stochastic processes and their applications

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Stochastic calculus with respect to fractional Brownian motion with Hurst parameter lesser than $\frac{1}{2}$

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Abstract

In this paper we introduce a stochastic integral with respect to the process $B_t = \int_0^t (t-s)^{-\alpha}\, \mathrm{d}W_s$ where $0 < \alpha < 1/2$, and W_t is a Brownian motion. Sufficient integrability conditions are deduced using the techniques of the Malliavin calculus and the notion of fractional derivative. We study continuity properties of the indefinite integral and we derive a maximal inequality.

1. Introduction

Fractional Brownian motion (fBm) of Hurst parameter $H \in (0,1)$ has been introduced by Mandelbrot and Van Ness (1968) as a centered Gaussian process $B^H = \{B_t^H,\ t \geqslant 0\}$ with covariance

$$E(B_s^H B_t^H) = \frac{V_H}{2}(s^{2H} + t^{2H} - |t-s|^{2H}),$$

where V_H is a normalizing constant given by

$$V_H = \frac{\Gamma(2-2H)\cos(\pi H)}{\pi H(1-2H)}.$$

It is a process starting from zero with stationary increments, $E(B_t^H - B_s^H)^2 = V_H |t-s|^{2H}$, which is self-similar, that is, $B_{\alpha t}^H$ has the same distribution as $\alpha^H B_t^H$. The constant H determines the sign of the covariance of the future and past increments. This covariance is positive when $H > \frac{1}{2}$ and negative when $H < \frac{1}{2}$. The case $H = \frac{1}{2}$ corresponds to the ordinary Brownian motion. Furthermore it exhibits a long-range dependence in the sense that the covariance between increments at a distance u decreases to zero as u^{2H-2}.

The self-similarity and long-range dependence properties make the fractional Brownian motion a suitable driving noise in different applications like mathematical finance

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and network traffic analysis. Recently, there have been several attempts to construct a stochastic calculus with respect to the fBm. Since for $H \neq \frac{1}{2}$ the fBm is not a semimartingale we cannot use the classical Itô calculus, and different approaches have been proposed.

In the case $\frac{1}{2} < H < 1$, Lin (1995) and Dai and Heyde (1996) have defined a stochastic integral $\int_0^t \phi_s \, dB_s^H$ as limit of Riemann sums. This integral does not satisfy the property $E(\int_0^t \phi_s \, dB_s^H) = 0$. For this reason, Duncan et al. (1998) have introduced a new stochastic integral with zero mean which is the limit of Riemann sums defined by means of the Wick product.

Since the fBm is a Gaussian process, one can apply the stochastic calculus of variations (see Nualart, 1995) and introduce the stochastic integral as the divergence operator, that is, the adjoint of the derivative operator. This idea has been developed by Decreusefond and Üstünel (1998a,b), in the general case $H \in (0,1)$. Given an integral representation of the form $B_t^H = \int_0^t K_H(t,s) \, dW_s$, where W is a Wiener process, the stochastic integral of a process ϕ constructed by this method turns out to be equal to $\int_0^t K_H(t,s)\phi_s \, dW_s$.

Using the notions of fractional integral and derivative, Zähle (1998) has introduced a path-wise stochastic integral. The integral of a process ϕ on a time interval $[0,T]$ is defined as

$$\int_0^T \phi_s \, dB_s^H = (-1)^\beta \int_0^T D_{0'}^\beta \phi(s) D_{T-}^{1-\beta} B^H(s) \, ds + \phi(0+) B_T^H,$$

provided ϕ is such that $\phi - \phi(0+)$ belongs to the space $I_{0+}^\beta(L^1([0,T])$ almost surely, where D^β denotes the fractional derivative of order β, and assuming $\beta > 1-H$. If the process ϕ has λ-Hölder continuous paths with $\lambda > 1-H$, then this integral can be interpreted as a Riemann–Stieltjes integral.

Our aim is to construct a stochastic integral with respect to the fBm with Hurst parameter $0 < H < \frac{1}{2}$, following the ideas introduced by Carmona and Coutin (1998). Let us describe the main ideas of this approach. It is well-known that the fBm $B^H = \{B_t^H, \ t \geqslant 0\}$ admits the following representation:

$$B_t^H = \frac{1}{\Gamma(1-\alpha)} \left\{ Z_t + \int_0^t (t-s)^{-\alpha} \, dW_s \right\},$$

where $\{W_s, s \in \mathbb{R}\}$ is a standard Brownian motion, $\alpha = \frac{1}{2} - H \in (0, \frac{1}{2})$, and $Z_t = \int_{-\infty}^0 [(t-s)^{-\alpha} - (-s)^{-\alpha}] \, dW_s$. Taking into account that the process Z_t has absolutely continuous trajectories, in order to develop a stochastic calculus with respect to B_t^H it suffices to consider the term $\int_0^t (t-s)^{-\alpha} \, dW_s$, that will be denoted by B_t along the paper.

The process $B_t = \int_0^t (t-s)^{-\alpha} \, dW_s$ can be approximated by the semimartingale $B_t^\varepsilon := \int_0^t (t-s+\varepsilon)^{-\alpha} \, dW_s$. Then, the stochastic integral $\int_0^T \phi_s \, dB_s$ will be defined as the limit of $\int_0^T \phi_s \, dB_s^\varepsilon$ as ε tends to zero. In order to find sufficient conditions for this limit to exist we decompose $\int_0^T \phi_s \, dB_s^\varepsilon$ as the sum of a divergence term (Skorohod integral with respect to W) plus a complementary term involving the derivative operator. This kind of decomposition has already been used to handle the Stratonovich and the forward and backward stochastic integrals with respect to the Brownian motion (see Nualart and Pardoux, 1988).

With this method we find the following sufficient conditions for the existence of $\int_0^T \phi_s \, dB_s$ (see Theorem 5):

(i) $\phi \in I_{T-}^{\alpha}(L^2([0,T];\mathbb{D}^{1,2}))$, that is, the right-fractional derivative of order α of ϕ exists in $[0,T]$ considering ϕ as a function with values in the Sobolev space $\mathbb{D}^{1,2}$, and

(ii) $\int_0^T \int_0^r |D_s\phi_r|(r-s)^{-\alpha-1} \, ds \, dr < \infty$ almost surely, where D denotes the derivative in the sense of Malliavin calculus.

Notice that these conditions differ from those imposed in the work by Zähle. On the one hand, the order of differentiability we require ($\alpha = \frac{1}{2} - H$) is lesser than the order needed in Zähle (1998) (greater than $1-H$). However, we assume additional conditions on the smoothness in the sense of Malliavin calculus. We prove in Proposition 8 that if ϕ satisfies (ii) and is Hölder-continuous in the norm of $\mathbb{D}^{1,2}$ of order $\lambda > \alpha$, then the integral $\int_0^T \phi_s \, dB_s$ is the limit in probability of Riemann sums. As a consequence, if in addition the paths of ϕ are Hölder continuous of order greater than $\frac{1}{2} + \alpha$ then our integral coincides with that defined in Zähle (1998).

The paper is organized as follows. In Section 2 we recall some preliminaries on the Malliavin calculus and on the fractional calculus. In Section 3 we prove the main theorems on the existence of the stochastic integral $\int_0^T \phi_s \, dB_s$. Section 4 is devoted to discuss the convergence of Riemann sums under suitable Hölder continuity assumptions. Finally, in Section 5 we study the existence and continuity of the indefinite integral $\int_0^t \phi_s \, dB_s$.

2. Some preliminaries

In this section we introduce some elements of stochastic calculus of variations and fractional calculus.

2.1. Malliavin calculus

We recall here the basic facts of Malliavin calculus required along the paper. For more precisions, see Nualart (1995).

Let $(\Omega, \mathscr{F}, P)$ be the canonical probability space of the one-dimensional Brownian motion $W = \{W_t, t \in [0,T]\}$. Let H be the Hilbert space $L^2([0,T])$. For any $h \in H$ we denote by $W(h)$ the Wiener integral $W(h) = \int_0^T h(t) \, dW_t$. Let $\mathscr{S}$ be the set of smooth and cylindrical random variables of the form

$$F = f(W(h_1), \ldots, W(h_n)), \tag{2.1}$$

where $n \geqslant 1$, $f \in C_b^{\infty}(\mathbb{R}^n)$ (f and all its derivatives are bounded), and $h_1, \ldots, h_n \in H$. Given a random variable F of the form (2.1), we define its derivative as the stochastic process $\{D_tF, \ t \in [0,T]\}$ given by

$$D_tF = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(W(h_1), \ldots, W(h_n))h_j(t), \quad t \in [0,T].$$

In this way the derivative DF is an element of $L^2([0,T]\times\Omega)\cong L^2(\Omega;H)$. More generally, we can define the iterated derivative operator on a cylindrical random variable by setting

$$D^n_{t_1,\dots,t_n}F = D_{t_1}\dots D_{t_n}F.$$

The iterated derivative operator D^n is a closable unbounded operator from $L^2(\Omega)$ into $L^2([0,T]^n\times\Omega)$ for each $n\geqslant 1$. We denote by $\mathbb{D}^{n,2}$ the closure of $\mathscr{S}$ with respect to the norm defined by

$$||F||^2_{n,2} = ||F||^2_{L^2(\Omega)} + \sum_{l=1}^{n} ||D^l F||^2_{L^2([0,T]^l\times\Omega)}.$$

We denote by δ the adjoint of the derivative operator D that is also called the Skorohod integral with respect to the Brownian motion W. The domain of δ (denoted by Dom δ) is the set of elements $u\in L^2([0,T]\times\Omega)$ such that there exists a constant c verifying

$$\left| E\int_0^T D_tFu_t\,\mathrm{d}t\right| \leqslant c||F||_2$$

for all $F\in\mathscr{S}$. If $u\in$ Dom δ, $\delta(u)$ is the element in $L^2(\Omega)$ defined by the duality relationship

$$E(\delta(u)F) = E\int_0^T D_tFu_t\,\mathrm{d}t, \quad F\in\mathscr{S}.$$

The Skorohod integral is an extension of the Itô integral in the sense that the set $L^2_a([0,T]\times\Omega)$ of square integrable and adapted processes is included into Dom δ and the operator δ restricted to $L^2_a([0,T]\times\Omega)$ coincides with the Itô stochastic integral (see Nualart and Pardoux, 1988). We will make use of the following notation: $\int_0^T u_t\,\mathrm{d}W_t = \delta(u)$.

Let $\mathbb{L}^{n,2} = L^2([0,T];\mathbb{D}^{n,2})$ equipped with the norm

$$||v||^2_{\mathbb{L}^{n,2}} = ||v||^2_{L^2([0,T]\times\Omega)} + \sum_{j=1}^{n} ||D^j v||^2_{L^2([0,T]^{j+1}\times\Omega)}.$$

We recall that $\mathbb{L}^{1,2}$ is included in the domain of δ, and for a process u in $\mathbb{L}^{1,2}$ we can compute the variance of the Skorohod integral of u as follows:

$$E(\delta(u)^2) = E\int_0^T u_t^2\,\mathrm{d}t + E\int_0^T\int_0^T D_su_tD_tu_s\,\mathrm{d}s\,\mathrm{d}t.$$

Let $\mathscr{S}_T$ be the set of processes of the form

$$u_t = \sum_{j=1}^{q} F_j h_j(t),$$

where $F_j\in\mathscr{S}$ and $h_j\in H$. We will denote by $\mathbb{L}^F$ the closure of $\mathscr{S}_T$ by the norm

$$||u||^2_F = E\int_0^T u_t^2\,\mathrm{d}t + E\int_{\{s\geqslant t\}} (D_su_t)^2\,\mathrm{d}s\,\mathrm{d}t + E\int_{\{r\vee s\geqslant t\}} (D_rD_su_t)^2\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}t.$$

That is, $\mathbb{L}^F$ is the class of stochastic processes $\{u_t, t\in[0,T]\}$ such that for each time t, the random variable u_t is twice differentiable with respect to the Wiener process in the two-dimensional future $\{(r,s),\ r\vee s\geqslant t\}$. Notice that if $u\in L_a^2([0,T]\times\Omega)$ then we have $D_su_t=0$ for almost all $s\geqslant t$. As a consequence, the space $L_a^2([0,T]\times\Omega)$ is contained in $\mathbb{L}^F$, and for any $u\in L_a^2([0,T]\times\Omega)$ we have $||u||_F^2=E\int_0^T u_t^2\,\mathrm{d}t$. On the other hand, in Alòs and Nualart (1998) it is proved that $\mathbb{L}^F\subset \operatorname{Dom}\delta$ and

$$E(\delta(u)^2)\leqslant 2||u||_F^2 \tag{2.2}$$

for all $u\in\mathbb{L}^F$.

Let us conclude this section by the following property:

Proposition 1. *Let $u\in\mathbb{L}^{1,2}$ and $F\in\mathbb{D}^{1,2}$ be such that $E[F^2\int_0^T u_t^2\,\mathrm{d}t]<\infty$. Then Fu is Skorohod integrable, and*

$$\int_0^T Fu_t\,\mathrm{d}W_t=F\int_0^T u_t\,\mathrm{d}W_t-\int_0^T D_tFu_t\,\mathrm{d}t, \tag{2.3}$$

provided that the right-hand side of (2.3) *is square integrable.*

2.2. Fractional integrals and derivatives

We here use the notation of Samko et al. (1993), which gives a very complete survey of fractional integrals and derivatives. Let $f\in L^1([0,T])$ and $\alpha>0$. The right-sided fractional Riemann–Liouville integral of f of order α on $[0,T]$ is given at almost all s by

$$I_{T-}^{\alpha}f(s)=\frac{1}{\Gamma(\alpha)}\int_s^T (r-s)^{\alpha-1}f(r)\,\mathrm{d}r.$$

These integrals extend the usual nth-order iterated integrals of f for $\alpha=n\in\mathbb{N}$. We have the first composition formula

$$I_{T-}^{\alpha}(I_{T-}^{\beta}f)=I_{T-}^{\alpha+\beta}f.$$

Fractional differentiation may be introduced as an inverse operation. Let us assume in the sequel that $0<\alpha<1$ and $p>1$. We will denote by $I_{T-}^{\alpha}(L^p)$ the class of functions f in $L^p([0,T])$ which may be represented as an I_{T-}^{α}-integral of some function $\phi\in L^p([0,T])$. If $f\in I_{T-}^{\alpha}(L^p)$, the function ϕ such that $f=I_{T-}^{\alpha}\phi$ is unique in L^p and it agrees with the right-sided Riemann-Liouville derivative of f of order α defined by

$$D_{T-}^{\alpha}f(s)=-\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}s}\int_s^T\frac{f(r)}{(r-s)^{\alpha}}\,\mathrm{d}r.$$

This derivative has the Weyl representation

$$D_{T-}^{\alpha}f(s)=\frac{1}{\Gamma(1-\alpha)}\left(\frac{f(s)}{(T-s)^{\alpha}}-\alpha\int_s^T\frac{f(r)-f(s)}{(r-s)^{\alpha+1}}\,\mathrm{d}r\right),$$

where the convergence of the integrals at the singularity $s=r$ holds in the L^p-sense.

Proposition 2 (Samko et al., 1993, Section 13)**.** *A function $f\in L^p([0,T])$ with $p>1$ belongs to $I_{T-}^{\alpha}(L^p)$ if and only if*

$$\int_0^T\frac{|f(s)|^p}{(T-s)^{\alpha p}}\,\mathrm{d}s<\infty,$$

and the integral

$$\int_{s+\varepsilon}^{T} \frac{f(r)-f(s)}{(r-s)^{\alpha+1}}\,dr$$

converges in $L^p([0,T])$ *as* ε *tends to zero, as a function of* s.

The space $I_{T-}^{\alpha}(L^p)$ is a Banach space with the norm

$$\|f\|_{\alpha,p} = (\|f\|_{L^p([0,T])}^p + \|D_{T-}^{\alpha} f\|_{L^p([0,T])}^p)^{1/p}.$$

For $p=2$, $I_{T-}^{\alpha}(L^2)$ is a Hilbert space with the scalar product

$$\langle f,g\rangle_{\alpha,2} = \langle f,g\rangle_2 + \langle D_{T-}^{\alpha} f, D_{T-}^{\alpha} g\rangle_2.$$

If $\alpha p<1$, then $I_{T-}^{\alpha}(L^p)\subset L^q$, where $1/q=1/p-\alpha$. If $\alpha p>1$, then any function in $I_{T-}^{\alpha}(L^p)$ is $(\alpha-1/p)$-Hölder continuous on $(0,T)$. We will work in the case $p=2$ and $0<\alpha<\frac{1}{2}$ and let us notice that the functions in $I_{T-}^{\alpha}(L^2)$ may not be continuous.

We can extend properly the above definitions and properties to the case of Hilbert-valued functions. If V is a Hilbert space we will denote the space of functions which can be written as the I_{T-}^{α}-integral of a function in $L^p([0,T];V)$ by $I_{T-}^{\alpha}(L^p([0,T];V))$. It is a Banach space with the norm

$$\|f\|_{\alpha,p,V} = (\|f\|_{L^p([0,T];V)}^p + \|D_{T-}^{\alpha} f\|_{L^p([0,T];V)}^p)^{1/p}.$$

Recall that by construction $I_{T-}^{\alpha}(D_{T-}^{\alpha} f)=f$. We also have $D_{T-}^{\alpha}(I_{T-}^{\alpha} f)=f$.

In a similar way we can introduce the left-sided fractional Riemann–Liouville integral and derivative by the formulas

$$I_{0+}^{\alpha} f(s) = \frac{1}{\Gamma(\alpha)}\int_0^s (s-r)^{\alpha-1} f(r)\,dr$$

and

$$D_{0+}^{\alpha} f(s) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{f(s)}{s^{\alpha}} + \alpha\int_0^s \frac{f(s)-f(r)}{(s-r)^{\alpha+1}}\,dr\right).$$

Notice that, if $W=\{W_t, t\in[0,T]\}$ is a one-dimensional Brownian motion, for $0<\alpha<\frac{1}{2}$ we have

$$\int_0^t (t-s)^{-\alpha}\,dW_s = \frac{W_t}{t^{\alpha}} + \alpha\int_0^t \frac{W_t-W_s}{(t-s)^{\alpha+1}}\,ds = \Gamma(1-\alpha) D_{0+}^{\alpha} W_t, \tag{2.4}$$

that is, the process $\int_0^t (t-s)^{-\alpha}\,dW_s$ is equal to the left-sided fractional derivative of the Brownian motion times the factor $\Gamma(1-\alpha)$.

3. Stochastic integral with respect to the fBm

Let $W=\{W_t, t\in[0,T]\}$ be a one-dimensional Brownian motion. We fix a parameter $0<\alpha<\frac{1}{2}$ and define

$$B_t = \int_0^t (t-s)^{-\alpha}\,dW_s.$$

We denote by $\mathscr{F}_t^W$ the σ-field generated by the family of random variables $\{W_s, 0\leqslant s\leqslant t\}$. Notice that $\mathscr{F}_t^B\subset\mathscr{F}_t^W$, where $\mathscr{F}_t^B$ is defined in a similar way.

For every $\varepsilon > 0$ we put

$$B_t^\varepsilon = \int_0^t (t-s+\varepsilon)^{-\alpha}\,\mathrm{d}W_s.$$

Notice that $\{B_t^\varepsilon, t \in [0,T]\}$ is a continuous semimartingale with the following stochastic differential, in the sense of Itô:

$$\mathrm{d}B_t^\varepsilon = \varepsilon^{-\alpha}\,\mathrm{d}W_t + \left(\int_0^t -\alpha(t-s+\varepsilon)^{-\alpha-1}\,\mathrm{d}W_s\right)\mathrm{d}t.$$

As a consequence, the stochastic integral $\int_0^T \phi_t\,\mathrm{d}B_t^\varepsilon$ is well defined for each square integrable and adapted process ϕ. If we assume that ϕ is differentiable in the sense of the Malliavin calculus, we can express the integral $\int_0^T \phi_t\,\mathrm{d}B_t^\varepsilon$ in a different form which is more convenient to take the limit of as ε tends to zero. Moreover, we will suppress for the moment the hypothesis of adaptability.

Lemma 3. *Let $\phi \in \mathbb{L}^{1,2}$. Then*

$$\int_0^T \phi_t\,\mathrm{d}B_t^\varepsilon = \int_0^T \left\{\phi_s(T-s+\varepsilon)^{-\alpha} - \alpha\int_s^T (\phi_r-\phi_s)(r-s+\varepsilon)^{-\alpha-1}\,\mathrm{d}r\right\}\mathrm{d}W_s$$
$$- \alpha\int_0^T\int_0^r D_s\phi_r(r-s+\varepsilon)^{-\alpha-1}\,\mathrm{d}s\,\mathrm{d}r.$$

Proof. We can write

$$\int_0^T \phi_t\,\mathrm{d}B_t^\varepsilon = \varepsilon^{-\alpha}\int_0^T \phi_t\,\mathrm{d}W_t - \alpha\int_0^T\left(\int_0^t (t-s+\varepsilon)^{-\alpha-1}\,\mathrm{d}W_s\right)\phi_t\,\mathrm{d}t,$$

where $\int_0^T \phi_t\,\mathrm{d}W_t$, since ϕ may not be adapted, is a Skorohod integral. Then

$$\begin{aligned}\int_0^T \phi_t\,\mathrm{d}B_t^\varepsilon &= \varepsilon^{-\alpha}\int_0^T \phi_t\,\mathrm{d}W_t - \alpha\int_0^T\left(\int_0^t \phi_t(t-s+\varepsilon)^{-\alpha-1}\,\mathrm{d}W_s\right)\mathrm{d}t\\ &\quad - \alpha\int_0^T\int_0^t D_s\phi_t(t-s+\varepsilon)^{-\alpha-1}\,\mathrm{d}s\,\mathrm{d}t\\ &= \varepsilon^{-\alpha}\int_0^T \phi_t\,\mathrm{d}W_t\\ &\quad - \alpha\int_0^T\left(\int_0^t (\phi_t-\phi_s)(t-s+\varepsilon)^{-\alpha-1}\,\mathrm{d}W_s\right)\mathrm{d}t\\ &\quad - \alpha\int_0^T\left(\int_0^t \phi_s(t-s+\varepsilon)^{-\alpha-1}\,\mathrm{d}W_s\right)\mathrm{d}t\\ &\quad - \alpha\int_0^T\int_0^t D_s\phi_t(t-s+\varepsilon)^{-\alpha-1}\,\mathrm{d}s\,\mathrm{d}t.\end{aligned}$$

Finally, applying Fubini's theorem for the Skorohod integral, we deduce the desired result, because

$$\begin{aligned}&\int_0^T\left(\int_0^t \phi_s(t-s+\varepsilon)^{-\alpha-1}\,\mathrm{d}W_s\right)\mathrm{d}t\\ &\quad = -\frac{1}{\alpha}\int_0^T \phi_s(T-s+\varepsilon)^{-\alpha}\,\mathrm{d}W_s + \frac{1}{\alpha}\varepsilon^{-\alpha}\int_0^T \phi_s\,\mathrm{d}W_s. \qquad \square\end{aligned}$$

This leads to the following definition.

Definition 4. Let $\phi \in \mathrm{Dom}\,\delta$. We say that ϕ is integrable with respect to B if $\int_0^T \phi_t \,\mathrm{d}B_t^\varepsilon$ converges in probability as ε tends to zero. The limit will be denoted by $\int_0^T \phi_t \,\mathrm{d}B_t$.

Though the hypothesis $\phi \in \mathrm{Dom}\,\delta$ is sufficient for the existence of the integral $\int_0^T \phi_t \,\mathrm{d}B_t^\varepsilon$, our aim is to obtain an explicit expression for the limit $\int_0^T \phi_t \,\mathrm{d}B_t$. This will be done in Theorem 5, where we assume $\phi \in \mathbb{L}^{1,2}$ and we make use of Lemma 3. Afterwards we will discuss the adapted case, and more generally, the case $\phi \in \mathbb{L}^F$.

Theorem 5. *Let ϕ be a process satisfying the following conditions*:

$$\phi \in I_{T-}^\alpha(\mathbb{L}^{1,2}), \tag{C1}$$

$$\int_0^T \int_0^r |D_s\phi_r|(r-s)^{-\alpha-1}\,\mathrm{d}s\,\mathrm{d}r < \infty \text{ a.s.} \tag{T1}$$

Then ϕ is integrable with respect to B and we have

$$\int_0^T \phi_t \,\mathrm{d}B_t = \Gamma(1-\alpha)\int_0^T D_{T-}^\alpha \phi_s \,\mathrm{d}W_s - \alpha\int_0^T \int_0^r D_s\phi_r(r-s)^{-\alpha-1}\,\mathrm{d}s\,\mathrm{d}r.$$

Remark 1. The stochastic integral appearing on the right-hand side of the above equality is a Skorohod integral.

Remark 2. Notice that the first summand in the above equality can be heuristically deduced from the fact that $B_t = \Gamma(1-\alpha)D_{0+}^\alpha W_t$ (see (2.4)), and the integration by parts formula (see Samko et al., 1993):

$$\int_0^T \phi_t D_{0+}^\alpha \psi_t \,\mathrm{d}t = \int_0^T D_{T-}^\alpha \phi_t \psi_t \,\mathrm{d}t,$$

provided that $\psi \in I_{0+}^\alpha(L^2)$.

Remark 3. If ϕ is a deterministic function, the above conditions reduce to $\phi \in I_{T-}^\alpha(L^2([0,T]))$, and we have

$$\int_0^T \phi_t \,\mathrm{d}B_t = \Gamma(1-\alpha)\int_0^T D_{T-}^\alpha \phi_s \,\mathrm{d}W_s.$$

As a consequence, the first Wiener chaos of the Gaussian process B_t is isometric to $I_{T-}^\alpha(L^2([0,T]))$ equipped with the scalar product

$$\langle \phi, \varphi \rangle = \Gamma(1-\alpha)^2 \int_0^T D_{T-}^\alpha \phi_s \, D_{T-}^\alpha \varphi_s \,\mathrm{d}s.$$

Remark 4. The space $I_{T-}^\alpha(\mathbb{L}^{1,2}) = I_{T-}^\alpha(L^2([0,T];\mathbb{D}^{1,2}))$ is isometric to the space $\mathbb{D}^{1,2}(I_{T-}^\alpha(L^2([0,T])))$, and it contains all the processes in $\mathbb{L}^{1,2}$ such that the derivative $\{D\phi_s, s\in[0,T]\}$ belongs to $I_{T-}^\alpha(L^2([0,T]))$. In other words, $I_{T-}^\alpha(\mathbb{L}^{1,2})$ is a Hilbert space with the norm

$$\left(E||\phi||_{\alpha,2}^2 + E\int_0^T ||D_r\phi||_{\alpha,2}^2 \,\mathrm{d}r\right)^{1/2} = (||\phi||_{1,2}^2 + ||D_{T-}^\alpha \phi||_{1,2}^2)^{1/2}.$$

Proof. Condition (T1) implies that

$$\int_0^T \int_0^r D_s\phi_r(r-s+\varepsilon)^{-\alpha-1}\,\mathrm{d}s\,\mathrm{d}r$$

converges in probability as ε tends to zero to

$$\int_0^T \int_0^r D_s\phi_r(r-s)^{-\alpha-1}\,\mathrm{d}s\,\mathrm{d}r.$$

Then it suffices to show that

$$\phi_s(T-s+\varepsilon)^{-\alpha} - \alpha\int_s^T (\phi_r-\phi_s)(r-s+\varepsilon)^{-\alpha-1}\,\mathrm{d}r \tag{3.1}$$

converges in the norm of $\mathbb{L}^{1,2}$ to

$$\phi_s(T-s)^{-\alpha} - \alpha\int_s^T (\phi_r-\phi_s)(r-s)^{-\alpha-1}\,\mathrm{d}r, \tag{3.2}$$

as ε tends to zero.

The fact that $\phi \in I^{\alpha}_{T-}(\mathbb{L}^{1,2})$ implies that there exists a process $\varphi \in \mathbb{L}^{1,2} = L^2([0,T];\mathbb{D}^{1,2})$ such that $\phi = I^{\alpha}_{T-}\varphi$, that is

$$\phi_s = \frac{1}{\Gamma(\alpha)}\int_s^T (\sigma-s)^{\alpha-1}\varphi_\sigma\,\mathrm{d}\sigma, \tag{3.3}$$

and (3.2) equals to $\Gamma(1-\alpha)\varphi_s$. Substituting (3.3) into (3.1) yields

$$\begin{aligned}
&\phi_s(T-s+\varepsilon)^{-\alpha} - \alpha\int_s^T (\phi_r-\phi_s)(r-s+\varepsilon)^{-\alpha-1}\,\mathrm{d}r\\
&\quad= \frac{1}{\Gamma(\alpha)}\left\{\int_s^T (\sigma-s)^{\alpha-1}(T-s+\varepsilon)^{-\alpha}\varphi_\sigma\,\mathrm{d}\sigma\right.\\
&\qquad -\alpha\int_s^T\int_r^T (\sigma-r)^{\alpha-1}(r-s+\varepsilon)^{-\alpha-1}\varphi_\sigma\,\mathrm{d}\sigma\,\mathrm{d}r\\
&\qquad \left.+\alpha\int_s^T\int_s^T (\sigma-s)^{\alpha-1}(r-s+\varepsilon)^{-\alpha-1}\varphi_\sigma\,\mathrm{d}\sigma\,\mathrm{d}r\right\}\\
&\quad= \frac{1}{\Gamma(\alpha)}\left\{\varepsilon^{-\alpha}\int_s^T (\sigma-s)^{\alpha-1}\varphi_\sigma\,\mathrm{d}\sigma\right.\\
&\qquad \left.-\alpha\int_s^T\left(\int_s^\sigma (\sigma-r)^{\alpha-1}(r-s+\varepsilon)^{-\alpha-1}\,\mathrm{d}r\right)\varphi_\sigma\,\mathrm{d}\sigma\right\}\\
&\quad= \frac{1}{\Gamma(\alpha)}\int_s^T \Lambda^{\varepsilon}(s,\sigma)\varphi_\sigma\,\mathrm{d}\sigma,
\end{aligned}$$

where

$$\Lambda^{\varepsilon}(s,\sigma) = \varepsilon^{-\alpha}(\sigma-s)^{\alpha-1} - \alpha\int_s^\sigma (\sigma-r)^{\alpha-1}(r-s+\varepsilon)^{-\alpha-1}\,\mathrm{d}r.$$

Notice that $\Lambda^{\varepsilon}(s,t) = \varepsilon^{-1}\Lambda((t-s)/\varepsilon)$, where

$$\Lambda(u) = u^{\alpha-1} - \alpha\int_0^u \frac{(u-r)^{\alpha-1}}{(r+1)^{\alpha+1}}\,\mathrm{d}r = \frac{u^{\alpha-1}}{u+1},$$

and the second equality follows after the change of variable $y=(u+1)/(r+1)-1$. Then $\Lambda(u)\geqslant 0$ and $\int_0^\infty \Lambda(u)\,\mathrm{d}u = \Gamma(\alpha)\Gamma(1-\alpha)$. By the generalized Minkowski inequality, we have that

$$\left(\int_0^T \left\|\int_0^{(T-s)/\varepsilon} \Lambda(u)\varphi_{s+\varepsilon u}\,\mathrm{d}u - \varphi_s \int_0^{(T-s)/\varepsilon} \Lambda(u)\mathrm{d}u\right\|_{1,2}^2 \mathrm{d}s\right)^{1/2}$$
$$\leqslant \int_0^\infty \Lambda(u)\left(\int_0^T ||\varphi_{s+\varepsilon u}-\varphi_s||_{1,2}^2\,\mathrm{d}s\right)^{1/2} \mathrm{d}u,$$

where we assume that $\varphi_s=0$ if $s\notin[0,T]$. For each fixed $u>0$, $\int_0^T ||\varphi_{s+\varepsilon u}-\varphi_s||_{1,2}^2\,\mathrm{d}s$ converges to zero as ε tends to zero because $\varphi\in L^2([0,T];\mathbb{D}^{1,2})$. So by dominated convergence

$$\int_s^T \frac{1}{\varepsilon}\Lambda\left(\frac{t-s}{\varepsilon}\right)\varphi_t\,\mathrm{d}t = \int_0^{(T-s)/\varepsilon} \Lambda(u)\varphi_{s+\varepsilon u}\,\mathrm{d}u$$

converges to $\Gamma(\alpha)\Gamma(1-\alpha)\varphi_s$ in $\mathbb{L}^{1,2}$ as ε tends to zero, which completes the proof. □

Condition (T1) is a trace-class hypothesis, which cannot be avoided as soon as we use Malliavin calculus. This condition is rather strong because it requires that $D_s\phi_r$ tends to zero as $s\uparrow r$ faster than $(r-s)^\alpha$. It does not hold for W_t or for the process B_t itself, but it holds for the process $\phi_t=\int_0^t (t-s)^\beta\,\mathrm{d}W_s$ if $\beta>\alpha$.

In the adapted case, we can replace (C1) by the following conditions:

$$\phi\in I_{T-}^\alpha(L^2([0,T]);L^2(\Omega))\cap L_a^2\cap\mathbb{L}^{1,2}, \tag{C2}$$

and

$$\lim_{\varepsilon\to 0} D_\theta \int_s^T (\phi_t-\phi_s)(t-s+\varepsilon)^{-\alpha-1}\,\mathrm{d}t = D_\theta\int_s^T(\phi_t-\phi_s)(t-s)^{-\alpha-1}\,\mathrm{d}t, \tag{C3}$$

where the convergence is in $L^2([0,T]^2\times\Omega)$. In fact, the convergence of $\int_0^T \phi_t\,\mathrm{d}B_t^\varepsilon$ to $\int_0^T \phi_t\,\mathrm{d}B_t$ in probability can be deduced by the following convergences in $L^2(\Omega)$, provided we have (T1):

$$\lim_{\varepsilon\to 0}\int_0^T \phi_s(T-s+\varepsilon)^{-\alpha}\,\mathrm{d}W_s = \int_0^T \phi_s(T-s)^{-\alpha}\,\mathrm{d}W_s, \tag{3.4}$$

$$\lim_{\varepsilon\to 0}\int_0^T\left(\int_s^T(\phi_r-\phi_s)(r-s+\varepsilon)^{-\alpha-1}\,\mathrm{d}r\right)\mathrm{d}W_s$$
$$=\int_0^T\left(\int_s^T(\phi_r-\phi_s)(r-s)^{-\alpha-1}\,\mathrm{d}r\right)\mathrm{d}W_s. \tag{3.5}$$

By (C2) and Proposition 2 we have $E\int_0^T \phi_s^2(T-s)^{-2\alpha}\,\mathrm{d}s<\infty$, and this together with the adaptability of ϕ implies (3.4). The convergence (3.5) can be proved as in Theorem 5 using the fact that $\int_s^T(\phi_r-\phi_s)(r-s+\varepsilon)^{-\alpha-1}\,\mathrm{d}r$ converges to $\int_s^T(\phi_r-\phi_s)(r-s)^{-\alpha-1}\,\mathrm{d}r$ in the norm of $\mathbb{L}^{1,2}$ as ε tends to zero, by conditions (C2) and (C3).

We also can weaken condition (C3) taking only derivatives of the process $\int_s^T(\phi_t-\phi_s)(t-s)^{-\alpha-1}\,\mathrm{d}t$ at times $\theta\leqslant s$. The counterpart is to impose a trace-class hypothesis a little bit stronger. We thus have the following theorem, which covers the adapted case.

Theorem 6. *Let ϕ be a process satisfying* (C2) *and the following additional conditions*:

$$\lim_{\varepsilon\to 0} \mathbf{1}_{\{\theta\leqslant s\}} D_\theta \int_s^T (\phi_t - \phi_s)(t-s+\varepsilon)^{-\alpha-1}\, dt$$
$$= \mathbf{1}_{\{\theta\leqslant s\}} D_\theta \int_s^T (\phi_t - \phi_s)(t-s)^{-\alpha-1}\, dt, \tag{C4}$$

where the limit holds in the $L^2([0,T]^2\times\Omega)$-sense,

$$E\int_0^T \left(\int_s^T \frac{|D_s\phi_r|}{(r-s)^{\alpha+1}}\, dr\right)^2 ds < \infty. \tag{T2}$$

Then the conclusion of Theorem 5 holds.

Proof. It is clear that (T2) implies (T1). Then it suffices to show that (C2),(C4) and (T2) imply (C3). We can write

$$E\int_0^T\int_0^T \left|\int_s^T D_\theta(\phi_t-\phi_s)[(t-s+\varepsilon)^{-\alpha-1}-(t-s)^{-\alpha-1}]\, dt\right|^2 ds\, d\theta$$
$$\leqslant E\int_{\{\theta\leqslant s\}} \left|\int_s^T D_\theta(\phi_t-\phi_s)[(t-s+\varepsilon)^{-\alpha-1}-(t-s)^{-\alpha-1}]\, dt\right|^2 ds\, d\theta$$
$$+E\int_{\{\theta\geqslant s\}} \left|\int_s^T D_\theta(\phi_t-\phi_s)[(t-s+\varepsilon)^{-\alpha-1}-(t-s)^{-\alpha-1}]\, dt\right|^2 ds\, d\theta.$$

On the one hand, the first term tends to zero owing to condition (C4). On the other hand, the dominated convergence theorem and (T2) imply that the second term converges to zero as well. □

Remaining in the adapted case, we can work in an exact analogous way with the space $\mathbb{L}^F$.

Theorem 7. *Let ϕ be a process satisfying*

$$\phi\in I_{T-}^{\alpha}(L^2([0,T]);L^2(\Omega))\cap \mathbb{L}^{2,2}\cap L_a^2 \tag{C5}$$

and

$$E\int_0^T \left(\int_s^T \frac{|D_s\phi_r|}{(r-s)^{\alpha+1}}\, dr\right)^2 ds + E\int_0^T\int_s^T \left(\int_\theta^T \frac{|D_\theta D_s\phi_r|}{(r-s)^{\alpha+1}}\, dr\right)^2 d\theta\, ds < \infty. \tag{T3}$$

Then the conclusion of Theorem 5 holds.

Proof. It suffices to show that

$$\lim_{\varepsilon\to 0}\int_s^T \frac{\phi_t-\phi_s}{(t-s+\varepsilon)^{\alpha+1}}\, dt = \int_s^T \frac{\phi_t-\phi_s}{(t-s)^{\alpha+1}}\, dt$$

as a process of the variable s, in the $\mathbb{L}^F$-sense. The proof of this convergence is similar to the proof of the previous theorem. □

4. Riemann sum convergence

In this section we study the approximation of the stochastic integral $\int_0^T \phi_t \, dB_t$ by Riemann sums. In order to show the convergence of the Riemann sums we require an additional Hölder continuity assumption on the process ϕ, similar to that introduced in Carmona and Coutin (1998). The proof of the convergence of the Riemann sums is based on Theorems 5 and 4.1.1. of Zähle (1998).

Let $\phi \in \mathbb{L}^{1,2}$, and consider a partition $\pi = \{0 = s_0 < s_1 \cdots < s_n = T\}$ of mesh $|\pi| = \max_{i=0\dots n-1} |s_{i+1} - s_i|$. Let us set

$$\phi_s^\pi = \sum_{i=0}^{n-1} \phi_{s_i} \mathbf{1}_{[s_i, s_{i+1})}(s).$$

We easily check that for $p = 2$ and $\alpha \in (0, \frac{1}{2})$, the process ϕ^π and its derivative satisfy the conditions of Proposition 2, and then we conclude that $\phi^\pi \in I_{T-}^\alpha(\mathbb{L}^{1,2})$. Furthermore, since it is clear that for ϕ^π condition (T1) holds, the assumptions of Theorem 5 are satisfied and we have

$$\int_0^T \phi_t^\pi \, dB_t = \Gamma(1-\alpha) \int_0^T D_{T-}^\alpha \phi_s^\pi \, dW_s - \alpha \int_0^T \int_0^r D_s \phi_r^\pi (r-s)^{-\alpha-1} \, ds \, dr. \tag{4.1}$$

Then we claim the following:

Proposition 8. *If* $\phi \in \mathbb{L}^{1,2}$ *satisfies* (T1) *and is Hölder-continuous in the norm of* $\mathbb{D}^{1,2}$ *of index* $\lambda > \alpha$, *that is*

$$\|\phi_s - \phi_t\|_{1,2} \leqslant H_\lambda |s-t|^\lambda$$

for some constant $H_\lambda > 0$, *and for all* $s, t \in [0, T]$, *then*

$$\lim_{|\pi| \to 0} \int_0^T \phi_s^\pi \, dB_s = \int_0^T \phi_s \, dB_s$$

in probability.

Proof. Condition (T1) is clearly sufficient to prove the convergence of the second term on the right-hand side of (4.1) to $-\alpha \int_0^T \int_0^r D_s \phi_r (r-s)^{-\alpha-1} \, ds \, dr$.

Then, it suffices to show that

$$\lim_{|\pi| \to 0} \| D_{T-}^\alpha \phi^\pi - D_{T-}^\alpha \phi \|_{\mathbb{L}^{1,2}} = 0.$$

We can write

$$\Gamma(1-\alpha) \|D_{T-}^\alpha \phi_s^\pi - D_{T-}^\alpha \phi_s\|_{1,2}$$
$$= \left\| (\phi_s^\pi - \phi_s)(T-s)^{-\alpha} - \alpha \int_s^T [(\phi_t^\pi - \phi_t) - (\phi_s^\pi - \phi_s)](t-s)^{-\alpha-1} \, dt \right\|_{1,2}$$

$$\leqslant ||\phi_s^\pi - \phi_s||_{1,2}(T-s)^{-\alpha} + \alpha||S_\pi(s)||_{1,2},$$

where

$$S_\pi(s) = \int_s^T [(\phi_t^\pi - \phi_t) - (\phi_s^\pi - \phi_s)](t-s)^{-\alpha-1}\,\mathrm{d}t.$$

On the one hand we have

$$\begin{aligned} ||\phi_s^\pi - \phi_s||_{1,2}(T-s)^{-\alpha} &\leqslant \sum_{i=0}^{n-1} ||\phi_{s_i} - \phi_s||_{1,2}\mathbf{1}_{[s_i,s_{i+1})}(s)(T-s)^{-\alpha} \\ &\leqslant \sum_{i=0}^{n-1} H_\lambda |s_i - s|^\lambda \mathbf{1}_{[s_i,s_{i+1})}(s)(T-s)^{-\alpha} \\ &\leqslant |\pi|^\lambda H_\lambda (T-s)^{-\alpha}, \end{aligned}$$

hence

$$\int_0^T ||\phi_s^\pi - \phi_s||_{1,2}^2 (T-s)^{-2\alpha}\,\mathrm{d}s \leqslant (1-2\alpha)H_\lambda^2 |\pi|^{2\lambda} T^{1-2\alpha},$$

which vanishes when $|\pi|$ tends to zero.

On the other hand, let us show that

$$\lim_{|\pi|\to 0} \int_0^T ||S_\pi(s)||_{1,2}^2\,\mathrm{d}s = 0.$$

We can write

$$\begin{aligned} S_\pi(s) = \sum_{i=0}^{n-1} \mathbf{1}_{[s_i,s_{i+1})}(s) \Bigg\{ &\int_s^{s_{i+1}} [(\phi_{s_i} - \phi_t) - (\phi_{s_i} - \phi_s)](t-s)^{-\alpha-1}\,\mathrm{d}t \\ &+ \sum_{k=i+1}^{n-1} \int_{s_k}^{s_{k+1}} [(\phi_{s_k} - \phi_t) - (\phi_{s_i} - \phi_s)](t-s)^{-\alpha-1}\,\mathrm{d}t \Bigg\}, \end{aligned}$$

and

$$||S_\pi(s)||_{1,2} \leqslant I_1(s) + I_2(s) + I_3(s),$$

where

$$I_1(s) = \sum_{i=0}^{n-1} \mathbf{1}_{[s_i,s_{i+1})}(s) \int_s^{s_{i+1}} ||\phi_s - \phi_t||_{1,2}(t-s)^{-\alpha-1}\,\mathrm{d}t,$$

$$I_2(s) = \sum_{i=0}^{n-1} \mathbf{1}_{[s_i,s_{i+1})}(s) \sum_{k=i+1}^{n-1} \int_{s_k}^{s_{k+1}} ||\phi_{s_k} - \phi_t||_{1,2}(t-s)^{-\alpha-1}\,\mathrm{d}t,$$

$$I_3(s) = \sum_{i=0}^{n-1} \mathbf{1}_{[s_i,s_{i+1})}(s) ||\phi_{s_i} - \phi_s||_{1,2} \sum_{k=i+1}^{n-1} \int_{s_k}^{s_{k+1}} (t-s)^{-\alpha-1}\,\mathrm{d}t.$$

Then we have

$$\begin{aligned} I_1(s) &\leqslant H_\lambda \sum_{i=0}^{n-1} \mathbf{1}_{[s_i,s_{i+1})}(s) \int_s^{s_{i+1}} (t-s)^{\lambda-\alpha-1}\,\mathrm{d}t \\ &\leqslant H_\lambda \frac{1}{\lambda-\alpha} \sum_{i=0}^{n-1} \mathbf{1}_{[s_i,s_{i+1})}(s)(s_{i+1}-s)^{\lambda-\alpha}, \end{aligned}$$

$$\int_0^T I_1(s)^2\,\mathrm{d}s \leqslant H_\lambda^2(\lambda-\alpha)^{-2}\frac{1}{2\lambda-2\alpha+1}\sum_{i=0}^{n-1}(s_{i+1}-s_i)^{2\lambda-2\alpha+1}$$
$$\leqslant H_\lambda^2(\lambda-\alpha)^{-2}\frac{1}{2\lambda-2\alpha+1}T|\pi|^{2\lambda-2\alpha}, \tag{4.2}$$

$$I_2(s) \leqslant H_\lambda \sum_{i=0}^{n-1}\mathbf{1}_{[s_i,s_{i+1})}(s)\sum_{k=i+1}^{n-1}(s_{k+1}-s_k)^\lambda\int_{s_k}^{s_{k+1}}(t-s)^{-\alpha-1}\,\mathrm{d}t$$
$$\leqslant H_\lambda \sum_{i=0}^{n-1}\mathbf{1}_{[s_i,s_{i+1})}(s)|\pi|^\lambda\frac{1}{\alpha}[(s_{i+1}-s)^{-\alpha}-(T-s)^{-\alpha}],$$

$$\int_0^T I_2(s)^2\,\mathrm{d}s \leqslant H_\lambda^2|\pi|^{2\lambda}\alpha^{-2}\sum_{i=0}^{n-1}\int_{s_i}^{s_{i+1}}[(s_{i+1}-s)^{-\alpha}-(T-s)^{-\alpha}]^2\,\mathrm{d}s$$
$$\leqslant H_\lambda^2|\pi|^{2\lambda}\frac{\alpha^{-2}}{1-2\alpha}\sum_{i=0}^{n-1}(s_{i+1}-s_i)^{1-2\alpha}$$
$$\leqslant TH_\lambda^2|\pi|^{2\lambda-2\alpha}\frac{\alpha^{-2}}{1-2\alpha}, \tag{4.3}$$

$$I_3(s) \leqslant H_\lambda \sum_{i=0}^{n-1}\mathbf{1}_{[s_i,s_{i+1})}(s)(s-s_i)^\lambda\int_{s_{i+1}}^{T}(t-s)^{-\alpha-1}\,\mathrm{d}t$$
$$\leqslant H_\lambda\frac{1}{\alpha}\sum_{i=0}^{n-1}\mathbf{1}_{[s_i,s_{i+1})}(s)(s-s_i)^\lambda(s_{i+1}-s)^{-\alpha},$$

and since

$$\int_a^b (s-a)^\lambda(b-s)^{-\alpha}\,\mathrm{d}s = (b-a)^{\lambda-\alpha+1}\frac{\Gamma(\lambda+1)\Gamma(1-\alpha)}{\Gamma(\lambda-\alpha+2)}, \tag{4.4}$$

we obtain

$$\int_0^T I_3(s)^2\,\mathrm{d}s \leqslant H_\lambda^2\alpha^{-2}\frac{\Gamma(2\lambda+1)\Gamma(1-2\alpha)}{\Gamma(2\lambda-2\alpha+2)}\sum_{i=0}^{n-1}(s_{i+1}-s_i)^{2\lambda-2\alpha+1}$$
$$\leqslant H_\lambda^2\alpha^{-2}\frac{\Gamma(2\lambda+1)\Gamma(1-2\alpha)}{\Gamma(2\lambda-2\alpha+2)}T|\pi|^{2\lambda-2\alpha}. \tag{4.5}$$

Hence, from (4.2), (4.3) and (4.5) we obtain the result.

5. The integral process

Define $\int_0^t \phi_s\,\mathrm{d}B_s = \int_0^T \phi_s \mathbf{1}_{[0,t]}(s)\,\mathrm{d}B_s$ if this stochastic integral exists. Our aim is to study the properties of the stochastic process $\{\int_0^t \phi_s\,\mathrm{d}B_s, t\in[0,T]\}$. After checking the existence of this process, we study its continuity properties and we also derive a maximal inequality.

About the existence, we have the following result:

Lemma 9. *Under the hypotheses of Theorem* 5, *for almost all* $t \in [0,T]$, $\int_0^t \phi_s \, \mathrm{d}B_s$ *exists and*

$$\int_0^t \phi_s \, \mathrm{d}B_s = \Gamma(1-\alpha) \int_0^t D_{t-}^{\alpha} \phi_s \, \mathrm{d}W_s - \alpha \int_0^t \int_0^r \frac{D_s \phi_r}{(r-s)^{\alpha+1}} \, \mathrm{d}s \, \mathrm{d}r.$$

Proof. If ϕ satisfies the trace-class hypothesis (T1), also does $\phi \mathbf{1}_{[0,t]}$. Let us prove that $\phi \mathbf{1}_{[0,t]}$ belongs to $I_{T-}^{\alpha}(\mathbb{L}^{1,2})$ for almost all $t \in [0,T]$. We will make use of the characterization of $I_{T-}^{\alpha}(\mathbb{L}^{1,2})$ given in Proposition 2. Clearly

$$\int_0^t \frac{||\phi_s||_{1,2}^2}{(T-s)^{2\alpha}} \, \mathrm{d}s < \infty$$

for all $t \in [0,T]$. Assuming $s < t$ we can write

$$\begin{aligned} \int_{s+\varepsilon}^T \frac{\phi_r \mathbf{1}_{[0,t]}(r) - \phi_s}{(r-s)^{\alpha+1}} \, \mathrm{d}r &= \int_{t \wedge (s+\varepsilon)}^t \frac{\phi_r - \phi_s}{(r-s)^{\alpha+1}} \, \mathrm{d}r - \int_{t \vee (s+\varepsilon)}^T \frac{\phi_s}{(r-s)^{\alpha+1}} \, \mathrm{d}r \\ &= \int_{t \wedge (s+\varepsilon)}^t \frac{\phi_r - \phi_s}{(r-s)^{\alpha+1}} \, \mathrm{d}r \\ &\quad + \frac{1}{\alpha} \phi_s [((t-s) \vee \varepsilon)^{-\alpha} - (T-s)^{-\alpha}]. \end{aligned}$$

As $\phi \in \mathbb{L}^{1,2}$ we have that for almost all $t \in [0,T]$, the process $\phi_s(t-s)^{-\alpha} \mathbf{1}_{[0,t]}(s)$ belongs also to $\mathbb{L}^{1,2}$, and as a consequence, $\phi_s((t-s) \vee \varepsilon)^{-\alpha} \mathbf{1}_{[0,t]}(s)$ converges to $\phi_s(t-s)^{-\alpha} \mathbf{1}_{[0,t]}(s)$ in $\mathbb{L}^{1,2}$ as ε tends to zero. On the other hand,

$$\int_{t \wedge (s+\varepsilon)}^t \frac{\phi_r - \phi_s}{(r-s)^{\alpha+1}} \, \mathrm{d}r = \int_{t \wedge (s+\varepsilon)}^T \frac{\phi_r - \phi_s}{(r-s)^{\alpha+1}} \, \mathrm{d}r - \int_t^T \frac{\phi_r - \phi_s}{(r-s)^{\alpha+1}} \, \mathrm{d}r,$$

which gives us that $\mathbf{1}_{[0,t]}(s) \int_{t \wedge (s+\varepsilon)}^t (\phi_r - \phi_s)(r-s)^{-\alpha-1} \, \mathrm{d}r$ converges in $\mathbb{L}^{1,2}$, for all $t \in [0,T]$, to

$$\mathbf{1}_{[0,t]}(s) \int_s^t \frac{\phi_r - \phi_s}{(r-s)^{\alpha+1}} \, \mathrm{d}r$$

as ε tends to zero. Hence, we have proved that in the $\mathbb{L}^{1,2}$ sense

$$\begin{aligned} &\lim_{\varepsilon \to 0} \int_{s+\varepsilon}^T \frac{\phi_r \mathbf{1}_{[0,t]}(r) - \phi_s \mathbf{1}_{[0,t]}(s)}{(r-s)^{\alpha+1}} \, \mathrm{d}r \\ &\quad = \mathbf{1}_{[0,t]}(s) \left\{ \int_s^t \frac{\phi_r - \phi_s}{(r-s)^{\alpha+1}} \, \mathrm{d}r + \frac{1}{\alpha} \phi_s [(t-s)^{-\alpha} - (T-s)^{-\alpha}] \right\}. \end{aligned}$$

Thus, $\mathbf{1}_{[0,t]}(s)\phi_s$ belongs to $I_{T-}^{\alpha}(\mathbb{L}^{1,2})$ and adding to the above expression the term $(1/\alpha)\phi_s(T-s)^{-\alpha}$, we obtain

$$D_{T-}^{\alpha}(\phi \mathbf{1}_{[0,t]})(s) = D_{t-}^{\alpha} \phi_s$$

for all $s \leqslant t$, which completes the proof. □

In a similar way we can prove Lemma 9 assuming the hypotheses of Theorem 7. Let us now study the continuity of the integral process

$$\left\{ \int_0^t \phi_s \, \mathrm{d}B_s, \; t \in [0,T] \right\}.$$

To simplify the exposition we restrict ourselves to the adapted case. We need the following lemma.

Lemma 10. *Suppose that* ϕ *is an adapted stochastic process satisfying the following condition*:

$$\int_0^T E|\phi_s|^p \, \mathrm{d}s < \infty \tag{H1(p)}$$

for some $p > 2/(1-2\alpha)$, *where* $0 < \alpha < \frac{1}{2}$. *Then the process*

$$\int_0^t (t-s)^{-\alpha} \phi_s \, \mathrm{d}W_s$$

has a version which is ε*-Hölder continuous for any* $\varepsilon < \frac{1}{2} - \alpha - 1/p$, *and we have the maximal inequality*

$$E\left(\sup_{0 \leqslant t \leqslant T} \left| \int_0^t (t-s)^{-\alpha} \phi_s \, \mathrm{d}W_s \right|^p \right) \leqslant C(p) \int_0^T E|\phi_t|^p \, \mathrm{d}t. \tag{5.1}$$

Proof. Taking $0 < \varepsilon < 1 - \alpha$ and using (4.4) we can write

$$\int_0^t (t-s)^{-\alpha} \phi_s \, \mathrm{d}W_s = C(\alpha, \varepsilon) \int_0^t \left(\int_s^t (t-r)^{\varepsilon - 1} (r-s)^{-\alpha - \varepsilon} \, \mathrm{d}r \right) \phi_s \, \mathrm{d}W_s,$$

where

$$C(\alpha, \varepsilon) = \frac{\Gamma(1-\alpha)}{\Gamma(\varepsilon)\Gamma(1-\alpha-\varepsilon)}.$$

Applying Fubini's stochastic theorem, and assuming $\varepsilon < 1 - 2\alpha$ we obtain

$$\int_0^t \phi_s (t-s)^{-\alpha} \, \mathrm{d}W_s = \int_0^t (t-r)^{\varepsilon-1} \left(\int_0^r (r-s)^{-\alpha-\varepsilon} \phi_s \, \mathrm{d}W_s \right) \mathrm{d}r. \tag{5.2}$$

Choosing now $\varepsilon \in (1/p, 1/2 - \alpha)$ and using Hölder's and Burkholder's inequalities, we get

$$\begin{aligned}
& E\left(\int_0^t \left| (t-r)^{\varepsilon-1} \int_0^r (r-s)^{-\alpha-\varepsilon} \phi_s \, \mathrm{d}W_s \right| \mathrm{d}r \right)^p \\
& \quad \leqslant \left(\frac{p-1}{\varepsilon p - 1} \right)^{p-1} t^{\varepsilon p - 1} \int_0^t E \left| \int_0^r (r-s)^{-\alpha-\varepsilon} \phi_s \, \mathrm{d}W_s \right|^p \mathrm{d}r \\
& \quad \leqslant C(p, \varepsilon, T) \int_o^t E \left| \int_0^r (r-s)^{-2(\alpha+\varepsilon)} \phi_s^2 \, \mathrm{d}s \right|^{p/2} \mathrm{d}r \\
& \quad \leqslant C(p, \varepsilon, T) \int_0^t E|\phi_s|^p \, \mathrm{d}s,
\end{aligned}$$

which is finite from hypothesis (H1(p)) and yields (5.1).

Set $X_t = \int_0^t (t-s)^{-\alpha} \phi_s \, \mathrm{d}W_s$, and $R_t = \int_0^t (t-s)^{-\alpha-\varepsilon} \phi_s \, \mathrm{d}W_s$. For any $r \leqslant t$ we can write using (5.2)

$$X_t - X_r = \int_0^r [(t-s)^{\varepsilon-1} - (r-s)^{\varepsilon-1}] R_s \, \mathrm{d}s + \int_r^t (t-s)^{\varepsilon-1} R_s \, \mathrm{d}s.$$

Hence,

$$|X_t - X_r| \leqslant 2\varepsilon(t-r)^{\varepsilon} \sup_{s\in[0,T]} |R_s|,$$

and $E(\sup_{s\in[0,T]}|R_s|^p) < \infty$ due to condition (H1)(p) provided $p > 2/(1-2(\alpha+\varepsilon))$, which is equivalent to $\varepsilon < \frac{1}{2} - \alpha - 1/p$. This completes the proof of the lemma. □

Let us consider the following hypotheses:

$$h_1 := \int_0^T \int_0^r \frac{E|\phi_r - \phi_s|^2}{(r-s)^{2(\alpha+1)}} \, ds \, dr < \infty, \tag{H2}$$

$$h_2 := \int_0^T \int_0^r \int_s^r \frac{E|D_\theta \phi_r|^2}{(r-s)^{2(\alpha+1)}} \, d\theta \, ds \, dr < \infty, \tag{H3}$$

$$h_3 := \int_0^T \int_0^r \int_0^r \int_s^r \frac{E|D_\sigma D_\theta \phi_r|^2}{(r-s)^{2(\alpha+1)}} \, d\theta \, d\sigma \, ds \, dr < \infty. \tag{H4}$$

We have the following result:

Theorem 11. *Suppose that ϕ is a process satisfying condition (C5) of Theorem 7, together with condition* (H1(p)) *for some $p > 2/(1-2\alpha)$, and hypotheses* (H2)–(H4). *Then the integral process $\{\int_0^t \phi_s \, dB_s,\ t \in [0,T]\}$ has a version which is ε-Hölder continuous for any $\varepsilon < \frac{1}{2} - \alpha - 1/p$.*

Proof. Owing to Lemma 9 we can write

$$\int_0^t \phi_s \, dB_s = \int_0^t (t-s)^{-\alpha} \phi_s \, dW_s - \alpha \int_0^t \left(\int_s^t \frac{\phi_r - \phi_s}{(r-s)^{\alpha+1}} \, dr \right) dW_s$$
$$-\alpha \int_0^t \int_0^r \frac{D_s \phi_r}{(r-s)^{\alpha+1}} \, ds \, dr. \tag{5.3}$$

For the first integral we apply Lemma 10. It is clear that the last term of this sum is $\frac{1}{2}$-Hölder-continuous due to the hypothesis (H3). So it remains to study the second term. Using the $\mathbb{L}^F$-estimate for the Skorohod integral (see the proof of (2.2) in Alòs and Nualart, 1998), we have

$$\int_0^t E\left| \int_0^r \frac{\phi_r - \phi_s}{(r-s)^{\alpha+1}} \, dW_s \right|^2 dr \leqslant 2 \left\{ \int_0^t \int_0^r \frac{E|\phi_r - \phi_s|^2}{(r-s)^{2(\alpha+1)}} \, ds \, dr \right.$$
$$\left. + \int_0^t \int_0^r \int_s^r \frac{E|D_\theta \phi_r|^2}{(r-s)^{2(\alpha+1)}} \, d\theta \, ds \, dr + \int_0^t \int_0^r \int_0^r \int_s^r \frac{E|D_\sigma D_\theta \phi_r|^2}{(r-s)^{2(\alpha+1)}} \, d\theta \, d\sigma \, ds \, dr \right\}, \tag{5.4}$$

which is finite from hypotheses (H2)–(H4). Using now Fubini's stochastic theorem, we have

$$\int_0^t \left(\int_s^t \frac{\phi_r - \phi_s}{(r-s)^{\alpha+1}} \, dr \right) dW_s = \int_0^t \left(\int_0^r \frac{\phi_r - \phi_s}{(r-s)^{\alpha+1}} \, dW_s \right) dr,$$

and this allows us to complete the proof. □

Corollary 12. *Under the hypotheses of Theorem* 11 *we can derive the following maximal inequality in probability*:

$$P\left\{\sup_{0\leqslant t\leqslant T}\left|\int_0^t \phi_s \,\mathrm{d}B_s\right| > \lambda\right\} \leqslant \left(\frac{\lambda}{3}\right)^{-p} C(p)\int_0^T E|\phi_t|^p \,\mathrm{d}t + \left(\frac{\lambda}{3\alpha}\right)^{-2} 2T(h_1+h_2+h_3) + \frac{\lambda}{3\alpha}\int_0^T\int_s^T \frac{E|D_s\phi_r|}{(r-s)^{\alpha+1}}\,\mathrm{d}r\,\mathrm{d}s$$

for all $\lambda > 0$.

Proof. Using the decomposition (5.3), the result is a straightforward consequence of Lemma 10 and the estimate (5.4). □

Remark 3. In a similar way we can obtain an estimate for the expectation

$$E\left(\sup_{0\leqslant t\leqslant T}\left|\int_0^t \phi_s \,\mathrm{d}B_s\right|^p\right),$$

with $p > 2/(1-2\alpha)$, using the estimates for the L^p-norm of the Skorohod integral $\int_0^t(\int_s^t (r-s)^{-\alpha-1}(\phi_r - \phi_s)\,\mathrm{d}r)\,\mathrm{d}W_s$ given in Alòs and Nualart (1998, Theorem 2.4.3).

Remark 4. In the nonadapted case we can establish results similar to Lemma 10 and to Theorem 11 using the L^p-estimates for the Skorohod integral given in Alòs and Nualart (1998) and Nualart (1995).

6. For Further Reading

The following references are also of interest to the reader. Feyel and de la Pradelle, 1996; Garsia et al. 1970; Kleptsyna et al. 1996.

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