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Ulam's type stability of impulsive ordinary differential equations[☆]

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ABSTRACT

In this paper, we introduce four Ulam's type stability concepts for impulsive ordinary differential equations. By applying the integral inequality of Gronwall type for piecewise continuous functions, Ulam's type stability results for impulsive ordinary differential equations are obtained. An example is also provided to illustrate our results.
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1. Introduction

In 1940, the stability of functional equations was originally raised by Ulam at Wisconsin University. The problem posed by Ulam was the following: "Under what conditions does there exist an additive mapping near an approximately additive mapping"? (for more details see [1]). The first answer to the question of Ulam [2] was given by Hyers in 1941 in the case of Banach spaces. Thereafter, this type of stability is called the Ulam–Hyers stability. In 1978, Rassias [3] provided a remarkable generalization of the Ulam–Hyers stability of mappings by considering variables.

As a matter of fact, the Ulam–Hyers stability and the Ulam–Hyers–Rassias stability have been taken up by a number of mathematicians and the study of this area has grown to be one of the central subjects in the mathematical analysis area. For more details on the recent advances on the Ulam–Hyers stability and the Ulam–Hyers–Rassias stability of differential equations, one can see the monographs [4–6] and the research papers [7–26]. However, to the best of our knowledge, Ulam's type stability results of impulsive ordinary differential equations have not been investigated yet.

In this paper, we discuss Ulam's type stability of the following impulsive ordinary differential equations

$$\begin{cases} x'(t) = f(t, x(t)), & t \in J' := J \setminus \{t_1, \dots, t_m\},\ J := [0, T],\ T > 0, \\ \Delta x(t_k) = I_k(x(t_k^-)), & k = 1, 2, \dots, m, \end{cases} \tag{1}$$

where $f : J \times R \to R$ is continuous, $I_k : R \to R$ and t_k satisfy $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T < +\infty$, $x(t_k^+) = \lim_{\epsilon \to 0^+} x(t_k + \epsilon)$ and $x(t_k^-) = \lim_{\epsilon \to 0^-} x(t_k + \epsilon)$ represent the right and left limits of $x(t)$ at $t = t_k$.

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We will introduce four Ulam's type stability definitions for Eq. (1) and mainly present the generalized Ulam–Hyers–Rassias stability results by virtue of the integral inequality of Gronwall type for piecewise continuous functions due to Samoilenko and Perestyuk [27].

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts. Throughout this paper, let $C(J, R)$ be the Banach space of all continuous functions from J into R with the norm $\|x\|_C := \sup\{|x(t)| : t \in J\}$ for $x \in C(J, R)$. We also introduce the Banach space $PC(J, R) := \{x : J \to R : x \in C((t_k, t_{k+1}], R),\ k = 0, \ldots, m$ and there exist $x(t_k^-)$ and $x(t_k^+)$, $k = 1, \ldots, m$, with $x(t_k^-) = x(t_k)\}$ with the norm $\|x\|_{PC} := \sup\{|x(t)| : t \in J\}$. Denote $PC^1(J, R) := \{x \in PC(J, R) : x' \in PC(J, R)\}$. Set $\|x\|_{PC^1} := \max\{\|x\|_{PC}, \|x'\|_{PC}\}$. It can be seen that endowed with the norm $\|\cdot\|_{PC^1}$, $PC^1(J, R)$ is also a Banach space.

Integral inequalities play an important role in the qualitative analysis of the solutions to ordinary differential equations. Samoilenko and Perestyuk [27] have first studied the following integral inequality of Gronwall type for piecewise continuous functions which will be used in the sequel.

Lemma 2.1. *Let for $t \geq t_0 \geq 0$ the following inequality hold*

$$x(t) \leq a(t) + b\int_{t_0}^{t} x(s)ds + \sum_{t_0<t_k<t} \beta_k x(t_k), \tag{2}$$

where $x, a \in PC([t_0, \infty), R^+)$, a is nondecreasing and $b, \beta_k > 0$.

Then, for $t \geq t_0$, the following inequality is valid:

$$x(t) \leq a(t) \prod_{t_0<t_k<t} (1+\beta_k) \exp\left(\int_{t_0}^{t} b(s)ds\right).$$

For more integral inequalities of Gronwall type for piecewise continuous functions, one can see [28,29].

3. Ulam's type stability concepts

In this section, we introduce Ulam's type stability concepts for Eq. (1). Let $\epsilon > 0$, $\psi \geq 0$ and $\varphi \in PC(J, R^+)$ is nondecreasing. We consider the following inequalities

$$\begin{cases} |y'(t) - f(t, y(t))| \leq \epsilon, & t \in J', \\ |\Delta y(t_k) - I_k(y(t_k^-))| \leq \epsilon, & k = 1, 2, \ldots, m, \end{cases} \tag{3}$$

$$\begin{cases} |y'(t) - f(t, y(t))| \leq \varphi(t), & t \in J', \\ |\Delta y(t_k) - I_k(y(t_k^-))| \leq \psi, & k = 1, 2, \ldots, m, \end{cases} \tag{4}$$

and

$$\begin{cases} |y'(t) - f(t, y(t))| \leq \epsilon\varphi(t), & t \in J', \\ |\Delta y(t_k) - I_k(y(t_k^-))| \leq \epsilon\psi, & k = 1, 2, \ldots, m. \end{cases} \tag{5}$$

Definition 3.1. Eq. (1) is Ulam–Hyers stable if there exists a real number $c_{f,m} > 0$ such that for each $\epsilon > 0$ and for each solution $y \in PC^1(J, R)$ of inequality (3) there exists a solution $x \in PC^1(J, R)$ of Eq. (1) with

$$|y(t) - x(t)| \leq c_{f,m}\epsilon, \quad t \in J.$$

Definition 3.2. Eq. (1) is generalized Ulam–Hyers stable if there exists $\theta_{f,m} \in C(R^+, R^+)$, $\theta_{f,m}(0) = 0$ such that for each solution $y \in PC^1(J, R)$ of inequality (3) there exists a solution $x \in PC^1(J, R)$ of Eq. (1) with

$$|y(t) - x(t)| \leq \theta_{f,m}(\epsilon), \quad t \in J.$$

Definition 3.3. Eq. (1) is Ulam–Hyers–Rassias stable with respect to (φ, ψ) if there exists $c_{f,m,\varphi} > 0$ such that for each $\epsilon > 0$ and for each solution $y \in PC^1(J, R)$ of inequality (5) there exists a solution $x \in PC^1(J, R)$ of Eq. (1) with

$$|y(t) - x(t)| \leq c_{f,m,\varphi}\epsilon(\varphi(t) + \psi), \quad t \in J.$$

Definition 3.4. Eq. (1) is generalized Ulam–Hyers–Rassias stable with respect to (φ, ψ) if there exists $c_{f,m,\varphi} > 0$ such that for each solution $y \in PC^1(J, R)$ of inequality (4) there exists a solution $x \in PC^1(J, R)$ of Eq. (1) with

$$|y(t) - x(t)| \leq c_{f,m,\varphi}(\varphi(t) + \psi), \quad t \in J.$$

Remark 3.5. It is clear that: (i) Definition 3.1 $\Longrightarrow$ Definition 3.2; (ii) Definition 3.3 $\Longrightarrow$ Definition 3.4; (iii) Definition 3.3 for $\varphi(t) = \psi = 1 \Longrightarrow$ Definition 3.1.

Remark 3.6. A function $y \in PC^1(J, R)$ is a solution of inequality (3) if and only if there is $g \in PC(J, R)$ and a sequence g_k, $k = 1, 2, \ldots, m$ (which depend on y) such that

(i) $|g(t)| \leq \epsilon, \ t \in J$ and $|g_k| \leq \epsilon, k = 1, 2, \ldots, m$;
(ii) $y'(t) = f(t, y(t)) + g(t), \ t \in J'$;
(iii) $\Delta y(t_k) = I_k(y(t_k^-)) + g_k, \ k = 1, 2, \ldots, m$.

One can have similar remarks for inequalities (4) and (5).

So, Ulam's type stabilities of the impulsive differential equations are some special types of data dependence of the solutions of impulsive differential equations.

Remark 3.7. If $y \in PC^1(J, R)$ is a solution of inequality (3) then y is a solution of the following integral inequality

$$\left| y(t) - y(0) - \sum_{i=1}^{k} I_i(y(t_i^-)) - \int_0^t f(s, y(s))ds \right| \leq (m + t)\epsilon, \quad t \in J. \tag{6}$$

Indeed, by Remark 3.6 we have that

$$\begin{cases} y'(t) = f(t, y(t)) + g(t), & t \in J', \\ \Delta y(t_k) = I_k(y(t_k^-)) + g_k, & k = 1, 2, \ldots, m. \end{cases}$$

Then

$$y(t) = y(0) + \sum_{i=1}^{k} I_i(y(t_i^-)) + \sum_{i=1}^{k} g_i + \int_0^t f(s, y(s))ds + \int_0^t g(s)ds, \quad t \in (t_k, t_{k+1}].$$

From this it follows that

$$\begin{aligned} \left| y(t) - y(0) - \sum_{i=1}^{k} I_i(y(t_i^-)) - \int_0^t f(s, y(s))ds \right| &\leq \sum_{i=1}^{m} |g_i| + \int_0^t |g(s)|ds \\ &\leq m\epsilon + \epsilon \int_0^t ds \\ &\leq (m + t)\epsilon. \end{aligned}$$

We have similar remarks for the solutions of inequalities (4) and (5).

4. Main results

Now, we are ready to state our main results in this paper.

Theorem 4.1. *Assume $f : J \times R \to R$ is continuous and there exists a constant $L_f > 0$ such that $|f(t, u) - f(t, v)| \leq L_f |u - v|$ for each $t \in J$ and all $u, v \in R$. Moreover, I_k: $R \to R$ and there exist constants $\rho_k > 0$ such that $|I_k(u) - I_k(v)| \leq \rho_k |u - v|$ for all $u, v \in R$ and $k = 1, 2, \ldots, m$. If there exists a $\lambda_\varphi > 0$ such that $\int_0^t \varphi(s)ds \leq \lambda_\varphi \varphi(t)$ for each $t \in J$ where $\varphi \in PC(J, R^+)$ is nondecreasing, then Eq. (1) is generalized Ulam–Hyers–Rassias stable with respect to (φ, ψ).*

Proof. Let $y \in PC^1(J, R)$ be a solution of inequality (4). Denote by x the unique solution of the impulsive Cauchy problem

$$\begin{cases} x'(t) = f(t, x(t)), & t \in J', \\ \Delta x(t_k) = I_k(x(t_k^-)), & k = 1, 2, \ldots, m, \\ x(0) = y(0). \end{cases} \tag{7}$$

Then we have

$$x(t)=\begin{cases} y(0)+\int_0^t f(s,x(s))ds, & \text{for } t\in[0,t_1],\\ y(0)+I_1(x(t_1^-))+\int_0^t f(s,x(s))ds, & \text{for } t\in(t_1,t_2],\\ y(0)+I_1(x(t_1^-))+I_2(x(t_2^-))+\int_0^t f(s,x(s))ds, & \text{for } t\in(t_2,t_3],\\ \vdots \\ y(0)+\sum_{k=1}^m I_k(x(t_k^-))+\int_0^t f(s,x(s))ds, & \text{for } t\in(t_m,T].\end{cases}$$

Like in (6), by differential inequality (4), for each $t\in(t_k,t_{k+1}]$, we have

$$\begin{aligned}\left|y(t)-y(0)-\sum_{i=1}^k I_k(y(t_i^-))-\int_0^t f(s,y(s))ds\right| &\le \sum_{i=1}^m |g_i|+\int_0^t \varphi(s)ds\\ &\le (m+\lambda_\varphi)(\varphi(t)+\psi),\quad t\in J.\end{aligned}$$

Hence for each $t\in(t_k,t_{k+1}]$, it follows

$$\begin{aligned}|y(t)-x(t)| &\le \left|y(t)-y(0)-\sum_{i=1}^k I_i(y(t_i^-))-\int_0^t f(s,y(s))ds\right|\\ &\quad+\sum_{i=1}^k |I_i(x(t_i^-))-I_i(y(t_i^-))|+\int_0^t |f(s,y(s))-f(s,x(s))|ds\\ &\le (m+\lambda_\varphi)(\varphi(t)+\psi)+L_f\int_0^t |y(s)-x(s)|\,ds+\sum_{i=1}^k \rho_i|y(t_i^-)-x(t_i^-)|.\end{aligned}$$

By Lemma 2.1, we obtain

$$\begin{aligned}|y(t)-x(t)| &\le (m+\lambda_\varphi)(\varphi(t)+\psi)\left(\prod_{0<t_k<t}(1+\rho_k)e^{L_f t}\right),\\ &\le c_{f,m,\varphi}(\varphi(t)+\psi),\quad t\in J,\end{aligned}$$

where

$$c_{f,m,\varphi}:=(m+\lambda_\varphi)\prod_{k=1}^m(1+\rho_k)e^{L_f T}>0.$$

Thus, Eq. (1) is generalized Ulam–Hyers–Rassias stable with respect to (φ,ψ). The proof is completed. □

Remark 4.2. (i) Under the assumptions of Theorem 4.1, we consider Eq. (1) and inequality (5). One can repeat the same process to verify that Eq. (1) is Ulam–Hyers–Rassias stable with respect to (φ,ψ).

(ii) Under the assumptions of Theorem 4.1, we consider Eq. (1) and inequality (3). One can repeat the same process to verify that Eq. (1) is Ulam–Hyers stable.

(iii) One can extend the above results to the case of Eq. (1) with $T=+\infty$.

Example 4.3. We consider the impulsive ordinary differential equation

$$\begin{cases} x'(t)=0,\quad t\in(0,1]\setminus\left\{\frac{1}{2}\right\},\\ \Delta x\left(\frac{1}{2}\right)=\dfrac{\left|x\left(\frac{1}{2}^-\right)\right|^{\frac{1}{2}}}{1+\left|x\left(\frac{1}{2}^-\right)\right|^{\frac{1}{2}}},\end{cases}\tag{8}$$

and the inequalities

$$\begin{cases} |y'(t)| \leq \epsilon, \quad t \in (0,1] \setminus \left\{\frac{1}{2}\right\}, \\ \left| \Delta y\left(\frac{1}{2}\right) - \frac{\left|y\left(\frac{1}{2}^-\right)\right|^{\frac{1}{2}}}{1+|y\left(\frac{1}{2}^-\right)|^{\frac{1}{2}}} \right| \leq \epsilon, \quad \epsilon > 0. \end{cases} \tag{9}$$

Let $y \in PC^1([0,1],R)$ be a solution of inequality (9). Then there exist $g \in PC^1([0,1],R)$ and $g_1 \in R$ such that:

(i) $|g(t)| \leq \epsilon, \quad t \in [0,1], \qquad |g_1| \leq \epsilon,$ (10)

(ii) $y'(t) = g(t), \quad t \in [0,1] \setminus \left\{\frac{1}{2}\right\},$ (11)

(iii) $\Delta y\left(\frac{1}{2}\right) = \frac{\left|y\left(\frac{1}{2}^-\right)\right|^{\frac{1}{2}}}{1+\left|y\left(\frac{1}{2}^-\right)\right|^{\frac{1}{2}}} + g_1.$ (12)

Integrating (11) from 0 to t via (12), we have

$$y(t) = y(0) + \chi_{(\frac{1}{2},1]}(t)\left(\frac{\left|y\left(\frac{1}{2}^-\right)\right|^{\frac{1}{2}}}{1+\left|y\left(\frac{1}{2}^-\right)\right|^{\frac{1}{2}}} + g_1\right) + \int_0^t g(s)ds,$$

for the characteristic function $\chi_{(\frac{1}{2},1]}(t)$ of $(\frac{1}{2},1]$.

Let us take the solution x of (8) given by

$$x(t) = y(0) + \chi_{(\frac{1}{2},1]}(t)\frac{\left|x\left(\frac{1}{2}^-\right)\right|^{\frac{1}{2}}}{1+\left|x\left(\frac{1}{2}^-\right)\right|^{\frac{1}{2}}}.$$

Then we have

$$\begin{aligned} |y(t)-x(t)| &= \left|\chi_{(\frac{1}{2},1]}(t)\left(\frac{\left|y\left(\frac{1}{2}^-\right)\right|^{\frac{1}{2}}}{1+\left|y\left(\frac{1}{2}^-\right)\right|^{\frac{1}{2}}} - \frac{\left|x\left(\frac{1}{2}^-\right)\right|^{\frac{1}{2}}}{1+\left|x\left(\frac{1}{2}^-\right)\right|^{\frac{1}{2}}} + g_1\right) + \int_0^t g(s)ds\right| \\ &\leq \chi_{(\frac{1}{2},1]}(t)\left|y\left(\frac{1}{2}^-\right) - x\left(\frac{1}{2}^-\right)\right|^{\frac{1}{2}} + |g_1| + \int_0^t |g(s)|ds \\ &\leq \chi_{(\frac{1}{2},1]}(t)\left|y\left(\frac{1}{2}^-\right) - x\left(\frac{1}{2}^-\right)\right|^{\frac{1}{2}} + \epsilon + \epsilon\int_0^t ds \\ &\leq \chi_{(\frac{1}{2},1]}(t)\left|y\left(\frac{1}{2}^-\right) - x\left(\frac{1}{2}^-\right)\right|^{\frac{1}{2}} + 2\epsilon, \quad t \in [0,1], \end{aligned}$$

which gives

$$|y(t)-x(t)| \leq 2\epsilon + \sqrt{2\epsilon}, \quad t \in [0,1].$$

Thus, Eq. (8) is generalized Ulam–Hyers stable, which is a special case of generalized Ulam–Hyers–Rassias stable.

Remark 4.4. We replace (9) by the following inequalities:

$$\begin{cases} |y'(t)| \le \varphi(t), \quad t \in (0,1] \setminus \left\{\frac{1}{2}\right\}, \ \varphi \in PC([0,1], R^+), \\ \left| \Delta y\left(\frac{1}{2}\right) - \frac{\left|y\left(\frac{1}{2}^-\right)\right|^{\frac{1}{2}}}{1+\left|y\left(\frac{1}{2}^-\right)\right|^{\frac{1}{2}}} \right| \le \psi, \quad \psi > 0. \end{cases} \tag{13}$$

Moreover, we replace (10) by

$$|g(t)| \le \psi, \quad t \in [0,1], \qquad |g_1| \le \psi.$$

One can repeat the same process in Example 4.3 to verify the main result in Theorem 4.1.

5. Other results

Motivated by the nonlinear impulsive terms in Example 4.3, we can prove generalized Ulam–Hyers–Rassias stability for Eq. (1) under assumptions that $f : J \times R \to R$ is continuous and there exists a constant $L_f > 0$ such that $|f(t,u) - f(t,v)| \le L_f|u-v|$ for each $t \in J$ and all $u, v \in R$. Moreover, I_k: $R \to R$ and there exist nondecreasing functions $\rho_k \in C(R^+, R^+)$, $\rho_k(0) = 0$ such that $|I_k(u) - I_k(v)| \le \rho_k(|u-v|)$ for all $u, v \in R$ and $k = 1, 2, \ldots, m$.

Indeed, following the proof of Theorem 4.1, we are led to the inequality

$$|v(t)| \le (m+T)\epsilon + L_f \int_0^t |v(s)|\, ds + \sum_{i=1}^k \rho_i(|v(t_i^-)|), \quad t \in (t_k, t_{k+1}] \tag{14}$$

for $v(t) := y(t) - x(t)$. Let $M_k := \sup_{t \in [t_k, t_{k+1}]} |v(t)|$ for $k = 0, \ldots, m$. Then (14) implies

$$|v(t)| \le (m+T)\epsilon + L_f \int_0^t |v(s)|\, ds + \sum_{i=1}^k \rho_i(M_{i-1}), \quad t \in (t_k, t_{k+1}]$$

and using the standard Gronwall inequality we get

$$M_k \le \left((m+T)\epsilon + \sum_{i=1}^k \rho_i(M_{i-1}) \right) e^{L_f T}. \tag{15}$$

Setting

$$\theta_0(\epsilon) = (m+T)\epsilon e^{L_f T},$$
$$\theta_k(\epsilon) = \left((m+T)\epsilon + \sum_{i=1}^k \rho_i(\theta_{i-1}(\epsilon)) \right) e^{L_f T}, \quad k = 1, \ldots, m.$$

Obviously, inequality (15) implies

$$M_k \le \theta_k(\epsilon), \quad k = 0, \ldots, m.$$

Hence

$$|v(t)| \le \theta_{f,m}(\epsilon) = \max\{\theta_k(\epsilon) : k = 0, \ldots, m\}.$$

Clearly $\theta_{f,m} \in C(R^+, R^+)$ and $\theta_{f,m}(0) = 0$. This proves our statement.

6. Final remark

Concerning the Lipschitz condition on f in Section 5, we may change it to the non-Lipschitz condition, i.e., there exists a nondecreasing function $G \in C(R^+, R^+)$ and $G(0) = 0$, $G(u) > 0$ for $u > 0$ such that $|f(t,u) - f(t,v)| \le G(|u-v|)$ for each $t \in J$ and all $u, v \in R$. The first difficulty arises with the existence and uniqueness of the impulsive Cauchy problem (7). But suppose that the corresponding solution exists on J. Then like above, we have

$$|v(t)| \le (m+T)\epsilon + \int_0^t G(|v(s)|)ds + \sum_{i=1}^k \rho_i(M_{i-1}), \quad t \in (t_k, t_{k+1}].$$

Considering the Bihari inequality instead of Gronwall one and repeating the same process in Section 5, one can obtain the same result after putting some additional conditions on G.

To see this, applying a Bihari inequality [30] we get

$$M_k \le \Omega^{-1}\left(\Omega\left(\left((m+T)\epsilon + \sum_{i=1}^{k} \rho_i(M_{i-1})\right)\right) + T\right), \quad t \in J,$$

where

$$\Omega(v) = \int_{v_0}^{v} \frac{d\sigma}{G(\sigma)}, \quad v_0 > 0, \tag{16}$$

and Ω^{-1} is the inverse of Ω and $\left(\Omega\left(\left((m+T)\epsilon + \sum_{i=1}^{k} \rho_i(M_{i-1})\right)\right) + t\right) \in D(\Omega^{-1})$ for all $t \in J$.

Setting

$$\theta_0(\epsilon) = \Omega^{-1}\left(\Omega\left((m+T)\epsilon\right) + T\right),$$

$$\theta_k(\epsilon) = \Omega^{-1}\left(\Omega\left(\left((m+T)\epsilon + \sum_{i=1}^{k} \rho_i(\theta_{i-1}(\epsilon))\right)\right) + T\right), \quad k = 1, \ldots, m.$$

Obviously,

$$M_k \le \theta_k(\epsilon), \quad k = 0, \ldots, m.$$

Hence

$$|v(t)| \le \theta_{f,m}(\epsilon) = \max\{\theta_k(\epsilon) : k = 0, \ldots, m\}.$$

To obtain the same result as in Section 5, we have to put some additional conditions (maybe very strong) on G such that $\Omega^{-1} \in C(R^+, R^+)$ with the property: there is a function $\omega \in C(R^+, R^+)$, $\omega(0) = 0$ such that $\Omega^{-1}(x+y) \le \omega(\Omega^{-1}(x))\omega(\Omega^{-1}(y))$ for any $x, y \in R^+$. For $\Omega^{-1}(v) = v_0 e^{Lv}$ we take $\omega(x) = \sqrt{v_0}x$ and then $G(x) = Lx$, so we obtain the Lipschitz case of Section 5.

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