \( P \) such that (3.4), (3.5) are satisfied. \( \alpha \) is a nonnegative continuous concave functional on \( P \) such that \( \alpha(x) \leq \theta(x) \) for all \( x \in \overline{P}(\gamma, L_2; \theta, r_2) \) and let \( T : \overline{P}(\gamma, L_2; \theta, r_2) \to \overline{P}(\gamma, L_2; \theta, r_2) \) be a completely continuous operator. Suppose

\[
(S_4) \ \{ x \in \overline{P}(\gamma, L_2; \theta, c; \alpha, b) \mid \alpha(x) > b \} \neq \emptyset \text{ and } \alpha(Tx) > b \text{ for } x \in \overline{P}(\gamma, L_2; \theta, c; \alpha, b);
\]

\[
(S_5) \ \gamma(Tx) < L_1, \ \theta(Tx) < r_1 \text{ for all } x \in \overline{P}(\gamma, L_1; \theta, r_1);
\]

\[
(S_6) \ \alpha(Tx) > b, \text{ for all } x \in \overline{P}(\gamma, L_2; \theta, r_2; \alpha, b) \text{ with } \theta(Tx) > c.
\]

Then \( T \) has at least three fixed points \( x_1, x_2, x_3 \) in \( \overline{P}(\gamma, L_2; \theta, r_2) \). Further,

\[
x_1 \in P(\gamma, L_1; \theta, r_1); \quad x_2 \in \{ \overline{P}(\gamma, L_2; \theta, r_2; \alpha, b) \mid \alpha(x) > b \}
\]

and

\[
x_3 \in \overline{P}(\gamma, L_2; \theta, r_2) \setminus (\overline{P}(\gamma, L_2; \theta, r_2; \alpha, b) \cup \overline{P}(\gamma, L_1; \theta, r_1)).
\]

**Lemma 3.** Denote \( \xi_0 = 0, \ \xi_{m-1} = 1, \ \alpha_0 = \alpha_{m-1} = \beta_0 = \beta_{m-1} = 0, \ \rho = (1 - \sum_{i=0}^{m-1} \alpha_i) \times (1 - \sum_{i=0}^{m-1} \beta_i) \), then the boundary value problem

\[
x'' + y(t) = 0,
\]

\[
x'(0) = \sum_{i=1}^{m-2} \alpha_i x'(\xi_i), \ x(1) = \sum_{i=1}^{m-2} \beta_i x(\xi_i)
\]

has a solution \( x(t) = \int_0^1 G(t, s) y(s) \, ds \), where

\[
G(t, s) = \frac{1}{\rho} \left\{ \begin{array}{l}
\sum_{k=0}^{m-1} \alpha_k \left[ (s - t) + \sum_{k=0}^{m-1} \beta_k (t - s) + \sum_{k=i}^{i-1} \beta_k (t - \xi_k) \right] \\
+ \left( 1 - \sum_{k=0}^{m-1} \alpha_k \right) \left[ (1 - s) + \sum_{k=i}^{m-1} \beta_k (s - \xi_k) \right], \quad t \leq s,
\end{array} \right.
\]

\[
\left( 1 - \sum_{k=0}^{m-1} \alpha_k \right) \left[ (1 - t) + \sum_{k=0}^{m-1} \beta_k (t - s) + \sum_{k=i}^{m-1} \beta_k (t - \xi_k) \right] \\
+ \sum_{k=i}^{m-1} \alpha_k \sum_{k=0}^{m-1} \beta_k (s - \xi_k), \quad t \geq s,
\]

for \( \xi_{i-1} < s < \xi_i, \ i = 1, 2, \ldots, m - 1 \).

**Proof.** Suppose \( \overline{G}(t, s) \) is the Green’s function of problem (3.6), (3.7). For \( \xi_{i-1} < s < \xi_i, \ i = 1, 2, \ldots, m - 1 \), we let

\[
\overline{G}(t, s) = \begin{cases} 
A + Bt, & t \leq s, \\
C + Dt, & t \geq s.
\end{cases}
\]
Considering the definition and properties of Green’s function together with (3.7), we have

\[
\begin{align*}
A + Bs &= C + Ds, \\
B - D &= -1, \\
B &= \sum_{k=0}^{i-1} \alpha_k B + \sum_{k=i}^{m-1} \alpha_k D, \\
C + D &= \sum_{k=0}^{i-1} \beta_k (A + B\xi_k) + \sum_{k=i}^{m-1} \beta_k (C + D\xi_k).
\end{align*}
\]

We get

\[
A = \frac{1}{\rho} \left[ \sum_{k=i}^{m-1} \alpha_k \left( \sum_{k=0}^{m-1} \beta_k \xi_k - 1 \right) \right. + \left. \left( 1 - \sum_{k=i}^{m-1} \alpha_k \right) \left( s - 1 + \sum_{k=i}^{m-1} \beta_k (\xi_k - s) \right) \right],
\]

\[
B = \frac{\sum_{k=i}^{m-1} \alpha_k}{1 - \sum_{k=0}^{m-1} \alpha_k},
\]

\[
C = \frac{1}{\rho} \left[ (1 - \sum_{k=0}^{i-1} \alpha_k) \left( \sum_{k=0}^{m-1} \beta_k s + \sum_{k=i}^{m-1} \beta_k \xi_k - 1 \right) - \sum_{k=i}^{m-1} \beta_k (s - \xi_k) \right],
\]

\[
D = \frac{1 - \sum_{k=0}^{i-1} \alpha_k}{1 - \sum_{k=0}^{m-1} \alpha_k},
\]

so

\[
\mathcal{G}(t, s) = \frac{1}{\rho} \left\{ \sum_{k=i}^{m-1} \alpha_k \left[ (t - s) + \sum_{k=0}^{m-1} \beta_k (s - t) + \sum_{k=0}^{i-1} \beta_k (\xi_k - t) \right] \\
+ \left( 1 - \sum_{k=0}^{i-1} \alpha_k \right) \left[ (s - 1) + \sum_{k=i}^{m-1} \beta_k (\xi_k - s) \right], \quad t \leq s,
\right.
\]

\[
\left. \left( 1 - \sum_{k=0}^{i-1} \alpha_k \right) \left[ (t - 1) + \sum_{k=0}^{i-1} \beta_k (s - t) + \sum_{k=i}^{m-1} \beta_k (\xi_k - t) \right] \\
+ \sum_{k=i}^{m-1} \alpha_k \sum_{k=0}^{i-1} \beta_k (\xi_k - s), \quad t \geq s. \right\}
\]

We show the expression of the Green’s function for problem (3.6), (3.7).

Let \( G(t, s) = -\mathcal{G}(t, s) \), the solution of boundary value problem (3.6), (3.7) is

\[
x(t) = \int_0^1 \mathcal{G}(t, s)(-y(s)) \, ds = \int_0^1 G(t, s)y(s) \, ds.
\]

**Lemma 4.** If condition \( (C_2) \) holds, we claim \( G(t, s) \geq 0, t, s \in [0, 1] \).

**Proof.** For \( \xi_{i-1} \leq s \leq \xi_i, i = 1, 2, \ldots, m - 1 \), if \( t \leq s \),

\[
(s - t) + \sum_{k=0}^{i-1} \beta_k (t - \xi_k) + \sum_{k=i}^{m-1} \beta_k (t - s)
\]

\[
\geq \sum_{k=0}^{m-1} \beta_k (s - t) + \sum_{k=0}^{i-1} \beta_k (t - \xi_k) + \sum_{k=i}^{m-1} \beta_k (t - s) = \sum_{k=0}^{i-1} \beta_k (s - \xi_k) \geq 0,
\]

\[
(1 - s) + \sum_{k=i}^{m-1} \beta_k (s - \xi_k) \geq \sum_{k=i}^{m-1} \beta_k (1 - \xi_k) \geq 0.
\]
If $t \geq s$,
\[
(1 - t) + \sum_{k=0}^{i-1} \beta_k (t - s) + \sum_{k=i}^{m-1} \beta_k (t - \xi_k) \geq \sum_{k=0}^{i-1} \beta_k (1 - s) + \sum_{k=i}^{m-1} \beta_k (1 - \xi_k) \geq 0,
\]
then $G(t, s) \geq 0$, $t, s \in [0, 1]$. □

Let $X = C[0, 1]$ be endowed with the ordering $x \leq y$ if $x(t) \leq y(t)$ for all $t \in [0, 1]$ and the norm
\[
\|x\| = \max \left\{ \max_{0 \leq t \leq 1} |x(t)|, \max_{0 \leq t \leq 1} |x'(t)| \right\}.
\]
From $x''(t) = -f(t, x, x') \leq 0$, we know that $x$ is concave on $[0, 1]$. We define the cone $P \subset X$ by
\[
P = \left\{ x \in X \middle| x(t) \geq 0, x'(0) = \sum_{i=1}^{m-2} \alpha_i x'(\xi_i), x(1) = \sum_{i=1}^{m-2} \beta_i x(\xi_i), x(t) \text{ is concave on } [0, 1] \right\} \subset X.
\]

**Lemma 5.** If $x \in P$, then $\max_{0 \leq t \leq 1} |x(t)| \leq l \max_{0 \leq t \leq 1} |x'(t)|$, where
\[
l = \frac{1 - \sum_{i=1}^{m-2} \beta_i \xi_i}{1 - \sum_{i=1}^{m-2} \beta_i} > 1
\]
is a constant.

**Proof.** From the concavity of $x(t)$, we claim $x'(0) \leq 0$. Otherwise, if $x'(0) > 0$, we have
\[
0 = x'(0) - x'(0) = x'(0) - \sum_{i=1}^{m-2} \alpha_i x'(\xi_i) > \sum_{i=1}^{m-2} \alpha_i (x'(0) - x'(\xi_i)) \geq 0,
\]
a contradiction. So $\max_{0 \leq t \leq 1} |x'(t)| = |x'(1)|$ and $\max_{0 \leq t \leq 1} |x(t)| = x(0)$.

Now we show $x(0) \leq l |x'(1)|$. From the concavity of $x(t)$, we have $\xi_i (x(1) - x(0)) \leq x(\xi_i) - x(0)$.

Multiplying both sides with $\beta_i$ and considering $x(1) = \sum_{i=1}^{m-2} \beta_i x(\xi_i)$, we have
\[
x(1) \geq \sum_{i=1}^{m-2} \frac{\beta_i (1 - \xi_i)}{1 - \sum_{i=1}^{m-2} \beta_i \xi_i} x(0).
\]
Considering the mean-value theorem, we get
\[
x(1) - x(\xi_i) = (1 - \xi_i)x'(\eta_i), \quad \eta \in (\xi_i, 1).
\]
From the concavity of $x$ similarly with above we know
\[
x(1) < \frac{\sum_{i=1}^{m-2} \beta_i (1 - \xi_i)}{1 - \sum_{i=1}^{m-2} \beta_i} |x'(1)|.
\]
Considering (3.9) together with (3.10) we have \( x(0) \leq l |x'(1)|\), where
\[
l = \frac{1 - \sum_{i=1}^{m-2} \beta_i \xi_i}{1 - \sum_{i=1}^{m-2} \beta_i}.
\]

**Lemma 6.** If \( x \in P \), we have
\[
\min_{\xi_{j-1} \leq t \leq \xi_j} x(t) \geq \delta \|x(t)\|,
\]
where \( \xi_j \in \{\xi_1, \xi_2, \ldots, \xi_{m-1}\} \), \( \delta = \frac{1 - \sum_{i=1}^{m-2} \beta_i \xi_i + \xi_j (\sum_{i=1}^{m-2} \beta_i - 1)}{1 - \sum_{i=1}^{m-2} \beta_i \xi_i} \) < 1 is a constant.

**Proof.** From Lemma 5 and the concavity of \( x(t) \), we have
\[
\min_{\xi_{j-1} \leq t \leq \xi_j} = x(\xi_j) \quad \text{and} \quad x(\xi_j) \geq (1 - \xi_j) x(0) + \xi_j x(1).
\]
Considering (3.9), we get the conclusion of Lemma 6. \( \square \)

### 4. Existence of three positive solutions

In this section, we impose growth conditions on \( f \) which allow us to apply Lemma 1 to establish the existence of three positive solutions of problem (1.5), (1.6).

Let the nonnegative continuous concave functional \( \alpha \), the nonnegative continuous convex functionals \( \theta \) and the nonnegative continuous functional \( \psi \) be defined on the cone by
\[
\gamma(x) = \max_{0 \leq t \leq 1} |x'(t)|, \quad \theta(x) = \psi(x) = \max_{0 \leq t \leq 1} |x(t)|, \quad \alpha(x) = \min_{\xi_{j-1} \leq t \leq \xi_j} |x(t)|.
\]
By Lemmas 5 and 6 the functionals defined above satisfy:
\[
\delta \theta(x) \leq \alpha(x) \leq \theta(x) = \psi(x), \quad \|x\| = \max\{\theta(x), \gamma(x)\} \leq l \gamma(x).
\]
Therefore condition (3.2) is satisfied.

Let
\[
m = \frac{1}{\rho} \sum_{k=j}^{m-1} \alpha_k \sum_{k=0}^{m-1} \beta_k (1 - \xi_k)(\xi_j - \xi_{j-1}), \quad M = \frac{1}{1 - \sum_{i=0}^{m-1} \alpha_i},
\]
\[
N = \max_{0 \leq t \leq 1} \int_0^1 G(t, s) \, ds, \quad \lambda = \min \left\{ \frac{m}{M}, \delta l \right\}.
\]
To present our main results, we assume there exist constants \( 0 < a, b, c, d \) with \( a < b < \lambda d \) such that
\[
(A_1) \quad f(t, u, v) \leq d/M (t, u, v) \in [0, 1] \times [0, ld] \times [-d, d];
\]
\[
(A_2) \quad f(t, u, v) > b/m (t, u, v) \in [\xi_{j-1}, \xi_j] \times [b, b/\delta] \times [-d, d];
\]
\[
(A_3) \quad f(t, u, v) < a/N (t, u, v) \in [0, 1] \times [0, a] \times [-d, d].
\]

**Theorem 1.** Under assumptions \((A_1)-(A_3)\), the boundary value problem (1.5)–(1.6) has at least three positive solutions \( x_1, x_2, x_3 \) satisfying
\[
\begin{align*}
\max_{0 \leq t \leq 1} |x'_i(t)w| & \leq d, \quad i = 1, 2, 3; \\
b & < \min_{\xi_{j-1} \leq t \leq \xi_j} |x_1(t)|; \\
a & < \max_{0 \leq t \leq 1} |x_2(t)|, \quad \min_{\xi_{j-1} \leq t \leq \xi_j} |x_2(t)| < b; \\
\max_{0 \leq t \leq 1} |x_3(t)| & \leq a. \tag{4.2}
\end{align*}
\]

**Proof.** Problem (1.5)–(1.6) has a solution \(x = x(t)\) if and only if \(x\) solves the operator equation

\[
x(t) = \int_0^1 G(t, s) f(s, x, x') \, ds = (Tx)(t). \tag{4.3}
\]

From (4.3) it is easy to get

\[
(Tx)'(t) = \int_0^t -f(s, x, x') \, ds - \frac{\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} f(s, x, x') \, ds}{1 - \sum_{i=1}^{m-2} \alpha_i}. \tag{4.4}
\]

If \(x \in \overline{P(\gamma, d)}\), then \(\gamma(x) = \max_{0 \leq t \leq 1} |x'(t)| \leq d\). With Lemma 5, assumption \((A_1)\) implies \(f(t, x, x') \leq d/M\).

On the other hand, for \(x \in P\),

\[
\gamma(Tx) = |(Tx)'(1)| = \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i} \left( \int_0^1 f(s, x, x') \, ds - \sum_{i=1}^{m-2} \alpha_i \int_{\xi_i} f(s, x, x') \, ds \right) \leq \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i} \int_0^1 f(s, x, x') \, ds \leq \frac{d}{M} = d.
\]

Hence \(T : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}\) and \(T\) is a completely continuous operator obviously.

To check condition \((S_1)\) of Lemma 1, we choose \(x(t) = b/\delta = c\). It is easy to see \(x(t) = b/\delta \in P(\gamma, \theta, \alpha, b, c, d)\) and \(\alpha(b/\delta) > b\). So \(\{x \in P(\gamma, \theta, \alpha, b, c, d) \mid \alpha(x) > b\}\) \(\neq \emptyset\).

If \(x \in P(\gamma, \theta, \alpha, b, c, d)\), we have \(b \leq x(t) \leq b/\delta, \ |x'(t)| < d\) for \(\xi_{j-1} \leq t \leq \xi_j\). From assumption \((A_2)\), we have

\[
f(t, x, x') \geq \frac{b}{m}.
\]

By definition of \(\alpha\) and the cone \(P\),

\[
\alpha(Tx) = (Tx)(\xi_j) = \int_0^{\xi_j} G(\xi_j, s) f(s, x, x') \, ds \geq \int_{\xi_{j-1}}^{\xi_j} G(\xi_j, s) f(s, x, x') \, ds
\]

\[
> \frac{b}{m} \int_{\xi_{j-1}}^{\xi_j} G(\xi_j, s) \, ds > \frac{b}{m} = b,
\]

so \(\alpha(Tx) > b, \forall x \in P(\gamma, \theta, \alpha, b, b/\delta, d)\).

Second, with (4.1) and \(b < \lambda d\), we have \(\alpha(Tx) \geq \delta \theta(Tx) > \delta b/\delta = b\) for all \(x \in P(\gamma, \alpha, b, d)\) with \(\theta(Tx) > b/\delta\).

Thus, condition \((S_2)\) of Lemma 1 is satisfied. Finally, we show that \((S_3)\) also holds.
Clearly, as \( \psi(0) = 0 < a \), we see \( 0 \notin R(\gamma, \psi, a, d) \). Suppose that \( x \in R(\gamma, \psi, a, d) \) with \( \psi(x) = a \), then by the assumption of \((A_3)\),

\[
\psi(Tx) = \max_{0 \leq t \leq 1} |(Tx)(t)| = \max_{0 \leq t \leq 1} \int_0^1 G(t, s) f(s, x, x') ds
\]

\[
< \frac{a}{N} \max_{0 \leq t \leq 1} \int_0^1 G(t, s) ds = a. \tag{4.5}
\]

So condition \((S_3)\) is satisfied. Thus, an application of Lemma 1 implies the boundary value problem \((1.5)-(1.6)\) has at least three positive solutions \(x_1, x_2, x_3\) satisfying (4.2). The proof is completed.

**Remark 4.1.** To apply Lemma 1, we only need \( T : P(\gamma, d) \to P(\gamma, d) \), therefore condition \((C_1)\) can be substituted with a weaker condition

\[(H_1) \quad f \in C([0, 1] \times [0, ld] \times [-d, d], [0, +\infty)).\]

Next we give another theorem which is an application of Lemma 2 in dealing with problem \((1.5)-(1.6)\).

**Theorem 2.** Suppose there exist constants \( r_2 \geq \frac{1}{3} b > b > r_1 > 0, L_2 \geq L_1 \geq 0 \) such that \( \frac{b}{m} \leq \min\{\frac{r_2}{N}, \frac{L_2}{M}\} \) and the following assumptions hold:

\[(A_4) \quad f(t, u, v) < \min\{\frac{r_2}{N}, \frac{L_1}{M}\}, \text{for } (t, u, v) \in [0, 1] \times [0, r_1] \times [-L_1, L_1];\]

\[(A_5) \quad f(t, u, v) > b/m, \text{for } (t, u, v) \in [\xi_{j-1}, \xi_j] \times [b, \frac{1}{3} b] \times [-L_2, L_2];\]

\[(A_6) \quad f(t, u, v) \leq \min\{\frac{r_2}{N}, \frac{L_2}{M}\}, \text{for } (t, u, v) \in [0, 1] \times [0, r_2] \times [-L_2, L_2].\]

Then boundary value problem \((1.5)-(1.6)\) has at least three positive solutions \(x_1, x_2, x_3\) satisfying

\[
\max_{0 \leq t \leq 1} x_1(t) \leq r_1, \quad \max_{0 \leq t \leq 1} |x_1'(t)| \leq L_1;
\]

\[
b < \min_{\xi_{j-1} \leq t \leq \xi_j} x_2(t) \leq \max_{0 \leq t \leq 1} x_2(t) \leq r_2, \quad \max_{0 \leq t \leq 1} |x_2'(t)| \leq L_2;
\]

\[
\max_{0 \leq t \leq 1} x_3(t) \leq \frac{b}{3}, \quad \max_{0 \leq t \leq 1} |x_3'(t)| \leq L_2. \tag{4.6}
\]

**Proof.** We define the cone \( P \) and functionals \( \alpha, \beta, \gamma \) as in Theorem 1 and also consider Eq. (4.3).

If \( x \in P(\gamma, L_2; \theta, r_2) \), assumption \((A_6)\) and (4.4), (4.5), imply that \( T : \overline{P}(\gamma, L_2; \theta, r_2) \to \overline{P}(\gamma, L_2; \theta, r_2) \) is a completely continuous operator.

We can check that conditions \((S_4), (S_5), (S_6)\) are satisfied similarly with the work we have done in this section above. Therefore boundary value problem \((1.5)-(1.6)\) has at least three positive solutions and (4.6) is satisfied. □
Remark 4.2. To apply Lemma 2, we only need \( T: \overline{P}(\gamma, L_{2}; \theta, r_{2}) \rightarrow \overline{P}(\gamma, L_{2}; \theta, r_{2}) \), therefore condition \((C_{1})\) can be substituted with a weaker condition

\[(H_{2}) \quad f \in C([0, 1] \times [0, r_{2}] \times [-L_{2}, L_{2}], [0, +\infty)). \]

5. Example

Finally we present an example to check our main results. Consider the boundary value problem

\[
x''(t) + f(t, x, x') = 0, \quad 0 \leq t \leq 1,
\]

\[
x'(0) = \frac{1}{4} x'(\frac{1}{3}) + \frac{1}{2} x'(\frac{2}{3}), \quad x(1) = \frac{1}{3} x'(\frac{1}{3}) + \frac{1}{2} x'(\frac{2}{3}),
\]

where

\[
f(t, u, v) = \begin{cases} \frac{1}{60} e^{t} + \frac{1}{10} u^{3} + \frac{1}{60} (\frac{v}{1200})^{3}, & 0 \leq u \leq 10, \\ \frac{1}{60} e^{t} + 100 + \frac{1}{60} (\frac{v}{1200})^{3}, & u \geq 10. \end{cases}
\]

Choose \( a = 1, b = 4, d = 1200, \xi_{j-1} = \frac{1}{3}, \xi_{j} = \frac{2}{3} \), we note that \( M = 4, m = \frac{14}{9}, l = \frac{10}{3}, \delta = \frac{4}{3}, \lambda = \frac{7}{18}, N = \frac{16}{3} \). We can check that conditions \((C_{2}), (H_{1})\) are satisfied and \( f(t, u, v) \) satisfies

\[
f(t, u, v) \leq 300, \quad (t, u, v) \in [0, 1] \times [0, 4000] \times [-1200, 1200];
\]

\[
f(t, u, v) \geq \frac{18}{7}, \quad (t, u, v) \in \left[ \frac{1}{3}, \frac{2}{3} \right] \times [4, 5] \times [-1200, 1200];
\]

\[
f(t, u, v) \leq \frac{3}{16}, \quad (t, u, v) \in [0, 1] \times [0, 1] \times [-1200, 1200].
\]

Then all assumptions of Theorem 1 are satisfied.

Thus problem (5.1)–(5.2) has at least three positive solutions \( x_{1}, x_{2}, x_{3} \) such that

\[
\max_{0 \leq t \leq 1} \left| x_{i}'(t) \right| \leq 1200, \quad i = 1, 2, 3; \quad \min_{\frac{1}{2} \leq t \leq \frac{2}{3}} x_{1}(t) > 4;
\]

\[
\max_{0 \leq t \leq 1} x_{2}(t) > 1, \quad \min_{\frac{1}{3} \leq t \leq \frac{2}{3}} x_{2}(t) < 4; \quad \max_{0 \leq t \leq 1} x_{3}(t) < 1.
\]

Now we consider boundary value problem (5.1)–(5.2) with Theorem 2.

Choose \( r_{1} = 1, r_{2} = 800, b = 4, L_{1} = 10, L_{2} = 1200 \), then \( \min \{ l_{N}, m_{M} \} = \frac{3}{16}, \frac{b}{m} = \frac{18}{7}, \)

\[
\min \{ \frac{r_{1}}{N}, \frac{r_{2}}{M} \} = 150.
\]

We can check that conditions \((C_{2}), (H_{2})\) are satisfied and \( f(t, u, v) \) satisfies

\[
f(t, u, v) < \frac{3}{16}, \quad \text{for } (t, u, v) \in [0, 1] \times [0, 1] \times [-10, 10];
\]

\[
f(t, u, v) > \frac{18}{7}, \quad \text{for } (t, u, v) \in \left[ \frac{1}{3}, \frac{2}{3} \right] \times [4, 5] \times [-1200, 1200];
\]

\[
f(t, u, v) < 150, \quad \text{for } (t, u, v) \in [0, 1] \times [0, 800] \times [-1200, 1200].
\]

Then all assumptions of Theorem 2 hold. Thus, problem (5.1)–(5.2) has at least three positive solutions \( x_{1}, x_{2}, x_{3} \) satisfying
\[
\max_{0 \leq t \leq 1} x_1(t) \leq 1, \quad \max_{0 \leq t \leq 1} |x_1'(t)| \leq 10;
\]
\[
4 < \min_{\frac{1}{3} \leq t \leq \frac{2}{3}} x_2(t) \leq \max_{0 \leq t \leq 1} x_2(t) \leq 800, \quad \max_{0 \leq t \leq 1} |x_2'(t)| \leq 1200;
\]
\[
\max_{0 \leq t \leq 1} x_3(t) \leq 5, \quad \max_{0 \leq t \leq 1} |x_3'(t)| \leq 1200.
\]

**Remark 5.1.** We see that the nonlinear term is involved in first-order derivative explicitly and can change the sign. The early results for multiplicity of positive solutions, to author’s best knowledge, are not applicable to the problem above. Meanwhile, as the nonlinear term does not satisfy the superlinear or sublinear condition, we cannot use the classic Krasnosel’skii fixed-point theorem to discuss even one positive solution of this problem, see [11].

**References**