Some Inequalities Involving Meromorphically Multivalent Functions

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While presenting an interesting improvement of a result of R. M. Robinson (Trans. Amer. Math. Soc. 61, 1947, 1–35), S. Owa, M. Nunokawa, and H. Saitoh (Ann. Polon. Math. 59, 1994, 159–162) obtained a number of inequalities for functions belonging to the class of normalized p-valent analytic functions. The main object of this paper is to derive analogous inequalities involving certain meromorphically multivalent functions.

1. INTRODUCTION

Let $A_p$ be the class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in \mathbb{N} = \{1, 2, 3, \ldots\}) \quad (1.1)$$
which are regular in the unit disk $U = \{z : |z| < 1\}$. We denote $\Sigma_p$ by the class of functions of the form

$$f(x) = \frac{1}{z^p} + \frac{a_0}{z^{p-1}} + \cdots + a_{k+p-1}z^k + \cdots \quad (p \in \mathbb{N}) \quad (1.2)$$

which are regular in the punctured disk $D = \{z : 0 < |z| < 1\}$.

In 1947, Robinson [5] proved the following

**Theorem A.** Let $S(z)$ and $T(z)$ be regular in $U$ and let $\Re(\log S(z)/S(z)) > 0 (z \in U)$. If $|T'/(z)/S'(z)| < 1 (z \in U)$ and $T(0) = 0$, then $|T(z)/S(z)| < 1 (z \in U)$.

Recently, Owa, Nunokawa, and Saitoh [4] improved Theorem A and investigated some interesting inequalities for functions belonging to the class $A_p$. In the present paper, we derive some inequalities involving meromorphically multivalent functions belonging to the class $\Sigma_p$. Our results are analogous to Theorem A and to the results obtained by Owa, Nunokawa, and Saitoh [4]. For various other interesting developments involving functions in the classes $A_p$ and $\Sigma_p$, the reader may be referred (for example) to the recent work of Srivastava and Owa [6].

To establish our results, we shall make use of the following lemmas.

**Lemma 1** [1, 2]. Let $\omega(z)$ be regular in $U$ with $\omega(0) = 0$. If $|\omega(z)|$ attains its maximum value in the circle $|z| = r < 1$ at a point $z_0 \in U$, then we can write

$$z_0 \omega'(z_0) = k \omega(z_0), \quad (1.3)$$

where $k$ is real and $k \geq 1$.

**Lemma 2** [3]. Let $p(z)$ be regular in $U$ with $p(0) = 1$. If there exists a point $z_0 \in U$ such that

$$\Re(p(z)) > 0 \quad (|z| < |z_0|),$$

$$\Re(p(z_0)) = 0, \quad \text{and} \quad p(z_0) \neq 0,$$

then $p(z_0) = ia$ ($a \neq 0$) and

$$\frac{z_0 p'(z_0)}{p(z_0)} = i \frac{k}{2} \left( a + \frac{1}{a} \right), \quad (1.4)$$

where $k$ is real and $k \geq 1$. 
2. THE MAIN RESULTS

By using Lemma 1, we first derive

**Theorem 1.** Let $S(z) \in \Sigma_m$, $T(z) \in \Sigma_n$ with $p = n - m \geq 1$. Let $S(z)$ satisfy $\text{Re}(S(z)/zS'(z)) < -\alpha$ ($0 \leq \alpha < 1/m$). If

$$\text{Re}\left(\frac{S(z)}{zS'(z)} \frac{zT'(z) + q}{T(z)}\right) > 1 + (q - p)\alpha \quad (z \in U) \quad (2.1)$$

with $q \geq p$, then

$$\left|\frac{T(z)}{S(z)}\right| < \frac{1}{|z|^{p+1}} \quad (z \in U). \quad (2.2)$$

**Proof.** Since $T(z)/S(z) = 1/z^p + \cdots \in \Sigma_p$, we define the function $\omega(z)$ by

$$T(z) = \frac{\omega(z)S(z)}{z^{p+1}}.$$

Then $\omega(z)$ is regular in $U$ with $\omega(0) = 0$. It follows from the definition of $\omega(z)$ that

$$\frac{T'(z)}{S'(z)} = \omega(z) \left\{1 + \left(\frac{z\omega'(z)}{\omega(z)} - p - 1\right)\frac{S(z)}{zS'(z)}\right\},$$

or

$$\frac{S(z)}{zS'(z)} \left(\frac{zT'(z)}{T(z)} + q\right) = \left(\frac{T'(z)}{S'(z)}\right) \left(\frac{S(z)}{T(z)}\right) + q \frac{S(z)}{zS'(z)}$$

$$= 1 + \left(\frac{z\omega'(z)}{\omega(z)} + q - p - 1\right)\frac{S(z)}{zS'(z)}. \quad (2.3)$$

If we suppose that there exists a point $z_0 \in U$ such that

$$\max_{|z| \leq |z_0|} |\omega(z)| = |\omega(z_0)| = 1,$$

then Lemma 1 gives us

$$z_0 \omega'(z_0) = k \omega(z_0) \quad (k \geq 1).$$
Therefore, we have

\[
\Re\left( \frac{S(z)}{z S'(z)} \left( \frac{zT'(z)}{T(z)} + q \right) \right) = 1 + (k + q - p - 1) \Re\left( \frac{S(z_0)}{z_0 S'(z_0)} \right)
\]

\[
\leq 1 - (k + q - p - 1) \alpha
\]

\[
\leq 1 - (q - p) \alpha.
\] (2.4)

This contradicts the hypothesis of Theorem 1, so that \(|\omega(z)| < 1\) for all \(z \in U\). This completes the proof of Theorem 1.

Next, by applying Lemma 2, we prove

**Theorem 2.** Let \(S(z) \in S_m, T(z) \in S_n\) with \(p = n - m \geq 1\). Let \(S(z)\) satisfy \(-\Re\{S(z)/z S'(z)\} > \alpha\) \((0 \leq \alpha < 1/2)\) and \(-\alpha/p \leq \Im\{-S(z)/(z S'(z))\} \leq \alpha/p\) \((0 \leq \alpha < 1/m)\). If

\[
-\Re\left( \frac{z^p T'(z)}{S'(z)} \right) > 0 \quad (z \in U),
\] (2.5)

then

\[
\Re\left( \frac{z^p T(z)}{S(z)} \right) > 0 \quad (z \in U).
\] (2.6)

**Proof.** Defining the function \(q(z)\) by \(T(z) = z^{-p} q(z) S(z)\), we see that \(q(z)\) is regular in \(U\) with \(q(0) = 1\). Note that

\[
\frac{z^p T'(z)}{S'(z)} = q(z) \left\{ 1 + \left( p - \frac{z q'(z)}{q(z)} \right) \left( -\frac{S(z)}{z S'(z)} \right) \right\}.
\] (2.7)

Suppose that there exists a point \(z_0 \in U\) such that

\[
\Re(q(z)) > 0 \quad (|z| < |z_0|),
\]

\[
\Re(q(z_0)) = 0, \quad \text{and} \quad q(z_0) \neq 0.
\]

Then, by applying Lemma 2, we have \(q(z_0) = ia\) \((a \neq 0)\) and

\[
\frac{z_0 q'(z_0)}{q(z_0)} = \frac{k}{2} \left( a + \frac{1}{a} \right) \quad (k \geq 1).
\]
Therefore, writing \((-S(z_0))/(z_0S'(z_0)) = \alpha_0 + i\beta_0\), we obtain
\[
-\Re \left( \frac{z^d T'(z_0)}{S'(z_0)} \right) = pa\beta_0 - \frac{k}{2}a(a + 1)\alpha_0 \leq pa\beta_0 - \frac{1 + a^2}{2}\alpha_0 < pa\beta_0 - \frac{1 + a^2}{2}\alpha. \tag{2.8}
\]

Since \(-\alpha/p \leq \beta_0 \leq \alpha/p\), if \(a > 0\), then
\[
pa\beta_0 - \frac{1 + a^2}{2}\alpha \leq a\alpha - \frac{1 + a^2}{2}\alpha = -\frac{\alpha}{2}(1 - \alpha)^2 \leq 0, \tag{2.9}
\]
and if \(a < 0\), then
\[
pa\beta_0 - \frac{1 + a^2}{2}\alpha \leq -a\alpha - \frac{1 + a^2}{2}\alpha = -\frac{\alpha}{2}(1 + a)^2 \leq 0. \tag{2.10}
\]

This contradicts the condition (2.5). Consequently, \(\Re(q(z)) > 0\) for all \(z \in U\), so that \(\Re((z^p T(z))/S(z)) > 0\) \((z \in U)\).

Lastly, by applying, the same technique as in the proof of Theorem 2, we obtain

**Theorem 3.** Let \(S(z) \in \Sigma_m, T(z) \in \Sigma_n\) with \(p = m - n \geq 1\). Let \(S(z)\) satisfy \(-\Re(S(z)/(zS'(z))) > \alpha\) \((0 \leq \alpha < 1/m)\) and \(-\alpha/p \leq \Im((-S(z))/(zS'(z))) \leq \alpha/p\) \((0 \leq \alpha < 1/m)\). If
\[
-\Re \left( \frac{T'(z)}{z^p S'(z)} \right) > 0 \quad (z \in U), \tag{2.11}
\]
then
\[
\Re \left( \frac{T(z)}{z^p S(z)} \right) > 0 \quad (z \in U).
\]
Remark. Theorem 2 and Theorem 3 are comparable with the results given earlier by Owa, Nunokawa, and Saitoh [4] for analytic functions in the class $A_p$.

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