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EQUATIONAL COMPLETION, MODEL INDUCED TRIPLES AND PRO-OBJECTS

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Introduction and preliminaries

We consider the following problem: Given a functor $I:\mathcal{M} \to \mathcal{S}$ (where $\mathcal{S}$ = sets), find the equational completion of I. That is to find an equational category $\mathcal{S}^{\mathbf{T}}$ together with a factorization of I through the underlying set functor (as $\mathcal{M} \to \mathcal{S}^{\mathbf{T}} \to \mathcal{S}$), which is "best possible" (so that if $\mathcal{M} \to \mathcal{S}^{\mathbf{T}_0} \to \mathcal{S}$ is another such factorization of I there exists a unique $U_0 : \mathcal{S}^{\mathbf{T}} \to \mathcal{S}^{\mathbf{T}_0}$ which makes everything commute). The existence (and a precise definition) of the equational completion if given by Linton [15] who shows the class of n-ary operations for the completion of I can be regarded as the class of all natural transformations from I^n to I where $n \in \mathcal{S}$. Thus the problem becomes, given $I:\mathcal{M} \to \mathcal{S}$, find some way of obtaining enough information about the natural transformations from I^n to I so that a reasonable description of $\mathcal{S}^{\mathbf{T}}$ can be given. A straightforward approach using the definition of a natural transformation is generally difficult because there often are many horribly infinitary operations for the equational completion. For example, if $\mathcal{M}$ = finite sets and $I:\mathcal{M} \to \mathcal{S}$ is the inclusion functor then a natural transformation from I^n to I corresponds to an ultrafilter on n (and the equational completion is the category of compact Hausdorff spaces). If $\mathcal{M}$ = fields the completion is the category of products of fields and continuous ring homomorphisms but the operations are difficult to des-

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cribe usefully. (We do not know how to describe the category of algebras that results if one considers only the finitary operations.) Another difficulty with, say, simply making a list of natural transformations from I^n to I for various n is that it is hard to tell when one has listed enough operations to determine $\mathcal{S}^{\mathbf{T}}$. One can frequently find enough operations to determine a subtheory $\mathbf{T}_0$ of $\mathbf{T}$ such that every $M \in \mathcal{M}$ has the structure of a $\mathbf{T}_0$-algebra, the $\mathbf{T}_0$-homomorphisms are precisely the admissible homomorphisms (in $\mathcal{S}^{\mathbf{T}}$) between members of $\mathcal{M}$, and such that the $\mathbf{T}_0$-subalgebras and finite products will exactly determine the behaviour of subobjects and finite products in $\mathcal{S}^{\mathbf{T}}$ of (lifted) members of $\mathcal{M}$. However $\mathbf{T}_0$ might still be much smaller than $\mathbf{T}$. (For example if $\mathcal{M}$ = finite groups, I the obvious underlying functor, then $\mathbf{T}_0$ = the theory of groups has the above properties, but the completion $\mathbf{T}$ is the theory of profinite groups.)

Nonetheless once such a subtheory $\mathbf{T}_0$ has been found (cf. the definitions of separating triple and of normal separating triple and the procedure in Example 4.1) then one can often make effective use of a topological approach to $\mathcal{S}^{\mathbf{T}}$ based on triples and embed $\mathcal{S}^{\mathbf{T}}$ in the topological $\mathbf{T}_0$-algebras.

Fairly complete descriptions (Theorems 3.1, 3.2 and 3.3) of $\mathcal{S}^{\mathbf{T}}$ as a subcategory of topological $\mathbf{T}_0$-algebras are given if each $M \in \mathcal{M}$ is finite or satisfies a descending chain type of condition. In these cases $\mathcal{S}^{\mathbf{T}}$ is related to the Pro-objects for $\mathcal{M}$ (which are briefly reviewed in Section 2). Examples involving pseudocompact rings, modules and algebras and the case $\mathcal{M}$ = countable sets show that the topological approach works well in specific cases too.

The equational completions that we shall examine turn out to be varietal, hence arise from triples on $\mathcal{S}$. If $\mathcal{S}^{\mathbf{T}}$ is the completion of $I: \mathcal{M} \to \mathcal{S}$ we shall regard $\mathbf{T}$ as a triple (rather than as a varietal equational theory) in which case $\mathbf{T}$ is the model induced triple arising from $I: \mathcal{M} \to \mathcal{S}$. (This follows from the argument sketched below and from the results of [1] where the dual notion, of a model induced cotriple is defined. These triples are called codensity triples in [14] and their relationship to equational completions is there established in a general setting. Therefore this paper gives some techniques for computing examples of the categories of algebras discovered by Linton, Appelgate and Tierney when we consider relatively specific cases. Incidentally we shall also relate $\mathcal{S}^{\mathbf{T}}$ to a model induced triple over Top (= Topological spaces and maps).)

Let us recall that if $\mathbf{T} = (T, \eta, \mu)$ is the model induced triple for $I: \mathcal{M} \to \mathcal{S}$ then $T(n) = \lim\, [(n, I) \to \mathcal{S}\,]$ where $n \in \mathcal{S}$ and (n, I) is the comma category of all functions $n \to I(M)$ where M ranges over the class of models (i.e. objects of $\mathcal{M}$). The functor $(n, I) \to \mathcal{S}$ assigns $I(M)$ to the function $n \to I(M)$. (This limit exists iff $\mathbf{T}$ is well-defined iff I is tractable (i.e. the collection n.t. (I^n, I) is small) iff the equational completion is varietal. A practical test for $I: \mathcal{M} \to \mathcal{S}$ to be tractable is given in the discussion preceeding 1.3 below.) For each $g: n \to I(M)$ in (n, I) we let $\langle g\rangle: T(n) \Rightarrow I(M)$ be the corresponding projection. Then η and μ are defined by $\langle g\rangle\eta = g$ and $\langle g\rangle\mu = \langle\langle g\rangle\rangle$. If $f: n \to m$ then $T(f)$ is defined by $\langle g\rangle\, T(f) = \langle gf\rangle$. Moreover every natural transformation $\lambda: I^n \to I$ gives rise to $\bar{\lambda} \in T(n)$ where $\langle g\rangle(\bar{\lambda}) = \lambda_M(g)$ for

$g:n \to I(M)$, that is for $g \in I^n(M)$. Thus every such natural transformation λ gives rise to an n-ary operation $\bar{\lambda}$ of T. This correspondence in one-one and onto and illustrates why the model induced triple gives rise to the equational completion. (We shall generally speak of n-ary operations of triples when strictly speaking we mean an n-ary operation of the corresponding varietal theory.)

In general if $\mathbf{T}$ is a triple on $\mathcal{A}$ then the statement $(A, \theta) \in \mathcal{A}^{\mathbf{T}}$ or (A, θ) is a $\mathbf{T}$-algebras means that $A \in \mathcal{A}$ and $\theta : T(A) \to A$ is a structure map. The morphisms of $\mathcal{A}^{\mathbf{T}}$ shall be referred to as $\mathbf{T}$-homomorphisms or as morphisms of $\mathcal{A}$ which are admissible. If $\mathbf{T}_0$ is a triple over sets then a topological $\mathbf{T}_0$-algebra with a topology such that the n-ary operations are continuous, using the product topology.

We shall use the term "limit" to refer to generalized inverse limits (i.e. the left roots in [6]). For emphasis, limits in $\mathcal{S}^{\mathbf{T}}$ shall sometimes be called $\mathbf{T}$-limits. Limits in Top shall sometimes be referred to as Top-limits. We say that a small category D is *filtered* if $d, e \in D$ imply there exists $c \in D$ and morphisms from c to d and from c to e. Also if $f, g \in D(c, d)$ then there exists h with $fh = gh$. A *filtered diagram* is a functor whose domain is filtered and its limit is a *filtered limit*. (Aside from the use of contravariant functors in [2], this definition is effectively equivalent to the definition of filtered limit in [2].) If $\mathcal{A}$ is a category then $\mathcal{A}^{\mathrm{op}}$ is the dual category and $\mathcal{A}(X, Y)$ is the set of morphisms from X to Y. If X is an object of $\mathcal{A}$ then X also denotes the identity morphism in $\mathcal{A}(X, X)$. If $\mathcal{A}$ and $\mathcal{B}$ are categories then $(\mathcal{A}, \mathcal{B})$ denotes the possibly illegitimate category of functors from $\mathcal{A}$ to $\mathcal{B}$. A subcategory $\mathcal{A}$ of $\mathcal{B}$ is *reflective* if the inclusion functor $\mathcal{A} \to \mathcal{B}$ has a left adjoint. (This follows Freyd [6].) A morphism is a *split epi* if it has a right inverse. The term *quotient map* is always used in the *topological* sense. Parentheses in expressions such as $T(n)$ are sometimes omitted when they are not needed particularly in complicated formulas (when only the crucial parentheses are included).

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§1. Some topological observations

Notation. From here on $I:\mathcal{M} \to \mathcal{S}$ shall denote a tractable functor and $\mathbf{T} = (T, \eta, \mu)$ shall be the model induced triple on $\mathcal{S}$. The category of $\mathbf{T}$-algebras shall be $\mathcal{S}^{\mathbf{T}}$ with $U_{\mathbf{T}}:\mathcal{S}^{\mathbf{T}} \to \mathcal{S}$ and $F_{\mathbf{T}}:\mathcal{S} \to \mathcal{S}^{\mathbf{T}}$ the underlying and free functors.

Given $M \in \mathcal{M}$ then $\bar{I}(M) \in \mathcal{S}^{\mathbf{T}}$ shall be defined by the structure map $\langle TM \rangle$: $TIM \to IM$ (here IM denotes the identity map on IM -- this notation for identity map is used throughout). This defines the lifted model functor $\bar{I}: \mathcal{M} \to \mathcal{S}^{\mathbf{T}}$.

If there is no danger of confusion and if $M \in \mathcal{M}$ we shall also use M to denote $I(M) \in \mathcal{S}$ and $\bar{I}(M) \in \mathcal{S}^{\mathbf{T}}$.

The topology on a $\mathbf{T}$*-algebra.* Let $M \in \mathcal{M}$. Then $T(n)$ can be topologized by the smallest topology rendering each $\langle g \rangle : T(n) \to M$ continuous, when each M is given the discrete topology. (Thus the $\mathcal{S}$-limit, $T(n)$, is regarded as a topological limit of discrete spaces.)

If $(n, \theta) \in \mathcal{S}^{\mathbf{T}}$ (so that $\theta : T(n) \to n$ is the structure map) we define $Q(n, \theta)$ as the quotient topology induced by θ. Then $Q : \mathcal{S}^{\mathbf{T}} \to$ Top is easily seen to be a functor. Let U: Top $\to \mathcal{S}$ be the underlying functor and $D : \mathcal{S} \to$ Top the left adjoint, which assigns the discrete topology to a set. Let $I_D : \mathcal{M} \to$ Top be defined by $I_D = DI (= Q\bar{I})$ and let $\widetilde{\mathbf{T}} = (\widetilde{T}, \widetilde{\eta}, \widetilde{\mu})$ be the model induced triple on Top. Let $\widetilde{U} : \mathrm{Top}^{\widetilde{\mathbf{T}}} \to$ Top, $\widetilde{F}$: Top $\to \mathrm{Top}^{\widetilde{\mathbf{T}}}$, $\widetilde{I} : \mathcal{M} \to \mathrm{Top}^{\widetilde{\mathbf{T}}}$ be the obvious functors. For convenience, if $X \in$ Top and $g : X \to M$ is continuous (that is $g : X \to I_D(M)$) we let $[g] : \widetilde{T}(X) \to M$ be the corresponding projection which makes $\widetilde{T}(X)$ the limit of $(X, I_D) \to$ Top. There exists a comparison functor $\varphi : \mathrm{Top}^{\widetilde{\mathbf{T}}} \to \mathcal{S}^{\mathbf{T}}$ since the adjointness $FD \dashv U\widetilde{U}$ generates the triple $\mathbf{T}$. Finally, if $M \in \mathcal{M}$ we shall also use M to denote $I_D(M) \in$ Top and $\widetilde{I}(M) \in \mathrm{Top}^{\widetilde{\mathbf{T}}}$ (as well as $I(M) \in \mathcal{S}$ and $\bar{I}(M) \in \mathcal{S}^{\mathbf{T}}$) so long as the context makes the exact meaning clear. To summarize the above and to record some immediate properties of Q we state:

1.1. Proposition. (*We suggest drawing a diagram of the above categories and functors. For clarity the category* $\mathcal{M}$ *and the functors* $I, \bar{I}, \widetilde{I}_D$ *and* I_D *and* $\widetilde{I}$ *might be omitted.*)

(a) $U_{\mathbf{T}}\varphi = U\widetilde{U}$ *and* $\varphi\widetilde{F}D = F_{\mathbf{T}}$ *and* φ *preserves limits.*

(b) *Q preserves quotients (meaning that if* $f : A \to B$ *is an onto* $\mathbf{T}$*-homomorphism then* $Q(f)$ *is a quotient map in the topological sense).*

Q preserves the topology of $T(n)$ *(meaning that* $Q(T_n, \mu)$ *is the limit topology on Tn mentioned above).*

However, Q need not assign the relative topology to a $\mathbf{T}$*-subalgebra nor does Q generally preserve limits not is* $Q\varphi$ *necessarily equal to* $\widetilde{U}$. *We can say that* $Q(\lim A_i)$ *is at least as large (has at least as many open sets) as* Top-lim $Q(A_i)$, *that* $Q\varphi(\widetilde{X})$ *is at least as large in* $\widetilde{U}(\widetilde{X})$ *for* $\widetilde{X} \in \mathrm{Top}^{\widetilde{\mathbf{T}}}$ *and that* $Q(A)$ *is at least as large as the relative topology on A induced by* $Q(B)$ *if* $A \subseteq B$ *is a* $\mathbf{T}$*-subalgebra.*

(c) *If* $X \in$ Top *then* $\exists \lambda (= \lambda_X) : TUX \to \widetilde{T}X$ *defined by* $[g]\, \lambda = \langle Ug \rangle$. *Then* λ *is continuous and can be regarded as a natural transformation from* $QF_{\mathbf{T}}U \to \widetilde{T}$. *If* $(X, \theta) \in \mathrm{Top}^{\widetilde{\mathbf{T}}}$ *then* $\varphi(X, \theta) = (UX, \theta\, \lambda_X)$. *Moreover if X is discrete then* λ_X *is the identity.*

Proof. (a) follows from the construction of φ, see [4].

As for (b) we first note that $\mu : T^2(n) \to T(n)$ is a quotient map as it is split epi in Top with $T(\eta)$ as right inverse. Thus $Q(Tn, \mu)$ has the limit topology. Now if $f : (X, \theta) \to (Y, \psi)$ is onto and admissible then $f\theta = \psi\, Tf$ and Tf is a quotient map (in fact Tf is split epi in Top as f is split epi in $\mathcal{S}$). Thus $f\theta$ is a quotient map and θ is continuous, so f is a quotient map.

Next $Q(\lim A_i)$ has at least as many open sets as Top-lim $Q(A_i)$ as the projections are Q-continuous. Similarly the Q topology on A is at least as large as the relative topology as the inclusion $A \to \underset{\sim}{B}$ is Q-continuous.

Finally let $\widetilde{X} = (X, \theta) \in \mathrm{Top}^{\mathbf{T}}$. By (c), $\varphi(X, \theta) = (UX, \theta \lambda_X)$. Clearly $\theta \lambda_X$ is continuous from TUX to $\widetilde{U}\widetilde{X}$. Hence $Q\varphi(X, \theta)$, the space with the quotient topology induced by $\theta \lambda_X$ has at least as many open sets as $\widetilde{U}\widetilde{X}$.

(c) Since $\eta : UX \to TUX$ is a front adjunction there is a unique $\lambda : TUX \to \widetilde{T}X$ which is $\mathbf{T}$-admissible and satisfies $\lambda\eta = \widetilde{\eta}$. Note that $|g| \lambda = \langle Ug \rangle$ follows. Also $\theta\lambda$ is the structure map of (X, θ) as $\theta\lambda$ is $\mathbf{T}$-admissible and $\theta\lambda\eta = X$.

1.2. Proposition. *(Continuity of the operations.) Let $\mathbf{T}_0$ be any equational theory generated by some finitary operations of $\mathbf{T}$. Then:*

(a) *If $(X, \theta) \in \mathcal{S}^{\mathbf{T}}$ and if $\theta^k : [T(X)]^k \to X^k$ is a quotient map (between the product topologies) for all finite k, then (X, θ), with the Q topology, is a topological $\mathbf{T}_0$-algebra. (In general, products of quotient maps need not be quotient maps. However if $\mathbf{T}$ admits a group operation or if every model is finite then θ^k will be a quotient map.)*

(b) *If ω is any k-ary operation of $\mathbf{T}$, for k finite, and if $A \in \mathcal{S}^{\mathbf{T}}$ then ω is continuous from $Q(A^k) \to Q(A)$ (where A^k is the product algebra in $\mathcal{S}^{\mathbf{T}}$. Note that $Q(A^k)$ may fail to be $Q(A)^k$.)*

(c) *Each $\widetilde{\mathbf{T}}$-algebra is a topological $\mathbf{T}_0$-algebra using the $\widetilde{U}$-topology, and is Hausdorff.*

Remark. The infinitary operations of $\mathbf{T}$ are usually not continuous.

Proof. (a) $T(X)$ is always a topological $\mathbf{T}_0$-algebra as it is a topological and algebraic limit of models which are (discrete) topological $\mathbf{T}_0$-algebras. Thus the operations of $\mathbf{T}_0$ are continuous from $[T(X)]^k$ to $T(X)$. If θ^k is a quotient map then clearly all $\mathbf{T}_0$-operations are continuous from $X^k \to X$ (by the naturality of operations).

If $\mathbf{T}$ admits a group operation then a well known argument shows that θ is an open mapping. (Usually the argument is stated in the presence of Hausdorffness which however is not needed.) If θ is an open onto mapping then θ^k is a quotient map for all k.

If every model if finite then the Q topology is always compact, Hausdorff (as shown in the proof of 3.1) and every continuous onto map is closed hence a quotient map.

(b) Notice that there exists a continuous map from $T(A^k) \to [T(A)]^k$ whose projections are $T(A^k \to A)$. But every k-ary operation gives rise to a continuous map from $T(A)^k \to T(A)$ [as in (a)] hence there exists a continuous map from $T(A^k) \to T(A)$. By taking quotients one can readily show that the operation is continuous from $Q(A^k)$ to $Q(A)$.

(c) Let $(X, \widetilde{\theta})$ be a $\widetilde{\mathbf{T}}$-algebra, where $\widetilde{\theta} : \widetilde{T}(X) \to X$, Then $\widetilde{\theta}$ induces a $\mathbf{T}_0$-structure on X (which coincides with the underlying $\mathbf{T}_0$-structure of $\varphi(X, \theta)$, see a similar discussion in 1.4). Moreover $\widetilde{U}(\widetilde{\theta})$ is split epi in Top as $\widetilde{\eta}$ is continuous. Hence $\widetilde{\theta}$ and $\widetilde{\theta}^k$ are quotient maps for all k and the above arguments apply.

Since $\widetilde{\theta}$ is split epi, $\widetilde{U}(X)$ is topologically equivalent to a subspace of $\widetilde{T}(X)$ hence is Hausdorff.

Definition. $\mathbf{T}_0 = (T_0, \eta_0, \mu_0)$ is a *separating triple* for $I: \mathcal{M} \to \mathcal{S}$ if each model has the structure of a $\mathbf{T}_0$-algebra; the models are closed under the formation of $\mathbf{T}_0$-subalgebras and the maps between models are precisely the $\mathbf{T}_0$-homomorphisms. In more precise terms $\mathbf{T}_0$ is a separating triple for I if I factors as $\mathcal{M} \to \mathcal{S}^{\mathbf{T}_0} \to \mathcal{S}$ where $\mathcal{M} \to \mathcal{S}^{\mathbf{T}_0}$ embeds $\mathcal{M}$ as a full subcategory closed under the formation of $\mathbf{T}_0$-subalgebras, and where $\mathcal{S}^{\mathbf{T}_0} \to \mathcal{S}$ is the underlying set functor.

We say that $\mathbf{T}_0$ is *finitary* if the corresponding equational theory is (i.e. if it is generated by the finitary operations).

Notation and remarks. If $\mathbf{T}_0$ is a separating triple and $n \in \mathcal{S}$ then we let $(n, I)_0$ be the full subcategory of (n, I) consisting of those $f: n \to M$ such that $f(n)$ generates M as a $\mathbf{T}_0$-algebra (i.e. the extension $T_0(n) \to M$ is onto). We observe:
(1) If $f_1 : n \to M_1$ and $f_2 : n \to M_2$ are in $(n, I)_0$ and if $e : f_1 \to f_2$ (that is $e: M_1 \to M_2$ and $ef_1 = f_2$), then e is onto as its range contains $f_2(n)$ which generates M_2. Also e is unique as it is determined on $f_1(n)$. Thus $(n, I)_0$ is partially ordered.
(2) $(n, I)_0$ is initial in (n, I) and $T(n) = \lim\ [(n, I)_0 \to \mathcal{S}]$. Moreover, $(n, I)_0$ is always small so $T(n)$ exists hence $I: \mathcal{M} \to \mathcal{S}$ is automatically tractable if a separating triple exists.
(3) Suppose that $\mathcal{M}$ has and I preserves finite products. Let $f_1 : n \to M_1$ and $f_2 : n \to M_2$ be in (n, I). Let $(f_1, f_2): n \to M_1 \times M_2$ be the obvious map. Let M be the $\mathbf{T}_0$-subalgebra generated by the range of (f_1, f_2). Denote by $f_1 \wedge f_2 : n \to M$ the map induced by (f_1, f_2). Then $f_1 \wedge f_2 \in (n, I)_0$ and is the inf of f_1 and f_2 whenever $f_1, f_2 \in (n, I)_0$.
(4) If $\zeta \in T(n)$ then the open sets of the form $\langle g\rangle^{-1}(m)$ for $g: n \to M$ in $(n, I)_0$ and $m = \langle g\rangle(\zeta)$ form a base for the neighborhoods at ζ. (Since $T(n)$ has the limit topology these neighborhoods form a subbase at ζ and they are also closed under finite intersections in view of the construction in (3) above.) This argument clearly applies to any filtered topological limit of discrete spaces.

1.3. Proposition. *(Consequences of a separating triple.) Let* $\mathbf{T}_0$ *be a separating triple for* $I: \mathcal{M} \to \mathcal{S}$ *(thus* I *is tracable by* (2) *above). Assume that* $\mathcal{M}$ *has and* I *preserves finite products. Then:*

(a) *There is a natural map* $t: T_0(X) \to T(X)$ *which has dense range for all* $X \in \mathcal{S}$. *If* $(X, \theta) \in \mathcal{S}^{\mathbf{T}}$ *then* $(X, \theta t) \in \mathcal{S}^{\mathbf{T}_0}$ *and is the underlying* $\mathbf{T}_0$*-algebra of* (X, θ).

(b) *If* (X, θ) *and* (Y, ψ) *are* $\mathbf{T}$*-algebras and* $Q(Y, \psi)$ *is Hausdorff then a* $\mathbf{T}$*-homomorphism from* (X, θ) *to* (Y, ψ) *is the same thing as a continuous* $\mathbf{T}_0$*-homomorphism.*

(c) *A closed* $\mathbf{T}_0$*-subalgebra of a* $\mathbf{T}$*-algebra is a* $\mathbf{T}$*-subalgebra.*

(d) $\mathrm{Top}^{\mathbf{T}}$ *is co-complete (as well as complete) so* $\varphi: \mathrm{Top}^{\mathbf{T}} \to \mathcal{S}^{\mathbf{T}}$ *has a left adjoint. Moreover if* $Q\varphi = \widetilde{U}$ *then* φ *is a full embedding.*

(e) *If* $\mathbf{T}_0$ *is finitary then* $\mathrm{Top}^{\tilde{\mathbf{T}}}$ *can be fully embedded into the topological* $\mathbf{T}_0$-*algebras (using the* $\tilde{U}$ *topology).*

Proof. Since $T(X)$ is a filtered limit of models over $(X, I)_0$ (which is filtered by (3) above) a basic neighbourhood of $\zeta \in T(X)$ has the form $\langle g \rangle^{-1}(m)$ where $m = \langle g \rangle(\zeta)$ and $g \in (X, I)_0$. Given $g: X \to M$ in $(X, I)_0$ let $\bar{g}: T_0(X) \to M$ be the extension (so $\bar{g}\eta_0 = g$). Define t by $\langle g \rangle t = \bar{g}$ for all $g \in (X, I)_0$. Since g is always onto, the range of t meets all basic neighbourhoods of each $\zeta \in TX$. Thus t has dense range. It is clear that $t: T_0 \to T$ can be regarded as a natural transformation. Finally we know that the underlying $\mathbf{T}_0$-algebra of TX is $\mathbf{T}_0$-lim $[(X, I)_0 \to \mathcal{S}^{\mathbf{T}_0}]$ as limits are preserved. Thus $t: T_0(X) \to T(X)$ and θ are (underlying) $\mathbf{T}_0$-homomorphisms. Moreover $t\,\eta_0 = \eta$ so $\theta\, t\, \eta_0 = X$ and θt is therefore the structure map of the $\mathbf{T}_0$-algebra underlying (X, θ).

As for (b) if $f: (X, \theta) \to (Y, \psi)$ is a continuous $\mathbf{T}_0$-homomorphism then $f\theta\, t = \psi\, T(f) t$ so $f\theta = \psi T(f)$ as t has dense range and $Q(Y, \psi)$ is Hausdorff.

As for (c) let (X, θ) be in $\mathcal{S}^{\mathbf{T}}$ and let $A \subseteq X$ be a closed $\mathbf{T}_0$-subalgebra. Let $i: A \to X$ be the inclusion and $\psi_0: T_0(A) \to A$ the $\mathbf{T}_0$-structure map. It suffices to show that the range of $\theta\, T(i)$ is continued in A. But $\theta\, T(i) t_A = \theta t_X T_0(i) = i \psi_0$, hence $\theta T(i) t_A$ has range in A. But the range of $\theta T(i) t_A$ is dense in the range of $\theta T(i)$ (as t_A has dense range) and A is closed so that range of $\theta T(i)$ is contained in A.

As for (d) we observe that if $(X, \theta) \in \mathrm{Top}^{\tilde{\mathbf{T}}}$ then $\tilde{U}(X, \theta) = X$ is Hausdorff as it is a retract (via $\tilde{\eta}$ and θ) of $\tilde{T}(X)$. Also $\lambda: TU(X) \to \tilde{T}(X)$ has dense range for the same reason that t does. Thus by the above arguments a $\tilde{U}$-closed $\mathbf{T}_0$-subalgebra of a $\mathbf{T}$-algebra is a $\tilde{\mathbf{T}}$-subalgebra. Thus $\mathrm{Top}^{\tilde{\mathbf{T}}}$ has coequalizers for if $g, h: Y \to Z$ are $\tilde{\mathbf{T}}$-homomorphisms then the set of all e with dense range and $eg = eh$ is a solution set. Thus $\mathrm{Top}^{\tilde{\mathbf{T}}}$ is cocomplete (by [14]) and φ has a left adjoint (by [4]).

Also the above arguments show that a $\tilde{\mathbf{T}}$-homomorphism is the same thing as a $\tilde{U}$-continuous $\mathbf{T}_0$-homomorphism. So φ is obviously full if $Q\varphi = \tilde{U}$. By construction φ is faithful.

As for (e) every $\tilde{\mathbf{T}}$-algebra can be regarded as a topological $\mathbf{T}_0$-algebra by 1.2(c). The analog of (b) shows that this is a full embedding. q.e.d.

We are particularly interested in the case when $\mathbf{T}_0$ admits a group operation (which will usually be denoted multiplicatively). In this case K is said to be a $\mathbf{T}_0$-*kernel* of $X \in \mathcal{S}^{\mathbf{T}_0}$ if there exists a $\mathbf{T}_0$-homomorphism f with $K = f^{-1}(1)$ (where 1 denotes the group identity). As usual a $\mathbf{T}_0$-quotient of X is an (onto) image of X in $\mathcal{S}^{\mathbf{T}_0}$. Thus a $\mathbf{T}_0$-quotient of X can be represented as X/K for K a $\mathbf{T}_0$-kernel. The following lemma is useful.

1.4. Lemma. *Let* $\mathbf{T}_0$ *be a triple over* $\mathcal{S}$ *which admits a group operation. Then the notions of* $\mathbf{T}_0$-*kernel and* $\mathbf{T}_0$-*quotient are defined above and satisfy the following properties:*

(a) *Let X and Y be* $\mathbf{T}_0$*-algebras and* $f: X \to Y$ *a* $\mathbf{T}_0$*-homomorphism. If* $K \subseteq Y$ *is a* $\mathbf{T}_0$*-kernel then so is* $f^{-1}(K)$*. If f is onto and if* $N \subseteq X$ *is a* $\mathbf{T}_0$*-kernel then so is* $f(N)$.

(b) *If K and N are* $\mathbf{T}_0$*-kernels of X then so is KN.*

(c) *Let X be a topological* $\mathbf{T}_0$*-algebra. Then KU is an open* $\mathbf{T}_0$*-kernel if U is an open* $\mathbf{T}_0$*-kernel and K is any* $\mathbf{T}_0$*-kernel. Moreover if K is a closed* $\mathbf{T}_0$*-kernel and if X admits a base* $\{U_\alpha\}$ *of open* $\mathbf{T}_0$*-kernel neighbourhoods of* 1 *then* $K = \cap KU_\alpha$.

Proof. Let N be a normal subgroup of the $\mathbf{T}_0$-algebra X. Then N is a $\mathbf{T}_0$-kernel iff the equivalence relation, mod N, is a $\mathbf{T}_0$-congruence (i.e. is compatible with every operation of $\mathbf{T}_0$. We are regarding $\mathbf{T}_0$ as an equational theory). (a) is an immediate consequence of this observation, and the argument below.

As for (b) let w be an n-ary operation of T_0 and let $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ be n-tuples of X with $x_i = y_i$ mod KN for all i. (Despite our notation we do not assume that n is finite.) Then there exist $k_i \in K$ and $n_i \in N$ with $x_i = k_i n_i y_i$ for all i. Let $z_i = n_i y_i$, then $w(x_i) = w(z_i)$ as K is a $\mathbf{T}_0$-kernel and $w(z_i) = w(y_i)$ as N is a $\mathbf{T}_0$-kernel.

(c) is obvious. q.e.d.

1.5. Proposition. *Let* $\mathbf{T}_0$ *be a separating triple with a group operation. Assume that* $\mathcal{M}$ *has and I preserves finite products. Then:*

(a) *The discrete members of* $\mathcal{S}^{\mathbf{T}}$ *are precisely the* $\mathbf{T}_0$*-quotients of models. (That is if M is any model and K any* $\mathbf{T}_0$*-kernels then* $M/K \in \mathcal{S}^{\mathbf{T}}$ *the projection* $M \to M/K$ *is a* $\mathbf{T}$*-homomorphism,* $Q(M/K)$ *is discrete and every discrete* $\mathbf{T}$*-algebra is of this form.)*

(b) *Let* $(X, \theta) \in \mathcal{S}^{\mathbf{T}}$. *(By 1.2 the Q topology makes* (X, θ) *a topological group.) The open* $\mathbf{T}_0$*-kernels form a basic system of neighbourhoods at the group identity* $1 \in X$.

Moreover $U \subseteq X$ *is an open* $\mathbf{T}_0$*-kernel iff U is a closed* $\mathbf{T}_0$*-kernel and* X/U *is* $\mathbf{T}_0$*-equivalent to a* $\mathbf{T}_0$*-quotient of a model, which is true iff U is a* $\mathbf{T}$*-kernel and* X/U *is* $\mathbf{T}$*-equivalent to a* $\mathbf{T}_0$*-quotient of a model.*

(c) *A closed* $\mathbf{T}_0$*-kernel of a* $\mathbf{T}$*-algebra is a* $\mathbf{T}$*-kernel.*

Remark. If we increase $\mathcal{M}$ by closing up under the formation of $\mathbf{T}_0$-quotients, this will not change $\mathcal{S}^{\mathbf{T}}$, in view of (a), and will not change the basic neighbourhoods of (b) so Q will not be affected either. Note that the hypotheses of this proposition are also preserved. Hence we shall often assume that $\mathcal{M}$ is closed under the formation of $\mathbf{T}_0$-quotients in cases when this entails no real loss of generality. Then the above statements are somewhat simplified.

Proof. (a) If M is a model and K is a $\mathbf{T}_0$-quotient of M then $M/K \in \mathcal{S}^{\mathbf{T}}$, the projection $M \to M/K$ is a $\mathbf{T}$-homomorphism and $Q(M/K)$ is discrete in view of the Lemma 1.6, below. Conversely, every discrete $\mathbf{T}$-algebra is of this form in view of (b) proven below.

(b) Let N be a neighbourhood of $1 \in X$. Then $\theta^{-1}(N)$ is a neighbourhood of $1 \in TX$ so there exists $h : X \to M$ with $\mathrm{Ker}\,\langle h\rangle \subseteq \theta^{-1}(N)$. Thus $\theta(\mathrm{Ker}\,\langle h\rangle) \subset N$ and $\theta(\mathrm{Ker}\,\langle h\rangle)$ is an open $\mathbf{T}_0$-kernel as θ is an onto, open mapping. Thus the open $\mathbf{T}_0$-kernels are a base at 1.

Next let $U \subseteq X$ be an open $\mathbf{T}_0$-kernel. Then $\theta^{-1}(U)$ is an open neighbourhood of $1 \in T(X)$. Using the fact that $T(X)$ is a filtered limit of models there exists $h : X \to M$ in $(X, I)_0$ with $\mathrm{Ker}\,\langle h\rangle \subseteq \theta^{-1}(U)$. Since $\langle h\rangle$ is onto, $\langle h\rangle\,\theta^{-1}(U)$ is a $\mathbf{T}_0$-kernel of M by 1.4(a). Let $K = \langle h\rangle\,\theta^{-1}(U)$. Let $p : M \to M/K$ be the projection, then p is a $\mathbf{T}$-homomorphism by (a). Note that $\mathrm{Ker}\,p\,\langle h\rangle = \theta^{-1}(U)$ (since $\mathrm{Ker}\,p\,\langle h\rangle = \langle h\rangle^{-1}\langle h\rangle\,(\theta^{-1}U)$ and $\mathrm{Ker}\,\langle h\rangle \subseteq \theta^{-1}(U)$. Thus $U = \theta\,(\mathrm{Ker}\,p\,\langle h\rangle)$ is a $\mathbf{T}$-kernel by applying 1.4(a) to $\mathbf{T}$. Hence X/U is equivalent to M/K as they are both quotients of TX by the same kernel.

Conversely let U be a closed $\mathbf{T}_0$-kernel with X/U $\mathbf{T}_0$-equivalent to a $\mathbf{T}_0$-quotient of a model. By (c) proved below, U is a $\mathbf{T}$-kernel so $X/U \in \mathcal{S}^{\mathbf{T}}$. Let $f : M/K \to X/U$ be the $\mathbf{T}_0$-equivalence where $M \in \mathcal{M}$. By (a), M/K is discrete hence f is continuous. Since U is closed and since $Q(X/U)$ has the quotient topology, $Q(X/U)$ is Hausdorff. Thus by 1.3(b), f is a $\mathbf{T}$-homomorphism. Since f is one-one and onto it is a $\mathbf{T}$-equivalence. Thus $Q(X/U) = Q(M/K)$ which is discrete and U is open. The other characterisation of open $\mathbf{T}_0$-kernels follows readily.

(c) Let K be a closed $\mathbf{T}_0$-kernel of X and let $\{U_\alpha\}$ be the set of open $\mathbf{T}_0$-kernels. Then $K = \cap KU_\alpha$ and each KU_α is an open $\mathbf{T}_0$-kernel (hence a $\mathbf{T}$-kernel) by 1.4(b) and (c). By considering products it follows that intersections of $\mathbf{T}$-kernels are $\mathbf{T}$-kernels, so K is a $\mathbf{T}$-kernel. q.e.d.

1.6. Lemma. *Let $\mathbf{T}_0$ be a separating triple and assume that $\mathcal{M}$ has and I preserves finite products. Let (X, θ) be a $\mathbf{T}$-algebra (then (X, θ) has a topology supplied by Q and an underlying $\mathbf{T}_0$-structure). Let (Y, ψ_0) be a $\mathbf{T}_0$-algebra and $f : X \to Y$ an onto $\mathbf{T}_0$-homomorphism. Assume that the quotient topology on Y induced by f is Hausdorff. Then there exists a unique structure map ψ with (Y, ψ) a $\mathbf{T}$-algebra and f a $\mathbf{T}$-homomorphism. It necessarily follows that (Y, ψ) is compatible with (Y, ψ_0) and that $Q(Y, \psi)$ coincides with the quotient topology induced by f.* (Note that in effect this lemma gives sufficient conditions for a $\mathbf{T}_0$-congruence to be a $\mathbf{T}$-congruence.)

Proof. Since f is a $\mathbf{T}_0$-homomorphism, $\psi_0 T_0(f) = f\theta\, t_X$. Let $e : Y \to X$ be any right inverse of f (so $fe = Y$) and define $\psi = f\theta\,T(e)$. Then $\psi t_Y = \psi_0 T_0(f) T_0(e) = \psi_0$. Moreover a density argument shows that $\psi T(f) = f\theta$. Using this fact and the facts that f and $T^2(f)$ are epi (note $T^2(e)$ is a right inverse for $T^2(f)$) and the fact that θ is a structure map it readily follows that ψ is a structure map. Using 1.1(b) the lemma follows. q.e.d.

Added in proof. The above Lemma (1.6) was independently obtained by Karl Lindblad a student of J.F. Kennison.

§2. Pro-objects and $\mathcal{M}$-objects

The (dual of the) notion of an $\mathcal{M}$-object was introduced by Appelgate and Tierney [1] using "atlases". An $\mathcal{M}$-object was shown to be the same thing as a **T**-algebra which is a limit of models (by dualizing [1]). A pro-object for $\mathcal{M}$ was described in [2] pp. 154–166. To within categorical equivalence, the pro-objects can be regarded as the category of all small filtered limits of representable in $(\mathcal{M}, \mathcal{S})^{op}$ (that is certain colimits in $(\mathcal{M}, \mathcal{S})$). We shall regard $\mathcal{M} \subseteq$ Pro-objects by identifying $M \in \mathcal{M}$ with the corresponding representable $\mathcal{M}(M, -) \in (\mathcal{M}, \mathcal{S})^{op}$. For emphasis limits in the category of pro-objects shall be denoted by "pro-lim". If $\{M_i\}$ is filtered then each map pro-lim $\{M_i\} \to M$ must factor through a projection as pro-lim $\{M_i\} \to M_i \to M$. Moreover suppose that the diagram $\{M_i\}$ arises explicitly as $\Delta : D \to M$ (where $M_i = \Delta(i)$). Suppose that f: pro-lim $\Delta \to M$ has two factorizations $f = f_1 p_i$ and $f = f_2 p_j$ through projections p_i and p_j. Then there exists $k \in D$ and $d_1 : k \to i$, $d_1 : k \to j$ in D such that $f_1(d_1) = f_2(d_2)$. (Conversely this condition implies $f_1 p_i = f_2 p_j$.) From this it is possible to obtain a purely formal definition of the pro-objects as formed filtered limits (see [2]).

We wish to know when the category of $\mathcal{M}$-objects is equivalent to the category of pro-objects for $\mathcal{M}$. In this section we show that this question is closely related to the question of when Q preserves limits of models. In the next section we give some conditions which imply that Q preserves limits of models.

We proceed to discuss pro-objects and to introduce some definitions which are technically useful. Following [1], observe that there is a lifted singular function $\bar{s} : \mathcal{S}^{\mathbf{T}} \to (\mathcal{M}, \mathcal{S})^{op}$ given by $\bar{s}(A) = \mathcal{S}^{\mathbf{T}}(A, -)$. This functor generally has a right adjoint $\bar{r}$ and $\bar{r}\bar{s}$ gives rise to the lifted model induced triple $\bar{\mathbf{T}} = (\bar{T}, \bar{\eta}, \bar{\mu})$ on $\mathcal{S}^{\mathbf{T}}$. (Thus $\bar{T}(A) = \lim\, [(A, \bar{I}) \to \mathcal{S}^{\mathbf{T}}]$ etc. $\bar{r}$ exists if this limit exists which is true if $\mathcal{M}$ is small or if there is a separating triple for I.) We now rephrase our question as: "When is $\bar{s}$ a full embedding of the $\mathcal{M}$-objects onto the pro-objects or onto some special class of pro-objects?"

In order to get a reasonable answer to this question we would like $\bar{s}$ to at least preserve the models. That is $\bar{s}(M)$ must be canonically equivalent to the representable, $\mathcal{M}(M, -)$. This happens precisely when $\bar{I} : \mathcal{M} \to \mathcal{S}^{\mathbf{T}}$ is full and faithful (which is usually true, for example the existence of a separating triple guarantees it). When $\bar{I}$ is full and faithful we shall identify $\bar{s}(M)$ with $\mathcal{M}(M, -)$ for all models M.

2.1. Proposition. *Let $\bar{I}$ be full and faithful. Then the following statements are equivalent:*

(1) *$\bar{s}$ fully and faithfully embeds the category of $\mathcal{M}$-objects into* Pro-$\mathcal{M}$ *(the category of pro-objects).*

(2) *Every $\mathcal{M}$-object N can be represented as $N =$* **T**-lim *M_i where $\{M_i\}$ is filtered and $\bar{s}(N) =$* pro-lim *$\bar{s}(M_i)$ (recall that* **T**-lim *M_i is the limit of $\{M_i\}$ in $\mathcal{S}^{\mathbf{T}}$ and* pro-lim *$\bar{s}(M_i)$ is the limit in* Pro-$\mathcal{M}$).

Moreover, either of the above equivalent statements imply that $\bar{\mathbf{T}}$ is idempotent and $\bar{\mathbf{T}}$ reflects $\mathcal{S}^{\mathbf{T}}$ onto the $\mathcal{M}$-objects.

Proof. (1) ⇒ (2): Assume that $\bar{s}$ is such an embedding, and let N be an $\mathcal{M}$-object. Then $\bar{s}(N) \in \text{Pro-}\mathcal{M}$ so $\bar{s}(N) = \text{pro-lim}\ \bar{s}(M_i)$. Since $\bar{s}$ is full and faithful it follows that $N = \mathbf{T}\text{-lim}\ M_i$.

(2) ⇒ (1): If X is a $\mathbf{T}$-algebra and $M \in \mathcal{M}$ then $\bar{s}$ maps $\mathcal{S}^{\mathbf{T}}(X, M)$ into the set of natural transformations from $\mathcal{M}(M, -)$ to $\mathcal{S}^{\mathbf{T}}(X, -)$. By Yoneda's lemma (in effect) it follows that the action of $\bar{s}$ on the hom set $\mathcal{S}^{\mathbf{T}}(X, M)$ is one-to-one and onto. Moreover, if N is a limit of models and $\bar{s}$ preserves this limit then $\bar{s}$ is still one-to-one and onto on the hom set $\mathcal{S}^{\mathbf{T}}(X, N)$. Thus (2) implies that $\bar{s}$ is full and faithful and also that $\bar{s}$ maps each $\mathcal{M}$-object into a pro-object.
Finally, if N is an $\mathcal{M}$-object and $N = \mathbf{T}\text{-lim}\ M_i$ with $\bar{s}(N) = \text{pro-lim}\ \bar{s}(M_i)$ then every map from N into a model must factor through a projection (as this is true in Pro-$\mathcal{M}$). This means that the projections are initial in the comma category $(N, \bar{I})$ so that $\bar{T}(N) = \lim\ (N, \bar{I}) = \lim M_i = N$. q.e.d.

Remarks. The next proposition will extend 2.1. We must find some technical remarks and definitions concerning different types of limits. Recall that a $\mathbf{T}_0$-algebra M has *d.c.c. for* $\mathbf{T}_0$*-subalgebras* if every strictly descending chain $M_1 \supsetneq M_2 \supsetneq \ldots$ of $\mathbf{T}_0$-subalgebras is necessarily finite. This is equivalent to the *minimum condition for* $\mathbf{T}_0$-subalgebras (that every filtered family of $\mathbf{T}_0$-subalgebras has a smallest member).

Remark (1). If $\mathbf{T}_0$ is a separating triple for $\mathcal{M}$ and if the $\mathcal{M}$-objects = pro-objects (via $\bar{s}$) then each $M \in \mathcal{M}$ has d.c.c. for $\mathbf{T}_0$-subalgebras. For assume $M_1 \supsetneq M_2 \supsetneq \ldots$ are $\mathbf{T}_0$-subalgebras of M. Let $M_\infty = \cap M_n$ and let $\langle n \rangle : M_\infty \to M_n$ be the inclusion. We also let $\langle m, n \rangle : M_m \to M_n$ be the inclusion for $m \geq n$. Then $M_\infty = \lim M_n$ in the category of $\mathcal{M}$-objects, hence also in the category of pro-objects as $\bar{s}$ is an equivalence. Therefore the identity map $M_\infty : M_\infty \to M_\infty$ must factor as $r\langle n \rangle$ for some $r : M_n \to M_\infty$ (as this is true in pro-objects.) Clearly $\langle n \rangle r \langle n \rangle = M_n \langle n \rangle$ hence $\langle n \rangle r$ and M_n must be equalized by some map in the diagram, that is $\langle n \rangle r \langle m, n \rangle = M_n \langle m,n \rangle$ for some $m \geq n$ (as this happens in pro-objects). This implies $M_m = M_\infty$.

We can avoid the d.c.c. for subalgebras if we wish to establish conditions for the weaker statement that $\mathcal{M}$-objects = Regular pro-objects (via $\bar{s}$). Regular pro-objects are defined below and from the largest possible class of pro-objects in which $\mathcal{M}$-objects can be embedded. As we shall show, given d.c.c. for subalgebras, all pro-objects are regular.

Definition. Let $I : \mathcal{M} \to \mathcal{S}$ be given. Let D be a filtered category. A diagram $\Delta : D \to \mathcal{M}$ is *regular* if (in addition to being filtered) $\Delta(d)$ is onto for all morphisms d of D (that is $I\Delta(d)$ is onto for all such d). A *regular limit* is the limit of a regular diagram. We are interested in regular limits in $\mathcal{S}^{\mathbf{T}}$ and also in pro-objects. A *regular pro-object* is a regular limit of representables (i.e. we regard $\mathcal{M} \subseteq (\mathcal{M}, \mathcal{S})^{\text{op}}$).

Remark (2). If $\Delta : D \to \mathcal{M}$ is regular then we may as well assume that D is partially ordered with $i \leq j$ iff there exists $d : i \to j$. Thus if we were to take the dual of D and allow Δ to become contravariant, then D^{op} is directed and Δ would be a classical inverse limit system all of whose maps are onto. But we do *not* do this. All functors and diagrams are *covariant* here. To prove the assertion note that if $d, d' : i \to j$ are morphisms of D then there exists e with $de = d'e$ hence $\Delta(d) = \Delta(d')$ as $\Delta(e)$ is onto. We can thus identify d and d' without affecting the value of $\lim \Delta$ in $\mathcal{S}^{\mathbf{T}}$ or in pro-objects.

Remark (3). Let $\mathbf{T}_0$ be a separating triple and let every $M \in \mathcal{M}$ have d.c.c. for $\mathbf{T}_0$-subalgebras. Then every pro-object would be regular. Given a filtered diagram $\Delta : D \to \mathcal{M}$, for each $i \in D$ let $\Delta'(i)$ be the smallest $\mathbf{T}_0$-subalgebra of $\Delta(i)$ which is the image of some model $\Delta(j)$ under some map $\Delta(d)$. Then Δ' is regular and $\lim \Delta = \lim \Delta'$ is obvious for $\mathcal{S}^{\mathbf{T}}$ and not difficult to show for pro-objects.

Remark (4). Let $\mathcal{M}$ have and I preserve finite products and let $\mathbf{T}_0$ be a separating triple. Then every $\mathcal{M}$-object can be represented as a regular limit of models. If $X = \lim \{M_\alpha | \alpha \in A\}$, let $M_F = \lim \{M_\alpha | \alpha \in F\}$ for each finite subset $F \subseteq A$. Let M'_F be the image of X in M_F. Then $\{M'_F\}$ is a regular diagram in $\mathcal{M}$ with $F_1 \leq F_2$ iff $F_1 \supseteq F_2$, and the obvious maps. Moreover $X = \lim M'_F$.

We need the following definition for technical reasons.

Definition. A regular diagram $\{M_i\}$ of models is $\mathbf{T}$-*regular* if every projection $p_i : X \to M_i$ is onto where $X = \lim M_i$ in $\mathcal{S}^{\mathbf{T}}$. (It obviously suffices to let $X = \lim M_i$ in $\mathcal{S}$ or in $\mathcal{S}^{\mathbf{T}_0}$ if $\mathbf{T}_0$ is a separating triple in order to test for $\mathbf{T}$-regularity.) A $\mathbf{T}$-regular diagram is clearly regular.

Let $I : \mathcal{M} \to \mathcal{S}^{\mathbf{T}}$ be full and faithful. Then a pro-object F is a $\mathbf{T}$-*regular pro-object* if $F = \text{pro-lim}\, \bar{s}(M_i)$ in $(\mathcal{M}, \mathcal{S})^{op}$ where $\{M_i\}$ is $\mathbf{T}$-regular.

Remark (5). The reason for the above definition is that, as we show in the proposition below, the $\mathbf{T}$-regular pro-objects are the largest class of pro-objects in which $\bar{s}$ can embed the $\mathcal{M}$-objects. Thus we have sharpened our comments about regular pro-objects. Also the $\mathbf{T}$-regular pro-objects often have a nice topological representation (see 2.4).

It would be interesting to know for which triples $\mathbf{T}$ does regular imply $\mathbf{T}$-regular. It is definitely not true for all $\mathbf{T}$ as it is not true for the identity triple (i.e. it is not true for sets as shown by Henkin in [10]). It is true in compact spaces and in the linearly compact triples as can be shown by the arguments in Section 3. We do not know if it is true for groups. Collecting our remarks we have

2.2. Proposition. *Assume that $\mathcal{M}$ has and I preserves finite products and that $\mathbf{T}_0$ is a separating triple. Then:*

(a) *If* $\bar{s}$ *:* $\mathcal{M}$*-objects* $\to$ *Pro-objects is an embedding of the* $\mathcal{M}$*-objects into a full subcategory then the range of* $\bar{s}$ *is contained in the* **T**-*regular pro-objects and is precisely the* **T**-*regular pro-objects iff* $\bar{s}$ *preserves* **T**-*regular limits.*

(b) *If* $\{M_i\}$ *is* **T**-*regular and* X = **T**-lim $\{M_i\}$ *then* $\bar{s}(X)$ = pro-lim $\bar{s}(M_i)$ *iff every map* $X \to M \in \mathcal{M}$ *factors as* $X \to M_i \to M$ *(through a projection). (If* $\{M_i\}$ *is filtered this condition is still necessary.)*

(c) *Every* $\mathcal{M}$*-object is the limit of a* **T**-*regular diagram.*

(d) $\bar{s}$: $\mathcal{M}$*-objects* $\to$ *Pro-objects is an equivalence of categories iff the condition in* (b) *holds for all* **T**-*regular diagrams; every regular diagram (of models) is* **T**-*regular and each model has d.c.c. for* $\mathbf{T}_0$*-subalgebras.*

Proof. We prove (b) first. Let $\Delta: D \to \mathcal{M}$ be **T**-regular and let X = **T**-limΔ. Assume that every map $X \to M \in \mathcal{M}$ factors as $X \to M_i \to M$ for some $i \in D$, where $M_i = \Delta(i)$ and $X \to M_i$ is the projection. We claim that $\bar{s}(X)$ takes on the same values (as a functor in $(\mathcal{M}, \mathcal{S})^{\text{op}}$) as the functor pro-lim $\{s(M_i)\}$. This boils down to showing that if $f_1 p_i = f_2 p_j$ (where $i, j \in D$ and p_i, p_j are projections) then there exists $k \in D$, and map $d_1 : k \to i$, $d_2 : k \to j$ with $f_1 \Delta(d_1) = f_2 \Delta(d_2)$.

But as D is filtered there exist k, d_1, d_2 such that $\Delta(d_1)p_k = p_i$ and $\Delta(d_2)p_k = p_j$. Then $f_1 \Delta(d_1)p_k = f_2 \Delta(d_1)p_k$ and p_k is epi as Δ is **T**-regular. The necessity in (b) is trivial. (c) is a consequence of the construction in Remark (4). As for (a) let N = **T**-lim M_i and assume $\bar{s}(N)$ = pro-lim $\bar{s}(M_i)$ where $\{M_i\}$ is filtered. Let M'_i = image of N under the projection $N \to M_i$. Then $\{M'_i\}$ is **T**-regular and N = **T**-lim $\{M'_i\}$ and s preserves this limit by (b). (The necessary condition for s to preserve $\{M_i\}$ implies the sufficient condition for s to preserve $\{M'_i\}$.) Thus $\bar{s}(N)$ is a **T**-regular pro-object. By 2.1 the range of $\bar{s}$ is contained in the **T**-regular pro-objects. If the range of $\bar{s}$ is the **T**-regular pro-objects and if $\{M_i\}$ is a **T**-regular diagram then $\bar{s}(\lim M_i)$ is the limit (in the range of $\bar{s}$) of $\{\bar{s}(M_i)\}$ which must be the pro-lim $\bar{s}(M_i)$ as this pro-limit is in the range of $\bar{s}$. The converse follows from (c) and 2.1.

Finally (d) is a consequence of (a), 2.1 and Remarks (1) and (3). q.e.d.

We shall now relate the above notions to the case when Q preserves limits of models. First we need:

2.3. Proposition. *(Consequences of* Q *preserving limits of models.) Let* $\mathcal{M}$ *have and* I *preserve finite products and let* $\mathbf{T}_0$ *be a separating triple. Assume that* Q *preserves limits of models. Then:*

(a) $\tilde{\mathbf{T}}$ *is idempotent and its range, the* $\mathcal{M}$*-objects, is thus a reflective subcategory of* $\mathcal{S}^{\mathbf{T}}$.

(b) *An* $\mathcal{M}$*-object is precisely a closed* $\mathbf{T}_0$*-subalgebra of a product of models. Each such closed subalgebra is given the relative topology by* Q.

(c) *The restriction* $Q|\mathcal{M}$*-objects has* φF *as left adjoint. This pair of adjoint functions generate* $\tilde{\mathbf{T}}$ *on* Top. *Thus there is a comparison functor*

$$\tau : \mathcal{M}\text{-objects} \to \text{Top}^{\tilde{\mathbf{T}}}.$$

Moreover, using φ and τ one can regard:

$$\mathcal{M}\text{-objects} \subseteq \mathrm{Top}^{\tilde{\mathbf{T}}} \subseteq \mathcal{S}^{\mathbf{T}}.$$

(d) $Q\varphi = \tilde{U}$.

Proof. (a) Let $X = \lim M_i$ be an $\mathcal{M}$-object and let $\bar{\eta}: X \to \bar{T}(X)$ be the unit of $\bar{\mathbf{T}}$ at X. By construction of $\bar{T}(X)$ there exists a retraction $\theta : \bar{T}(X) \to X$. Since $\bar{\eta}$ and θ are $\mathbf{T}$-homomorphisms they are Q-continuous hence $\bar{\eta}(X)$ is closed in the Q-topology on $\bar{T}(X)$. But $\bar{\eta}(X)$ is dense in the limit topology (by using the type of argument used in 1.4). By hypothesis, we see $\bar{\eta}(X) = \bar{T}(X)$ hence $X = \bar{T}(X)$ and $\bar{\mathbf{T}}$ is idempotent. The rest of (a) is obvious.

(b) Let $X \subseteq \prod M_\alpha | \alpha \in A$ be a closed $\mathbf{T}_0$-subalgebra of a product of models. Let D be the class of all finite subsets $F \subseteq A$ partically ordered by $F_1 \leq F_2$ iff $F_1 \supseteq F_2$. (Then D happens to be filtered.) Given $F \in D$ let M_F be the projection of X on $\prod M_\alpha | \alpha \in F$. Let $L = \mathbf{T}\text{-}\lim \{M_F\}$ then we can construct L so that $X \subseteq L \subseteq \prod M_\alpha$. By the now familiar argument, X is dense in L but X is closed so $X = L$. Thus X is an $\mathcal{M}$-object. (Note that in view of the hypothesis the topologies on L and $\prod M_\alpha$ are unambigious.) The converse is obvious.

(c) Let $X \in \mathrm{Top}$ be given. Then $\tilde{F}(X)$ is a $\tilde{\mathbf{T}}$-limit of models and since $Q\varphi$ preserves limits of models, $Q\varphi\tilde{F}(X) = \tilde{T}(X)$. We claim that $\tilde{\eta} : X \to \tilde{T}(X)$ is a front adjunction. Let $N = \lim M_i$ be an $\mathcal{M}$-object and let $h : X \to Q(N)$ be continuous. By construction of $\tilde{T}(X)$ there exists a unique $\mathbf{T}$-homomorphism $f : \tilde{T}(X) \to N$ such that $f\tilde{\eta} = h$ (by considering projections). It follows from this proof that $\tilde{\mathbf{T}}$ is the triple generated by the adjointness between φF and $Q|(\mathcal{M}\text{-objects})$. Hence the comparison function τ exists. The last part of (c) is the statement that φ is a full embedding (which will follow from 1.4(d) and 2.3(d) (proven below)) and that $\varphi\tau$ is the inclusion if $\mathcal{M}$-objects into $\mathcal{S}^{\mathbf{T}}$. But φ and τ both preserve models and both preserve limits (as they are comparison functions). Hence $\varphi\tau$ preserves $\mathcal{M}$-objects.

(d) Let $(X, \theta) \in \mathrm{Top}^{\tilde{\mathbf{T}}}$. Then $\tilde{T}(X)$ has an unambigious topology. $\theta : \tilde{T}(X) \to X$ is a quotient map (onto the $\tilde{U}$ topology) since $\theta\tilde{\eta} = X$. But θ in Q-continuous so every Q-open subset of X is $\tilde{U}$-open. On the other hand $\theta\lambda$ is $\tilde{U}$-continuous, but by definition is a quotient map from $T(X)$ onto $Q(X, \theta\lambda) = Q\varphi(X, \theta)$. Thus a $\tilde{U}$-open subset is Q-open. q.e.d.

Definition. Let $\mathbf{T}_0$ be a finitary triple over $\mathcal{S}$ which admits a group operation. Then the notion of a complete topological $\mathbf{T}_0$-*algebra* is clear (using the usual notion of a complete topological group).

If $\mathcal{M}$ is any collection of (discrete) $\mathbf{T}_0$-algebras we say that the topological $\mathbf{T}_0$-algebra, X, is $\mathcal{M}$-*generated* if X has a basic system of neighbourhoods $\{U_\alpha\}$ at 1 such that each U_α is an open $\mathbf{T}_0$-kernel and X/U_α is algebraically equivalent to a member of $\mathcal{M}$.

2.4. Lemma. *Let $\mathcal{M}$ and $\mathbf{T}_0$ be as in the above definition and assume that $\mathcal{M}$ has and I preserves finite products. Then the category of* **T**-*regular pro-objects is equivalent to the category of* **T**-*regular limits (in the category of topological $\mathbf{T}_0$-algebras) of discrete models. This is precisely the category of complete, Hausdorff, $\mathcal{M}$-generated topological $\mathbf{T}_0$-algebras.*

Proof. Let $\{M_i\}$ be a **T**-regular diagram of models. Let $\lim M_i$ be the limit in topological $\mathbf{T}_0$-algebras (so that it has the limit topology). Let f: $\lim M_i \to M$ be a continuous $\mathbf{T}_0$-homomorphism. Then Ker f is open so there exists a projection p_i with Ker $p_i \subseteq$ Ker f (as {Ker p_i} is a base at 1). Thus f factors through p_i as p_i is onto. The proof of 2.2(b) now applies.

If X is complete and Hausdorff and has a base $\{U_\alpha\}$ of open $\mathbf{T}_0$-kernels with $X/U_\alpha \in \mathcal{M}$ then $\{X/U_\alpha\}$ is a **T**-regular diagram whose limit is X. q.e.d.

2.5. Proposition. *Assume that $\mathcal{M}$ has and I preserves finite products. Let $\mathbf{T}_0$ be a finitary separating triple which admits a group operation. Further, for convenience (see 1.5) assume that the models are closed under the formation of $\mathbf{T}_0$-quotients. Then:*

(a) *The* **T**-*regular pro-objects is equivalent to the category of complete $\tilde{\mathbf{T}}$-algebras.*

(b) *If Q preserves limits of models, then the following five categories are canonically equivalent (see 2.2, 2.3*(c) *and* (a) *above):*

$\mathcal{M}$-*objects* = *complete* $\tilde{\mathbf{T}}$-*algebras*
= *complete, Hausdorff* **T**-*algebras*
= **T**-*regular pro-objects*
= *complete, Hausdorff, $\mathcal{M}$-generated $\mathbf{T}_0$-algebras.*

(c) *Conversely, if $\mathcal{M}$-objects* = **T**-*regular pro-objects (via $\bar{s}$) then Q preserves limits of models.*

Proof. (a) The proof of 1.5(b) can be applied to show that all $\tilde{\mathbf{T}}$-algebras are $\mathcal{M}$-generated (the additional use of 1.3(e) makes this easier to see). By 1.3(e) and 2.4 and the observation that all models are $\tilde{\mathbf{T}}$-algebras and that $\mathrm{Top}^{\tilde{\mathbf{T}}}$ has all limits, which are preserved by $\tilde{U}$, (a) follows.

(b) It easily follows from 2.4, 1.3(b) and (a) above, that all five of these categories are equivalent to the category of complete, Hausdorff, $\mathcal{M}$-generated topological $\mathbf{T}_0$-algebras. That these equivalence are "canonical" follows from noting that all of the equivalence preserve the models and limits and the topology (by 2.3(d) and the fact that Q and $\tilde{U}$ preserve topological limits of models). That the $\mathcal{M}$-objects are equivalent to the **T**-regular pro-object *via s* follows from 2.3(a), 2.2 and the proof of 2.4.

(c) Conversely, assume that the $\mathcal{M}$-objects are equivalent to the **T**-regular pro-objects via $\bar{s}$. Then the condition in 2.2(b) holds for every **T**-regular diagram.

Now let X = **T**-lim $\{M_i\}$. Claim that $Q(X)$ = Top-lim $\{M_i\}$. We may as well assume that $\{M_i\}$ is **T**-regular (as we can replace $\{M_i\}$ by the diagram $\{M'_F\}$ used in Remark (4) without changing X or the limit topology. By construction $\{M'_F\}$ is **T**-regular).

By 1.5(b), $Q(X)$ is the smallest topology rendering each $\mathbf{T}$-homomorphism $X \to M$ continuous. By assumption each such $X \to M$ factors as $X \to M_i \to M$ where $X \to M_i$ is a projection. Hence Top-lin$\{M_i\}$renders each such $X \to M$ continuous and the claim follows (e.g. by 1.1(b)). q.e.d.

§3. The compact and linearly compact cases

Definition. Following Manes [16, p. 104] we say that a full subcategory $B \subseteq \mathcal{S}^{\mathbf{T}}$ is a *Birkhoff subcategory of* $\mathcal{S}^{\mathbf{T}}$ if B is closed under the formation of products, $\mathbf{T}$-subalgebras and $\mathbf{T}$-quotients. Manes proves that Birkhoff subcategories are tripleable over $\mathcal{S}$. Conversely, if $\mathcal{S}^{\mathbf{T}'}$ is a full tripleable subcategory of $\mathcal{S}^{\mathbf{T}}$ and if the induced map $T(X) \to T'(X)$ is onto for all X then $\mathcal{S}^{\mathbf{T}'}$ is a Birkhoff subcategory. From an equational point of view, a Birkhoff subcategory is determined by equational identities (so that the Birkhoff subcategory is the class of all algebras satisfying the identities). In the category of rings, for example, the subcategory of all commutative rings is the Birkhoff subcategory determined by the identity $xy = yx$.

3.1. Theorem *(the compact case). Let* $\mathcal{M}$ *have and I preserve finite products. Let* $\mathbf{T}_0$ *be a finitary separating triple and assume that every model is finite. Then* $\mathcal{S}^{\mathbf{T}}$ *is the smallest Birkhoff subcategory (containing the models) of the compact, Hausdorff* $\mathbf{T}_0$*-algebras. Moreover* Q *preserves limits and* $Q\varphi = \tilde{U}$ *hence* φ *is a full embedding. Finally:*

$\mathcal{M}$*-objects = Pro-objects* $\subseteq \mathrm{Top}^{\tilde{\mathbf{T}}} \subseteq \mathcal{S}^{\mathbf{T}}$.

If $\mathbf{T}_0$ *admits a group operation these categories coincide.*

Proof. Since the category C of compact Hausdorff $\mathbf{T}_0$-algebras is equational (see [16]) and contains the models, there exists a limit preserving forgetful functor $V: \mathcal{S}^{\mathbf{T}} \to C$ (by definition of the equational completion or since every operation of C is an operation of $\mathcal{S}^{\mathbf{T}}$). If $(X, \theta) \in \mathcal{S}^{\mathbf{T}}$ then $V(\theta): VT(X) \to VX$ is a closed mapping hence a quotient map so the C-topology on VX is precisely $Q(X, \theta)$. Thus, by 1.3(b), V is a full embedding. Moreover if $F_C: \mathcal{S} \to C$ is the free functor then by 1.4(c) the canonical map $F_C(X) \to T(X)$ is dense hence onto. Thus $\mathcal{S}^{\mathbf{T}}$ is a Birkhoff subcategory and clearly the smallest one containing the models by the fact that $\mathcal{S}^{\mathbf{T}}$ is an equational completion. By 1.1(b) the Q-topology on a $\mathbf{T}$-limit is at least as big as the topological limit but both are compact, Hausdorff hence they coincide. Thus Q preserves limit so $Q\varphi = \tilde{U}$ by 2.3(d) and φ is a full embedding by 1.3(d).

It remains to show that $\mathcal{M}$-objects = Pro-objects in view of 2.3(c). (The statement concerning the case when $\mathbf{T}_0$ has a group operation is a special case of Theorem 3.3 proved below.) Using 2.2(b) let M_i be $\mathbf{T}$-regular and let $X = \mathbf{T}\text{-lim}\, M_i$. Let $f: X \to M$ be given. Each $x \in X$ has a neighbourhood of the form $p_i^{-1}(m_i)$ on which f is constant as M is discrete. Using compactness and filteredness there exists a projection p_i such that f is constant on all sets of the form $p_i^{-1}(m)$. Since p_i is onto, f factors through p_i by considering congruence relations.

We shall pursue 2.2(d). Note that $\bar{\mathbf{T}}$ is idempotent by 2.3(a) and d.c.c. for $\mathbf{T}_0$-subalgebras of models is trivial as all models are finite. Thus it suffices to show that if M_i is regular then it is **T**-regular. By a previous remark we may as well assume that i varies in a filtered, partially ordered set. Let $X = \mathbf{T}\text{-lim}\, M_i$ and let $p_i : X \to M_i$ be the projection. Regard $X \subseteq \prod M_i$ and let $\bar{p}_i : \prod M_i \to M_i$ be the projection for the product. For each $d : M_j \to M_k$ in the diagram, let:

$$A_d = \{y \in \prod M_i | d\,\bar{p}_j(y) = \bar{p}_k(y)\}\,.$$

Then A_d is closed and we can regard $X = \cap A_d$. Let i_0 and $m \in M_{i_0}$ be chosen. Then $(\bar{p}_{i_0})^{-1}(m) \cap A_d$ has f.i.p. (as can readily be shown by using the fact that the diagram is filtered). By compactness of $\prod M_i$ there is an x in the intersection so $x \in \cap A_d = X$ and $p_{i_0}(x) = m$. Hence the projections are onto. q.e.d.

Definition. Let $\mathbf{T}_0$ be a finitary triple over sets which admits a group operation. Then a $\mathbf{T}_0$-*kernel coset* is a set of the form Ux where U is a $\mathbf{T}_0$-kernel. The notion of a *topological* $\mathbf{T}_0$-*algebra* is well known. A topological $\mathbf{T}_0$-algebra is *linearly compact* if every family of closed $\mathbf{T}_0$-kernel cosets with f.i.p. (finite intersection property) has non-void intersection.

Notice that a discrete $\mathbf{T}_0$-algebra is linearly compact if it has d.c.c. for $\mathbf{T}_0$-kernels. The converse is not true (see [18]). The term "pseudocompact" is used for a special case of linear compactness in [5].

Definition. $\mathbf{T}_0$ is a *normal separating triple* for $I : \mathcal{M} \to \mathcal{S}$ if $\mathbf{T}_0$ is a finitary triple with a group operation; the models are closed under $\mathbf{T}_0$-quotients (see 1.5); and $\mathcal{M}$ has and I preserves finite products.

$I : \mathcal{M} \to \mathcal{S}$ satisfies the *linearly compact conditions* (abreviated LCC) with respect to $\mathbf{T}_0$ if $\mathbf{T}_0$ is a normal separating triple and each $M \in \mathcal{M}$ is linearly compact in the discrete topology.

Definition. Let $\mathbf{T}_0$ admit a group operation. Let $\mathcal{M}$ be a given class of (discrete) $\mathbf{T}_0$-algebras. Let X be a topological $\mathbf{T}_0$-algebra. Then X is *strongly* $\mathcal{M}$-*generated* if it is $\mathcal{M}$-generated and if $K \subseteq X$ is open whenever K is a closed $\mathbf{T}_0$-kernel with X/K algebraically equivalent to some $M \in \mathcal{M}$.

Remarks. Linearly compact modules have often been examined in the literature (e.g. see [5, 12, 13, 18], however their tripleability properties over $\mathcal{S}$ and sometimes Top seem not to have been established before). It is convenient here to list some of the elementary results that extend to the general case of $\mathbf{T}_0$-algebras and which are useful below or in considering examples.

(1) Discrete linearly compact $\mathbf{T}_0$-algebras are closed under the formation of finite products and $\mathbf{T}_0$-quotients, but not necessarily under the formation of $\mathbf{T}_0$-subalgebras. (In fact, infinite nested intersections of linearly compact $\mathbf{T}_0$-subalgebras need not be linearly compact.)

(2) Hausdorff linearly compact $\mathbf{T}_0$-algebras having a base of open $\mathbf{T}_0$-kernels at 1 are closed under the formation of regular limits and arbitrary products and $\mathbf{T}_0$-quotients (by closed $\mathbf{T}_0$-kernels).

(3) If $\mathcal{M}$ is any full subcategory of discrete linearly compact $\mathbf{T}_0$-algebras then, for $\mathcal{M}$-generated, Hausdorff topological $\mathbf{T}_0$-algebras, completeness is equivalent to linear compactness. Also the complete (or linearly compact) Hausdorff $\mathcal{M}$-generated topological $\mathbf{T}_0$-algebras are category equivalent to the regular pro-objects for $\mathcal{M}$.

These remarks can easily be proved either by generalizing the proofs in [12] and [18] or by using the methods in this section. We shall however sketch the proofs. If A and B are discrete, linearly compact $\mathbf{T}_0$-algebras, let $\mathcal{F}$ be a maximal filter of $\mathbf{T}_0$-kernels cosets of $A \times B$. Let $\mathcal{F}_A = \{p_A(F) | F \in \mathcal{F}\}$. There exists $a \in \cap\ \mathcal{F}_A$ hence $p_A{}^{-1}(a) \in \mathcal{F}$ as it meets all members of $\mathcal{F}$ and $\mathcal{F}$ is maximal. Similarly there exists $b \in B$ with $p_B{}^{-1}(b) \in \mathcal{F}$ so $(a, b) \in \cap\ \mathcal{F}$. Thus $A \times B$ is linearly compact. A similar argument (replacing $p_A(F)$ by its closure) can be used to prove (2). Alternatively one can generalize the proof of 3.13. Most of (3) follows from 2.4 (that regular limits are $\mathbf{T}$-regular is essentially known and also proved as 3.11 below). That linear compactness implies completeness follows since maximal filters containing small sets have subfilters of closed $\mathbf{T}_0$-kernel cosets.

Remark. Several of the other interesting properties for linear compactness (cf. [18]) can be generalized to $\mathbf{T}_0$-algebras. Also some of the lemmas below such as 3.9, 3.10, 3.12 and 3.14 can be generalized to Hausdorff linearly compact $\mathbf{T}_0$-algebras having a base of open $\mathbf{T}_0$-kernels.

3.2. Theorem *(the linearly compact case). Assume that $I: \mathcal{M} \to \mathcal{S}$ satisfies LCC with respect to* $\mathbf{T}_0$. *Then:*

(a) $\mathcal{S}^{\mathbf{T}}$ *is equivalent (using the Q-topology) to the category of strongly $\mathcal{M}$-generated complete, Hausdorff* $\mathbf{T}_0$*-algebras. Every member of* $\mathcal{S}^{\mathbf{T}}$ *is linearly compact. Moreover $\tilde{\mathbf{T}}$ is the identity triple on* $\mathcal{S}^{\mathbf{T}}$ *so that* $\mathcal{S}^{\mathbf{T}} = \tilde{\mathcal{M}}$*-objects.*

(b) $\mathrm{Top}^{\tilde{\mathbf{T}}}$ = *Regular Pro-objects. Alternatively,* $\mathrm{Top}^{\tilde{\mathbf{T}}}$ *is equivalent (using the $\tilde{U}$-topology) to the category of $\mathcal{M}$-generated complete Hausdorff* $\mathbf{T}_0$*-algebras. Every member of* $\mathrm{Top}^{\tilde{\mathbf{T}}}$ *is linearly compact. Note that we can regard* $\mathcal{S}^{\mathbf{T}} \subseteq \mathrm{Top}^{\tilde{\mathbf{T}}} \subseteq$ Top $\mathbf{T}_0$*-algebras. In this case* $\varphi: \mathrm{Top}^{\tilde{\mathbf{T}}} \to \mathcal{S}^{\mathbf{T}}$ *is a coreflective functor (that is the right adjoint of the inclusion* $\mathcal{S}^{\mathbf{T}} \to \mathrm{Top}^{\tilde{\mathbf{T}}}$*). Explicitly φ preserves the underlying* $\mathbf{T}_0$*-algebra structure and simply increases the number of open sets. If* $X \in \mathrm{Top}^{\tilde{\mathbf{T}}}$ *then* *the set of all closed* $\mathbf{T}_0$*-kernels K such that X/K is algebraically a model becomes a base at* 1 *for* $\varphi(X)$. *The case when* φ *is a category equivalence is discussed in* 3.3.

(c) *A is a* **T**-*subalgebra of* $X \in \mathcal{S}^{\mathbf{T}}$ *iff A is a closed* $\mathbf{T}_0$*-subalgebra (in the Q-topology).* (*$Q(A)$ may presumably fail to be the relative topology, but see* 3.3.)

3.3. Theorem *(the d.c.c. case). Let* $\mathbf{T}_0$ *be a normal separating triple for* $I : \mathcal{M} \to \mathcal{S}$. *Assume that each model has d.c.c. with respect to* $\mathbf{T}_0$*-kernels (thus LCC is satisfied). Then* (*in addition to the results of* 3.2):

(a) φ *is a category equivalence.*

(b) *Q preserves all limits.*

(c) *If* $A \subseteq X$ *is a* **T**-*subalgebra then* $Q(A)$ *has the relative topology.*

(d) $Q\varphi = \tilde{U}$. (*Thus* $\mathcal{S}^{\mathbf{T}}$ *is tripleable over* Top *via Q.*)

(e) *If* $f : A \to B$ *is an onto* **T**-*homomorphism then f admits a continuous section (that is Q(f) is a split epi).* This was proved by Serre [17] for pro-finite groups and by Brumer [5], for pseudocompact modules (see Example 4.3).

Moreover if $\mathbf{T}_0$ *is a normal separating triple, then* (b) *and* (c) *together imply the d.c.c. assumption on* $\mathbf{T}_0$*-kernels. This d.c.c. assumption is, in the presence of LCC equivalent to* (a), (b) *and* (d), *individually.*

The proofs of 3.2 and 3.3 shall be postponed until we establish some lemmas concerning certain regular limits of models. For technical reasons we set up the following notation and definitions.

Notation. In what follows D shall always denote a small, filtered, partially ordered category and $\Delta : D \to \mathcal{M}$ shall be regular. If $g \in D$ then we generally let $M_g = \Delta(g)$ and if $g \leq h$ in D then the induced map $M_g \to M_h$ shall be denoted as $\delta(g, h)$ or just δ if there is no danger of confusion.

The canonical projection $\lim \Delta \to M_g$ shall be denoted by $\langle g \rangle$. This extends the notation for the important case where $D = (X, \Gamma)_0$ and Δ is chosen so that $\lim \Delta = T(X)$ etc.

Definition. Let $\Delta : D \to \mathcal{M}$ be as above. Given $g, h \in D$ and $a \in M_h$ we choose $k \leq g, h$ and let $\delta_1 = \delta(k, h)$ and $\delta_2 = \delta(k, g)$. We define $\tau_{g,h}(a) = \delta_2(\delta_1^{-1}(a))$. Then $\tau_{g,h}(a) \subseteq M_g$ is independent of the choice of k. (For $b \in M_g$ is in $\tau_{g,h}(a)$ iff there exists $x \in M_k$ with $\delta_1(x) = a$ and $\delta_2(x) = b$. If $k' \leq k$ then there exists such an $x \in M_k$ iff there exists a suitable $x' \in M_{k'}$ as $\delta : M_{k'} \to M_k$ is onto. Since D is filtered, given k_1 and k_2 there exists $k' \leq k_1, k_2$ and the independence of the choice of k is obvious.)

We note that if $\zeta \in \lim \Delta$ and $\langle h \rangle(\zeta) = a$ and $\langle g \rangle(\zeta) = b$ then $b \in \tau_{g,h}(a)$. We shall establish a general result which proves the converse.

Definition. Let $\Delta : D \to \mathcal{M}$ be as above. Then (H, γ) is a Δ-*prescription* (or simply a *prescription*) if $H \subseteq D$ and if γ is a function such that $\gamma(h) \in M_h$ for all $h \in H$. We call (H_0, γ_0) a *subprescription* of (H, γ) [and (H, γ) an *extension* of (H_0, γ_0)] if $H_0 \subseteq H$ and $\gamma | H_0 = \gamma_0$. A *solution* of (H, γ) is a point $\zeta \in \lim \Delta$ such that $\langle h \rangle(\zeta) = \gamma(h)$ for all $h \in H$.

(H, γ) is *consistent* if it has a solution and is *finitely consistent* if every finite subprescription is consistent.

Definition. Let (H, γ) be a Δ-prescription and let $g \in D$ be given. Then we define

$$\hat{\gamma}(g) = \cap \{\tau_{g,h}(a) \mid h \in H,\ a = \gamma(h)\}.$$

We say that (H, γ) is *formally consistent* if $\hat{\gamma}(g) \neq \emptyset$ for all $g \in D$. We call (H, γ) *finitely formally consistent* if every finite subprescription is formally consistent. Observe that a (finitely) consistent prescription in (finitely) formally consistent.

Definition. $\Delta : D \to \mathcal{M}$ is *very regular* if (Δ is regular and) D has infs (denoted $g \wedge h$) and the induced map $\Delta(g \wedge h) \to \Delta(g) \times \Delta(h)$ is one-one. (For example if $D = (X, I)_0$ and $T(X) = \lim \Delta$ etc. then Δ is very regular, see discussion preceeding 1.3 assuming there exists a well-behaved $\mathbf{T}_0$.)

Definition. Let $\Delta : D \to \mathcal{M}$ be very regular. Then the Δ-prescription (H, γ) is $\wedge$*-closed* if $h_1 \wedge h_2 \in H$ whenever $h_1, h_2 \in H$ and $\gamma(h_1 \wedge h_2)$ is the unique element of $\Delta(h_1 \wedge h_2)$ which maps into $(\gamma(h_1), \gamma(h_2)) \in \Delta(h_1) \times \Delta(h_2)$.

Most of the following lemmas have straightforward proofs which are omitted or sketched.

3.4. Lemma. *Let $\Delta : D \to \mathcal{M}$ be very regular. If (H, γ) is $\wedge$-closed then (H, γ) is finitely formally consistent.*

3.5. Lemma. *Given LCC, a finitely formally consistent diagram is formally consistent.* (Note $\tau_{g,h}(a)$ is a $\mathbf{T}_0$-kernel coset by applying 1.4.)

3.6. Lemma. *Let $f, g, h \in D$ with $f \leq g$. Let $a \in M_h$. Then $\delta : M_f \to M_g$ maps $\tau_{f,h}(a)$ onto $\tau_{g,h}(a)$. Therefore if (H, γ) is a prescription and $f \leq g$ then δ map $\hat{\gamma}(f)$ into $\hat{\gamma}(g)$.*

3.7. Lemma. *Assume LCC and that $\Delta : D \to \mathcal{M}$ is very regular. Let (H, γ) be $\wedge$-closed and let $f \leq g$. Then δ map $\hat{\gamma}(f)$ onto $\hat{\gamma}(g)$.*

Proof. Given $s \in \hat{\gamma}(g)$ then $\delta^{-1}(s) \cap \tau_{f,h}(a)$ (where $h \in H$ and $a = \gamma(h)$) has f.i.p. (by 3.6 and as H has finite infs). Apply LCC. q.e.d.

3.8. Lemma. *Assume LCC and that $\Delta : D \to \mathcal{M}$ is very regular. Let (H, γ) be $\wedge$-closed. Let $g \in D$ and $b \in \hat{\gamma}(g)$ be given. Let $G = \{g \wedge h \mid h \in H\} \cup \{g\} \cup H$. Then there exists a unique β such that (G, β) is a $\wedge$-closed extension of (H, γ) and $\beta(g) = b$.*

Proof. Given $h \in H$ let $\delta_1 : \Delta(g \wedge h) \to \Delta(g)$ and $\delta_2 : \Delta(g \wedge h) \to \Delta(h)$. By 3.7 there exists $x \in \hat{\gamma}(g \wedge h)$ such that $\delta_1(x) = b$. Then $\delta_2(x)$ must be $\gamma(h)$ (by 3.6) and x is uniquely determined as Δ is very regular. Define $\beta(g \wedge h) = x$. Since x is determined by $\delta_1(x) = b$ we see that this definition is unambigious in case $g \wedge h = g \wedge (h')$. By the same argument if we further define $\beta(g) = b$ and $\beta(h) = \gamma(h)$ for $h \in H$ we see that β is still well-defined and (G, β) is $\wedge$-closed.

3.9. Lemma. *Assume LCC and that $\Delta : D \to \mathcal{M}$ is very regular. Then every $\wedge$-closed prescription is consistent.*

Proof. Let (H, γ) be $\wedge$-closed. By Zorn's Lemma there exists a maximal $\wedge$-closed extension (H^*, γ^*). By 3.8 we seen that $H^* = D$ hence γ^* defines a member of $\lim \Delta$ which is a solution of (H, γ). q.e.d.

3.10. Lemma. *Assume LCC (and let $\Delta : D \to \mathcal{M}$ be regular). Then every finitely consistent Δ-prescription is consistent.*

Proof. Given $h_1, \ldots, h_n \in D$ choose $f \leq h_i$ for $i = 1, \ldots, n$ and let $\Delta(f) \to \Delta(h_1) \times \ldots \times \Delta(h_n)$ be the obvious map. Then the image of $\Delta(f)$ is independent of the choice of f (for the same reason that $\tau_{g,h}(a)$ is independent of the choice of k).

We now extend D to D' by adjoining formal finite infs to D and extend Δ to Δ' so that $\Delta'(h_1 \wedge \ldots \wedge h_n)$ is the above defined image of $\Delta(f)$. Then D is initial in D' so $\lim \Delta = \lim \Delta'$ and $\Delta' : D' \to \mathcal{M}$ is very regular by construction.

If (H, γ) is a finitely consistent Δ-prescription then (H, γ) can readily be extended to a $\wedge$-closed Δ'-prescription which is consistent by 3.9. Thus (H, γ) is consistent. q.e.d.

3.11. Corollary. *Assume LCC. Then every regular diagram is* **T***-regular.*

Proof. Using Remark (2), preceeding 2.2, it suffices to show that the regular diagram $\Delta : D \to \mathcal{M}$ is **T**-regular where D is partially ordered. Let $h \in D$ and $a \in M_h$ be given. Let $H = \{h\}$ and $a = \gamma(h)$. Then (H, γ) is a $\wedge$-closed prescription for the extended diagram $\Delta' : D' \to \mathcal{M}$ constructed above. By 3.9 there exists $\zeta \in \lim \Delta'$ such that $\langle h \rangle(\zeta) = a$ so Δ is **T**-regular. q.e.d.

Definition. Given LCC then the *$T(X)$-diagram* (for $X \in \mathcal{S}$) shall refer to the canonical functor $(X, I)_0 \to \mathcal{M}$. This diagram is very regular (see the discussion proceding 1.3). A prescription for this diagram shall be called a *$T(X)$-prescription*.

3.12. Proposition. *Assume LCC. Let $(X, \theta) \in \mathcal{S}^{\mathbf{T}}$ and let H be a set of* T*-homomorphisms from X into models. For each $h: X \to M$ in H let $\gamma(h) \in M$. If (H, γ) is finitely consistent (i.e. given $H_0 \subseteq H$, a finite subset, there exists $x \in X$ with $h(x) = \gamma(h)$ for all $h \in H_0$) then (H, γ) is consistent (i.e. there exists $x \in X$ with $h(x) = \gamma(h)$ for all $h \in H$).*

Proof. We may as well assume that each $h \in H$ is onto. Then $H \subseteq (X, I)_0$ and (H, γ) may be regarded as a $T(X)$-prescription. If $x \in X$ is such that $h(x) = \gamma(h)$ for all $h \in H_0$ then $\langle h \rangle (\eta(x)) = \gamma(h)$ for all $h \in H_0$ so (H, γ) is obviously finitely consistent. By 3.10 there exists $\zeta \in T(X)$ with $\langle h \rangle(\zeta) = \gamma(h)$ for all $h \in H$. Then $x = \theta(\zeta)$ has the desired property. (Note $\langle h \rangle = h\theta$ since h is a T-homomorphism for all $h \in H$.) q.e.d.

3.13. Proposition. *Assume LCC. Let $(X, \theta) \in \mathcal{S}^{\mathbf{T}}$. Then (using the Q-topology) (X, θ) is a linearly compact $\mathbf{T}_0$-algebra. It follows that (X, θ) is complete from the remark preceeding Theorem* 3.2.

Proof. By 1.2, (X, θ) with the Q-topology can be regarded as a topological $\mathbf{T}_0$-algebra. Let $\{U_i\}$ be a collection of closed $\mathbf{T}_0$-kernels of X and assume that the cosets $\{U_i x_i\}$ have f.i.p. We must show $\cap \{U_i x_i\} \neq \emptyset$. Since each closed $\mathbf{T}_0$-kernel is an intersection of open $\mathbf{T}_0$-kernels (see 1.4 which applies in view of 1.5(b)) it clearly suffices to assume that each U_i is an open $\mathbf{T}_0$-kernel. By 1.5(b) it follows that $X/U = M_i$ is a model. Let $h_i: X \to M_i$ be the corresponding projection. Let $H = \{h_i\}$ and $\gamma(h_i) = h_i(x_i)$. Then (H, γ) satisfies the hypotheses of 3.12 as finite consistency is here equivalent to the f.i.p. condition. The conclusion of 3.12 implies $\cap \{U_i x_i\} \neq \emptyset$. q.e.d.

3.14. Lemma. *Assume LCC. Let (X, θ) and (Y, ψ) be* T*-algebras with $(X, \theta) =$* T-limM_i *(a limit of models). Let $g: (X, \theta) \to (Y, \psi)$ be a* T*-homomorphism. Then* Ker g *is closed in the limit topology on X.*

Proof. We first claim that Ker $g = \theta$(Ker Tg). For if $\zeta \in$ Ker Tg then $g(\theta\,\zeta) = \psi(Tg)(\zeta) = 1$ hence $\theta(\zeta) \in$ Ker g. Conversely, let $x \in$ Ker g. Then $x = \theta(\eta x (\eta 1)^{-1})$ and $\eta x (\eta 1)^{-1} \in$ Ker Tg as $(Tg)\eta x = \eta_Y(gx)$ and $(Tg)(\eta 1)^{-1} = [\eta_Y(g1)]^{-1}$ and $gx = g1 = 1$.

We next claim that θ(Ker Tg) is closed in the limit topology. Assume that t is in the closure of θ(Ker Tg) in the limit topology. Then for any finite set of indices say $\{1, 2, ..., n\}$ (so labelled for convenience), there exists $\zeta \in$ Ker Tg with $p_i\theta(\zeta) = p_i(t)$ for $i = 1, ..., n$ (where $p_i: X \to M_i$ is the projection). But $p_i\theta(\zeta) = \langle p_i \rangle(\zeta)$ as p_i is admissible and $\zeta \in$ Ker Tg iff $\langle f \rangle (Tg)(\zeta) = \langle gh \rangle(\zeta) = 1$ for all $f: Y \to M$. By applying 3.12 to $T(X)$ we can find $\zeta \in TX$ such that $\langle p_i \rangle(\zeta) = p_i(t)$ for all i and $\langle gf \rangle(\zeta) = 1$ for all $f: Y \to M$. Then $\theta(\zeta) = t$ and $\zeta \in$ Ker Tg so $t \in \theta$ (Ker Tg). q.e.d.

3.15. Corollary. *Assume LCC. Let $(X, \theta) \in \mathcal{S}^{\mathbf{T}}$. Then $Q(X, \theta)$ is Hausdorff.*

Proof. $\theta: TX \to X$ is a **T**-homomorphism and TX is a limit of models hence Ker θ is closed in TX. Thus $1 \in X$ is closed in the quotient topology, $Q(X, \theta)$, which implies $Q(X, \theta)$ is Hausdorff as X is a topological group. q.e.d.

3.16. Corollary. *Assume LCC. Then* $\bar{\mathbf{T}}$ *is idempotent and trivial (that is* $\bar{T}$ *is the identity functor). Thus* $\mathcal{S}^{\mathbf{T}} = \mathcal{M}$*-objects and every* $X \in \mathcal{S}^{\mathbf{T}}$ *is a canonical limit of models.*

Proof. Let $X \in \mathcal{S}^{\mathbf{T}}$. Then $\bar{\eta}: X \to \bar{T}(X)$ is one-one in view of 1.5(b) since $Q(X)$ is Hausdorff. But by applying 3.11 and by representing $\bar{T}(X)$ as a regular filtered limit of models we see that $\bar{\eta}(X)$ is dense in $\bar{T}(X)$ (when $\bar{T}(X)$ is given the limit topology which need not be $Q\bar{T}(X)$). By 3.13, X is complete hence is closed in the limit topology on $\bar{T}(X)$. Thus η is onto, hence is an equivalence. q.e.d.

Proof of Theorem 3.2. (a) Let $U_0: \mathcal{S}^{\mathbf{T}} \to \mathcal{S}^{\mathbf{T}_0}$ be the forgetful functor and define Q_0 from $\mathcal{S}^{\mathbf{T}}$ into topological $\mathbf{T}_0$-algebras by using the Q topology together with U_0 for the $\mathbf{T}_0$-structure. By 1.2, 1.3 and 3.15 we see that Q_0 is a full embedding. If $X \in \mathcal{S}^{\mathbf{T}}$ then $Q_0(X)$ is complete, Hausdorff, $\mathcal{M}$-generated and linearly compact by 1.5 and the above results. Moreover suppose $K \subseteq X$ is a closed $\mathbf{T}_0$-kernel with X/K algebraically equivalent to a model. Then $X \to X/K$ is a **T**-homomorphism by 1.5(c), so $Q(X) \to Q(X/K)$ is continuous. But $Q(X/K)$ is discrete so K is open. Hence X is strongly $\mathcal{M}$-generated. Also $\mathcal{S}^{\mathbf{T}} = \mathcal{M}$-objects by 3.16. Conversely let X be a complete, Hausdorff strongly $\mathcal{M}$-generated $\mathbf{T}_0$-algebra. Let $\{U_\alpha\}$ be the base of all open $\mathbf{T}_0$-kernels for which $X/U_\alpha \in \mathcal{M}$. Then $\{X/U_\alpha\}$ is a regular filtered, partially ordered diagram in the obvious way. Since X is Hausdorff and complete $X = \text{Top-lim}\{X/U_\alpha\}$ (or more precisely the limit in topological $\mathbf{T}_0$-algebras). We claim that $Q(\mathbf{T}\text{-lim}\{X/U_\alpha\}) = \text{Top-lim}\{X/U_\alpha\}$ which shows $X = Q_0(\mathbf{T}\text{-lim}\{X/U_\alpha\})$, hence we can regard $X \in \mathcal{S}^{\mathbf{T}}$. Let $f: \mathbf{T}\text{-lim}\{X/U_\alpha\} \to M \in \mathcal{M}$ be a **T**-homomorphism. Then Ker f closed in the limit topology by 3.14 hence is open as X is strongly $\mathcal{M}$-generated. The claim now follows, from 1.5.

(b) In view of the remarks preceeding the statement of Theorem 3.2 $\tilde{T}(X)$ is linearly compact for all $X \in$ Top hence every $\tilde{\mathbf{T}}$-algebra is linearly compact, therefore complete. Since every regular diagram is **T**-regular for $\mathcal{M}$, we see that much of (b) follows from 2.4 and 2.5. That φ is a coreflection and behaves as stated follows from 1.5 and 3.14 and the observation that φ preserves models and their limits and everything is a limit of models.

(c) Let $A \subseteq X$ be a **T**-subalgebra of the **T**-algebra X. Let $\alpha: A \to X$ be the inclusion and let $x \in X$ be in the closure of $\alpha(A)$. Let H be the set of all **T**-homomorphism $h: A \to M$ which factor through α as $h = g\alpha$ where $g: X \to M$ is onto and **T**-admissible. Then for each $h \in H$ let $\gamma(h) = g(x)$ where g is chosen so that $h = g\alpha$ (note that $g(x)$ is uniquely determined even though g is not). Claim that (H, γ) is finitely consistent. Let $h_1, \ldots, h_n \in H$ be given and chose g_i with $h_i = g_i\alpha$. Then $\cap g_i^{-1}(g_i(x))$ is an open neighbourhood of x so there exists $a \in A$ with $g_i(\alpha a) = g_i(x)$ or $h_i(a) = \gamma(h_i)$ for $i = 1, ..., n$. By applying 3.12 to A there exists $a \in A$ with $g(a) = g(x)$ for all $g: X \to M$. Since X is Hausdorff $a = x$ which shows that A is closed. q.e.d.

Proof of Theorem 3.3. (a) We have LCC so Theorem 3.2 applies and it clearly suffices to show that every $\mathbf{T}$-algebra is strongly $\mathcal{M}$-generated. Let $X \in \mathrm{Top}^{\mathbf{T}}$ and K a closed $\mathbf{T}_0$-kernel with X/K algebraically equivalent to a model be given. Then there exists a smallest $\mathbf{T}_0$-kernel U of X for which U is open and $K \subseteq U$ by applying the d.c.c. (or the minimum condition) for X/K. By 1.4(c) we have $K = U$ so K is open.

(b), (c) and (d). By (a) above and the fact that φ preserves limits and models we see that the $\tilde{U}$-topology on $\lim M_i$ coincides with the Q-topology. Thus Q preserves limits of models hence 2.3 applies. Q preserves all limits as Q has a left adjoint by 3.2(a) and 2.3(c). Also (c) follows straightforwardly from 3.2(c) and 2.3(b), and (d) is the same as 2.3(d).

(e) The idea of the proof is to obtain a "continuous version" of 3.12. Let $f: A \to B$ be a given onto $\mathbf{T}$-homomorphism. Note that by 3.16 we have $A = \overline{T}(A) = \lim \{M_g | g \in (A, \overline{I})_0\}$ where $g: A \to M_g$ is in $(A, \overline{I})_0$ iff g is admissible and onto. We define (H, γ) to be a $\wedge$*-closed continuous-B-prescription* if $H \subseteq (A, \overline{I})_0$ and for each $h \in H$, $\gamma(h)$ is a continuous function from B to M_h. Moreover if $b \in B$ we let (H, γ_b) be the ordinary prescription with $\gamma_b(h) = \gamma(h)(b)$. It is further required that (H, γ_b) be $\wedge$-closed for all $b \in B$.

Now let (H, γ) be such a continuous prescription and $g: A \to M_g$ be arbitrary in $(A, \overline{I})_0$ with $g \notin H$. Choose $h \in H$ such that $g(\mathrm{Ker}\ h)$ is a minimum in M_g. Let $M_0 = M_g/g(\mathrm{Ker}\ h)$ and let $p: M_g \to M_0$ be the projection. Observe that there exists $r: M_h \to M_0$ with $rh = pg$. It is readily shown, from the choice of h, that $\hat{\gamma}_b(g)$ is a coset of precisely $g(\mathrm{Ker}\ h)$ for each $b \in B$. Moreover, if $m \in \hat{\gamma}_b(g)$ then $p(m) = r(\gamma_b(h))$ (in view of 3.7) so $p^{-1}(r\gamma_b(h)) = \hat{\gamma}_b(g)$ as they are both cosets of $g(\mathrm{Ker}\, h)$ and have points in common. (Note $r\gamma_b(h)$ is a single point as $\gamma(h)$ is a function.) Thus if $s: M_0 \to M_g$ is any section (that is $ps = M_0$ and s is *not* required to be a $\mathbf{T}$-homomorphism) then (H, γ) can be extended to a $\wedge$-closed continuous extension (G, β) where $\beta(g) = sr\gamma(h)$ as the method used in 3.8 can be applied pointwise. By Zorn's Lemma, every $\wedge$-closed continuous B-prescription can be extended to one with $H = (A, \overline{I})_0$ which gives rise to a continuous map $\mu: B \to A$ (that is $\mu: Q(B) \to Q(A)$ but is not necessarily $\mathbf{T}$-admissible) such that $h\mu = \gamma(h)$ for all $h \in (A, \overline{I})_0$. Now observe that $B = \overline{T}(B) = \lim \{M_k | k \in (B, \overline{I})_0\}$. Let $H_0 = \{kf | k \in (B, \overline{I})_0\}$ and define γ_0 so that $\gamma_0(kf) = k$ (which is determined as f is onto).

Clearly (H_0, γ_0) is a $\wedge$-closed continuous B-prescription so there exists a continuous $s: B \to A$ such that $kfs = k$ for all $k \in (B, \overline{I})_0$ which implies $fs = B$. Finally the d.c.c. hypothesis is implied by (b) and (c). Let $M \in \mathcal{M}$ and $A_1 \supseteq A_2 \supseteq ...$ be a descending chain of $\mathbf{T}_0$-kernels. By (b) and (c) the image of M in $\prod M/A_n$ has the relative topology induced by the product topology. But the image of M must be discrete by 1.5(a) which implies that the descending chain is ultimately constant. Moreover, given LCC it is readily shown that (a) $\Rightarrow$ (b) $\Rightarrow$ (d) by the above arguments. Using

3.2 we can readily obtain (d) ⇒ (a) and (b) ⇒ (c). Thus (a), (b) and (d) each individually imply (b) and (c). q.e.d.

§4. Examples

Most of the applications of Theorem 3.1 are immediate (e.g. the equational completion of finite sets is the category of compact Hausdorff spaces and the corresponding $\mathrm{Top}^{\mathbf{T}}$ is the subcategory of totally disconnected members of $\mathcal{S}^{\mathbf{T}}$. Also the equational completion of finite groups is pro-finite groups which is tripleable over Top, cf. [7] and [8]). Applications of Theorems 3.2 and 3.3 sometimes require some preliminary work with individual operations to get a normal separating triple as 4.1 and 4.2 show. A number of interesting papers have been written on linearly were an equational completion led to Theorems 3.2 and 3.3. Example 4.4 shows that even if Q grossly fails to preserve limits the topology can still be very useful in des- even if Q grossly faisl to preserve limits the topology can still be very useful in describing the equational completion. In 4.5 the subject is briefly discussed from the point of view of operations.

Notation. Unless otherwise specified, rings are assumed to have units and ring homomorphisms and modules are unitary. In contrast to the previous sections, the group operations here happen to be Abelian and are denoted by "+". Also "0" instead of "1" is used for the group identity, so one must exercise care in applying the previous results.

4.1. Fields. Let $\mathcal{F}_0$ be the category of fields and (unitary) ring homomorphisms. As it stands $\mathcal{F}_0$ does not admit an obvious separating triple (as subrings of fields are not always fields) nor does $\mathcal{F}_0$ have finite products. We first observe that if $\mathcal{F}$ is the category of all finite products of fields (and unitary ring homomorphisms) then $\mathcal{F}$ and $\mathcal{F}_0$ have the same equational completion. (It can readily be shown that if $\prod K_i \in \mathcal{F}$ and if $F \in \mathcal{F}_0$ then any homomorphism $\prod K_i \to F$ factors through a projection $\prod K_i \to K_i \to F$. Hence every operation of $\mathcal{F}_0$ is also an operation of $\mathcal{F}$.)

We next observe that the theory of commutative regular rings is a separating triple for $\mathcal{F}$. In any ring R we say that r is the *semi-inverse* of s if $rsr = r$, $srs = s$ and $sr = rs$. If the *semi-inverse* exists it is determined by the ring structure. (If t is another semi-inverse of r then $tr^2 = r = sr^2$ and $t = t^3r^2 = t^2sr^2 = s^3r^2 = s$.) It follows that every ring homomorphism preserves semi-inverses when they exist, hence the semi-inverse is an operation in any category of rings with semi-inverses. A *commutative* ring has semi-inverses for every element iff it is regular (for each r there exists an r' with $rr'r = r$).

Every field is clearly regular (with 0 its own semi-inverse) and every member of $\mathcal{F}$ is regular. Moreover it can be easily verified that a subregular ring of a finite product of fields is in $\mathcal{F}$. (Look at idenpotents.) Also $\mathcal{F}$ is closed under the formation of quotients (this can be done by considering idempotents or by observing that the members of $\mathcal{F}$ are precisely the commutative regular Artin rings). Thus the theory of commutative regular rings is a normal separating triple for $\mathcal{F}$.

We now can apply Theorems 3.2 and 3.3 (as every member of $\mathcal{F}$ has d.c.c. for ideals). Thus the equational completion of $\mathcal{F}$ is the category of regular pro-objects. It is also the category of topological limits of members of $\mathcal{F}$, hence also of $\mathcal{F}_0$. Clearly if $X \in \mathcal{S}^T$ then $X = \lim K_i$ and it can be arranged that K_i is a field and the projection map $X \to K_i$ is onto, for all i. It follows that every map in the diagram $\{K_i\}$ is onto hence is an isomorphism. Thus X is a product of fields. Thus the equational completion of the category of fields is the category of all products of fields and ring homomorphisms which are continuous in the product topology.

(The same argument works if one allows non-unitary homomorphisms and/or skew-fields or ordered fields (which have a lattice operation).)

The category of fields obviously has a large number of weird operations. For example one can define $w(x, y) = x + y$ in characteristic 0 fields but $w(x, y) = xy$ elsewhere. In fact if we wish to describe an arbitrary n-ary operation, w, we have to consider all collections of n-tuples $(x_1, ..., x_n)$ in fields F. One obvious restriction on $w(x_1, ..., x_n)$ is that it must lie in the subfield generated by $x_1, ..., x_n$ and this is essentially the only restriction on w. Formally, define an *n-pointed field* as a field F together with an n-tuple $(x_1, ..., x_n)$ such that no proper subfield of F contains each x_i. Let $\mathcal{F}(n)$ be a representative collection of n-pointed fields such that each n-pointed field is isomorphic to a unique member of $\mathcal{F}(n)$ (where isomorphisms must preserve the n-tuple). Then one can define an operation w by choosing $w(x_1, ..., x_n) \in F$ entirely at random for each $[F, (x_1, ..., x_n)] \in \mathcal{F}(m)$. (Notice that F has no non-trivial automorphism preserving the n-tuple.) In other words $T(n) = \Pi\mathcal{F}(n)$ (despite the notation we have not assumed that n is finite). It clearly follows that the theory of fields does not have rank.

It would be of some interest perhaps to describe the theory generated by finitary field operations. Consider for example the unary operation r such that $r(x) = 1$ iff x is rational (in the usual characteristic 0 sense) and $r(x) = 0$ otherwise. (This defines r on fields and by an obvious extension r is defined for all products of fields.) Not every ring homomorphism $f: Q^N \to K$ preserves r (where Q^N is a countable product of rationals and K is any field). In fact f is continuous iff f preserves r (in this case). This refutes the conjecture that the topology is used only to handle infinitary operations. Also the theory generated by the finitary operations is richer than the theory of commutative regular rings.

4.2. Artin rings. In [5] a *pseudocompact ring* is defined as a complete Hausdorff ring R which admits a base at 0 of two-sided open ideals I for which R/I is an Artin ring. It is easily seen that in our terminology this is the same thing as an "Artin-generated" Hausdorff, linearly compact ring or, alternatively a regular pro-object for Artin rings.

In view of Theorems 3.2 and 3.3 it seems reasonable to conjecture that the category of pseudocompact rings is the equational completion of the category of Artin rings. We shall show that this is not quite the case but that for certain classes $\mathcal{A}$ of Artin rings the $\mathcal{A}$-generated pseudocompact rings is the equational completion of $\mathcal{A}$.

Notice that the Artin rings are closed under finite products but not under the formation of subrings so the theory of rings is not a separating triple. Moreover as implied by the example below there is no way of enriching the theory of rings so as to obtain a separating triple. There are however separating triples for certain classes of Artin rings.

If the category of all pseudocompact rings were equational the underlying set functor would preserve and create limits hence the discrete pseudocompact rings (i.e. the Artin rings) would have to be closed under finite limits and arbitrary intersections. The following counter-example shows this is not so. Let A be the Artin ring of all 2 by 2 real matrices. For convenience, for the remainder of the paragraph, let $\langle r, s\rangle$ denote the 2 by 2 matrix $(a_{ij}) \in A$ for which $a_{11} = a_{22} = r$ and $a_{21} = 0$ and $a_{12} = s$. Let $P = \langle 1, 1\rangle$. Define $\Phi(X) = PXP^{-1}$ then $\Phi : A \to A$ is a ring homomorphism. Let E be the equalizer of Φ and the identity on A. Then E is the ring of all matrices of the form $\langle a, b\rangle$. Let C be the complex numbers and let $\theta : E \to C$ be defined by $\theta\langle a, b\rangle = a$. Then the intersection of all equalizers of θ and $\gamma\theta$ (for γ an automorphism of C) is the ring E' of all $\langle a, b\rangle$ for which a is rational and b is real. Then E' is not an Artin ring. (Observe that the reals are an infinite dimensional vector space of the rationals. If V is any linear subspace, then the set of all $\langle 0, b\rangle$ with $b \in V$ is an ideal of E'. Clearly the ideals of E' do not satisfy the d.c.c. nor even the a.c.c..)

We now construct a triple $\mathbf{T}_0$ whose algebras shall be rings R endowed with a 3-tuple of unary operations that map x into (x_1, x_2, x_3) such that

$$x = x_1 + x_2$$

$$x_1 x_2 = x_2 x_1 = 0$$

x_3 is the semi-inverse of x_1.
(that is $x_1 x_3 x_1 = x_1$, $x_3 x_1 x_3 = x_3$ and $x_1 x_3 = x_3 x_1$).

A $\mathbf{T}_0$-algebra R shall be called *canonical* if x_2 is nilpotent for all $x \in R$. For a canonical $\mathbf{T}_0$-algebra the operations are uniquely determined by the ring structure (and the requirement that x_2 be nilpotent). Thus the $\mathbf{T}_0$-homomorphisms between canonical $\mathbf{T}_0$-algebras are precisely the ring homomorphisms. (To prove this, let R be a canonical $\mathbf{T}_0$-algebra and assume $x = r+p$ where p is nilpotent, r has semi-inverse s and $rp = pr = 0$. Claim $r = x_1$, $p = x_2$ and $s = x_3$. Observe that for sufficiently large m we have $x^m = r^m = x_1{}^m$. This implies $s^m = x_3{}^m$. Hence $x_1 = x_1{}^{m+1} x_3{}^m = r^{m+1} s^m = r$. Therefore, $s = x_3$ and $p = x_2$.)

A ring shall be called *canonical* if it can be endowed with a canonical $\mathbf{T}_0$-structure. Notice that canonical rings are closed under finite products, quotients, the formation of $\mathbf{T}_0$-subalgebras (but *not* under infinite products). Moreover every simple

Artin ring (i.e. ring of all linear transformations of a finite dimensional vector space) is canonical in view of Fitting's Lemma.

We now construct a class $\mathcal{A}$ of Artin rings. Start with all canonical rings which contain a minimal field (i.e. either the rationals or Z_p for some prime p) as a (unitary) subring and which are finite dimensional as vector spaces over the minimal field. To this collection of rings adjoin all finite products to get $\mathcal{A}$. Then every member of $\mathcal{A}$ is an Artin ring and $\mathbf{T}_0$ is a separating triple for $\mathcal{A}$. Thus the $\mathcal{A}$-generated pseudocompact rings is the equational completion of $\mathcal{A}$ and is also the category of pro-object for $\mathcal{A}$ by 3.3 and 2.2. (Each object of $\mathcal{A}$ satisfies d.c.c. for both $\mathbf{T}_0$-subalgebras and $\mathbf{T}_0$-kernels.)

One can also adjoin all fields and skew fields to $\mathcal{A}$ and close up under finite products. Then the resulting rings are still Artin and $\mathbf{T}_0$- is still a separating triple (so 3.3 still applies) but d.c.c. for $\mathbf{T}_0$-subalgebras is lost. If any additional simple Artin rings are adjoined then the argument of the above counter-example will apply.

4.3. Linearly compact modules. Let R be any ring. Then the (discrete) linearly compact R-modules have an equational completion that can be computed by 3.2. The theory of R-modules is still a separating triple even if one treats the more general case of topological R-modules for a topological ring R (as done in [18]) provided the models are always discrete, topological linearly compact R-modules. A number of interesting properties of linearly compact modules can be found in [13] and [18]. The case in which R is a pseudocompact ring is treated in [5]. Then a discrete topological R-module has the d.c.c. iff it is of finite length. The case of pseudocompact algebras is also discussed in [5] and provides another straightforward example for Theorem 3.3.

4.4. Countable sets (a realcompact case). Let $\mathcal{M}$ be the category of countable sets and all functions. Let $I: \mathcal{M} \to \mathcal{S}$ be the inclusion. Then the theory of sets is a separating triple. Moreover, as we shall point out, the Q topologies are all Hausdorff, hence the equational completion is a full subcategory of Top. The main step in computing this full subcategory are:

Definition. A topological space is *nearly discrete* if it is Hausdorff and if every non-measurable collection of open sets has open intersection.

Lemma. (a) *A nearly discrete space is realcompact iff every open cover has a non-measurable subcover.*

(b) *The nearly discrete spaces are coreflective in the category of Hausdorff spaces. The nearly discrete, realcompact spaces are closed under the formation of (coreflected) limits.*

(c) *The category of nearly discrete, realcompact spaces is tripleable over $\mathcal{S}$.*

Proof (sketch). Most of the proof follows by mimicking the corresponding proofs for compact Hausdorff spaces. The coreflection into nearly discrete spaces is accomplished by letting the class of all non-measurable intersections of open sets be a base for the new topology. If X is realcompact then it is a closed subset of R^n for some n. Moreover X is still closed in the nearly discrete topology on R^n so X has the covering property (which is inherited by nearly discrete products and closed subsets). Conversely the covering property implies realcompactness using the z-ultrafilter characterization of realcompact spaces (see ref. [9]). q.e.d.

Corollary. *The equational completion of the category of countable sets is the category of realcompact, nearly discrete topological spaces. (If every cardinal is non-measurable this is the category of discrete spaces or simply $\mathcal{S}$ in effect.)*

Proof (sketch). If X is realcompact then X can be embedded as a closed subset of a product of real lines [9]. If, in addition, X is nearly discrete, then it easily follows that X is equivalent to a closed subset of a (coreflected) product of discrete countable spaces. Using the fact that the continuous image of a realcompact space is (in the category of nearly discrete spaces) realcompact, hence closed, X is the canonical limit of discrete countable spaces. The result now follows straightforwardly. q.e.d.

4.5. What do the operations look like? We have dealt with the general theory of equational completions primarily from the point of view of triples. At this point we collect a few observations about the operations. Recall that the set of n-ary operations is $T(n)$, hence we have a topological space of n-ary operations. Thus the corresponding equational theory $\mathcal{T}$ (say, the category with $\mathcal{T}(n, k) = T(n)^k$) can be regarded as a category with topological hom sets. However, it is *not* a topological category in the sense of Beck [3] since composition $T(n)^k \times T(k) \to T(n)$ although continuous for all finite k (which can be shown by using the type of argument given in 1.2) is nonetheless almost never continuous for infinite k. However, for any fixed k-tuple on a $\mathbf{T}$-algebra X the evaluation map $T(k) \to X$ is continuous (as it is $\mathbf{T}$-admissible).

We know that $T(n) = \lim(n, I)$. Moreover if $\mathbf{T}_0$ is any normal separating triple and if LCC is satisfied then $T(n) = \lim(n, I)_0$ and for every full, filtered subdiagram $\Delta \subseteq (n, I)_0$ the canonical map $T(n) \to \lim \Delta$ is onto (using the argument of 3.10). Thus one can obtain an n-ary operation by prescribing its behaviour on a set Δ of n-tuples and lifting it from $\lim \Delta$ to $T(n)$ (the lifting depends on a choice). Moreover, if the models all satisfy d.c.c. for $\mathbf{T}_0$-kernels then by (b) and (e) of Theorem 3.3, the entire space, $\lim \Delta$, can be *continuously* lifted to a topologically equivalent closed subspace of $T(n)$.

Our final observation is that the equational completions generally do not have rank. For example suppose that the hypotheses of 3.3 are satisfied and that $\mathbf{T}$ has (infinite) rank r. Let $X \subseteq Y$ be any $\mathbf{T}$-subalgebra and let s be the (cardinal) successor of r. Let $A \subseteq Y^s$ be the set of all points which project into X for all but at most r

projections. Then A would be a non-closed **T**-subalgebra contradicting 3.2(c). The same contradiction works for the compact case. It also works for countable sets iff at least one measurable cardinal exists. We note that here the n-ary operations are the ω-ultrafilters on n.

Note. The discontinuity of the property infinitary operations has led us to restrict ourselves at times to the case where the separating triple is finitary. However, all hypotheses of the form "$\mathbf{T}_0$ is finitary" can be eliminated if we agree that a topological $\mathbf{T}_0$-algebra is to be a $\mathbf{T}_0$-algebra with topology such that the *finitary* operations are continuous.

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