Spin Models on Triangle-Free Connected Graphs

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Spin models were introduced by V. Jones (Pac. J. Math. 137 (1989), 311–334) to construct invariants of knots and links. A spin model is defined as a pair $S = (X, w)$ of a finite set $X$ and a function $w: X \times X \rightarrow \mathbb{C}$ satisfying several axioms. Let $G = (X, E)$ be a connected graph with the usual metric $d: X \times X \rightarrow \{0, 1, \ldots, d\}$, where $d$ denotes the diameter of $G$. It is shown that, if $G$ has no 3-cycle, and if $S = (X, t \cdot \delta)$ is a spin model for a mapping $t: \{0, 1, \ldots, d\} \rightarrow \mathbb{C}$ satisfying some conditions (which hold if $t$ is injective), then $G$ is an almost bipartite distance-regular graph.

1. Introduction

Spin models for invariants of links were introduced by Jones [10]. A spin model is a pair $S = (X, w)$ of a finite set $X$, $|X| = n > 0$, and a complex-valued function $w$ on $X \times X$ such that (for all $x, y, z \in X$)

(S1) $w(x, y) = w(y, x) \neq 0$,
(S2) $\sum_{x \in X} w(y, x) w(z, x)^{-1} = n \delta_{y, z}$,
(S3) $\sum_{u \in X} w(x, u) w(y, u) w(z, u)^{-1} = \sqrt{n} w(x, y) w(x, z)^{-1} w(y, z)^{-1}$.

Equation (S3) with $x = z$ becomes

$$\sum_{u \in X} w(y, u) = \sqrt{n} w(x, y)^{-1}. \tag{1}$$

This shows that $w(x, x) = \alpha$ is a constant, not depending on the choice of $x \in X$, which is called the modulus of $S$. We remark that our $w$ corresponds to $w_+$ in [10], and our modulus is the inverse of the modulus introduced in that paper. See [1–3, 6–8, 10] for spin models and related invariants of links.

Let $G = (X, E)$ be a connected graph with the usual metric $d$ on $X$, i.e., $d(x, y), x, y \in X$, is the length of a shortest path connecting $x$ and $y$. The set of vertices $x$ at distance $i$ from a fixed vertex $z$ is denoted by $I_i(z)$. For two

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vertices $z, x \in X$ at distance $i$, let $c(z, x)$ (resp. $a(z, x)$, $b(z, x)$) denote the number of neighbours of $x$ in $\Gamma_{i-1}(z)$ (resp. $\Gamma_i(z)$, $\Gamma_{i+1}(z)$). $\Gamma$ is said to be \textit{distance-regular} if $c(z, x), a(z, x), b(z, x)$ depend only on the distance $\delta(z, x) = i$, rather than on the individual vertices $x, z$ with $\delta(z, x) = i$. In this case, the numbers $c_i = c(z, x), a_i = a(z, x), b_i = b(z, x)$ are called the \textit{intersection numbers} of $\Gamma$. As is easily shown a distance-regular graph $\Gamma$ is bipartite if and only if $a_i = 0$ for all $i$. $\Gamma$ is said to be \textit{almost bipartite} when $a_1 = \cdots = a_{d-1} = 0$. See [4, 5] for precise informations about distance-regular graphs.

We say a spin model $S = (X, w)$ is constructed on a connected graph $\Gamma = (X, E)$ if $w = t \cdot \tilde{c}$ for some mapping $t: \{0, 1, \ldots, d\} \to \mathbb{C}$, where $d$ denotes the diameter of $\Gamma$. In this case we say also $\Gamma = (X, E)$ affords $S = (X, w)$. Some new spin models were constructed on distance-regular graphs without 3-cycles. Examples are Jaeger's model on the Higman–Sims graph [7] and models on Hadamard graphs which we call Hadamard models [11]. Related link invariants of these spin models were determined by Jaeger [7–9]. Bipartite distance-regular graphs which afford spin models were determined in [13].

In this paper, we prove the following result.

**Theorem 1.1.** Let $\Gamma = (X, E)$ be a connected graph of diameter $d$ having on 3-cycle, and let $t: \{0, 1, \ldots, d\} \to \mathbb{C}$ be a mapping with $t_1 \neq t_i$ and $t_{i-2} \neq t_i \neq t_{i-1}$ ($i = 2, \ldots, d$), where $t_i = t(i)$. If $S = (X, t \cdot \tilde{c})$ is a spin model, then $\Gamma$ is an almost bipartite distance-regular graph.

We remark that Jaeger’s Higman–Sims model and Hadamard models satisfy the assumptions of Theorem 1.1.

The proof of Theorem 1.1 is based on the matrix $N_{Y, Z}$ which was introduced in [12]. In Section 2, we review $N_{Y, Z}$ and give a slight extension of Lemma 1.1 of [12], which will be used in Section 3 to obtain some formulas. The proof of Theorem 1.1 will be given in Sections 4 and 5.

### 2. The Matrix $N_{Y, Z}$

Let $S = (X, w), |X| = n$, be a spin model. Let $W$ denote the $n \times n$-matrix, indexed by $X \times X$, having $(x, y)$-entry $w(x, y)$, $x, y \in X$, and let $u_{y, z}$ denote the $n$-dimensional column vector, indexed by $X$, whose $x$-entry is $w(y, x) w(z, x)^{-1}$. Then Eq. (S3) can be written as

$$W u_{y, z} = \sqrt{n} w(y, z)^{-1} u_{y, z},$$

so that $u_{y, z}$ is an eigenvector of $W$ corresponding to the eigenvalue $\sqrt{n} w(y, z)^{-1}$. As is easily shown from (S2), for a fixed $y \in X$, the $n$ vectors
u_y z \in X$ are linearly independent, and hence form a basis of the space of $n$-dimensional column vectors. Therefore the values $\sqrt{n} w(y, z)^{-1}$, $z \in X$, are exactly all the eigenvalues of $W$, and the multiplicity $m(\theta)$ of an eigenvalue $\sqrt{n} \theta^{-1}$ agrees with the number of elements $z \in X$ such that $w(y, z) = \theta$:

$$m(\theta) = \# \{ z \in X \mid w(y, z) = \theta \}.$$  

Put

$$m'(\theta) = \# \{ z \in X \mid w(y, z) = \theta, z \neq y \}.$$  

Then $m'(\theta) = m(\theta)$ if $\theta \neq \alpha$, and $m'(\theta) = m(\theta) - 1$ if $\theta = \alpha$, where $\alpha$ denotes the modulus of $S$. For a subset $Y$ of $X$ and a subset $Z$ of $X \times X$, we define a matrix $N_{Y, Z}$, indexed by $Y \times Z$, whose $(x, (y, z))$-entry is

$$(N_{Y, Z})_{(x, (y, z))} = w(x, y) w(x, z)^{-1}, \quad x \in Y, \quad (y, z) \in Z.$$  

We remark that the $(x, (y, z))$-entry of the matrix $N_{Y, Z}$ is the $x$-entry of the vector $u_y z$.

**Lemma 2.1.** Let $S = (X, w)$ be a spin model, and let $\theta$ be a value of $w$. Let $Y$ be a subset of $X$ and let $Z$ be a subset of $X \times X$ with $Z \cap A = \emptyset$, where $A = \{ (x, x) \mid x \in X \}$. If $Z \subset w^{-1}(\theta)$ and $|Y| = |Z| > m'(\theta)$, then $\det N_{Y, Z} = 0$.

**Proof.** Since the rank of $N_{Y, Z}$ cannot exceed the rank of $N_{X, Z}$, it will be enough to show that the rank of $N_{X, Z}$ is at most $m'(\theta)$. First consider the case $\theta \neq \alpha$, where $\alpha$ is the modulus of $S$; in this case we have $m'(\theta) = m(\theta)$. Since every column vector of $N_{X, Z}$ is an eigenvector of $W$ corresponding to the eigenvalue $\sqrt{n} \theta^{-1}$, the dimension of the subspace spanned by the column vectors of $N_{X, Z}$ cannot exceed $m(\theta)$. So the rank of $N_{X, Z}$ is at most $m(\theta) = m'(\theta)$. Next consider the case $\theta = \alpha$. Let $V_z$ denote the eigenspace of $W$ corresponding to the eigenvalue $\sqrt{n} \alpha^{-1}$. Then $V_z$ has dimension $m(\alpha)$. Let $V_0$ be the subspace spanned by $j$, where $j$ denotes the vector with all entries 1. Then $V_0$ is included in $V_z$ by (1). Now consider the orthogonal complement $V_0^\perp$ of $V_0$ in $V_z$. Then $u_y z \in V_0^\perp$ for all $(y, z) \in Z$, since we have $\sum_{x \in X} w(y, x) w(z, x)^{-1} = 0$ by (S2) and $y \neq z$. Therefore the rank of $N_{X, Z}$ cannot exceed the dimension of $V_0^\perp$ which is equal to $m(\theta) - 1 = m'(\theta)$.  

**3. Some Formulas**

Throughout this paper, let $\Gamma = (X, E)$ be a connected graph of diameter $d > 1$ on $|X| = n$ vertices with no 3-cycle, let $\tau : \{0, 1, \ldots, d\} \to C$ be a
mapping such that $t_0 \neq t_2$ and $t_1 \neq t_i$ for $i > 1$, and assume $S = (X, w)$ is a spin model, where $w = t \cdot \delta$. Note that $\Gamma$ is regular of valency $k = m'(t_1)$ by our assumption $t_i \neq t_i$ for $i > 1$. We put $s_i = t_{i-1}^{-1} t_i$ for $i = 1, \ldots, d$.

**Lemma 3.1.** Let $x, z \in X$, and let $y_1, \ldots, y_k$ be the neighbours of $x$. Put $w(x, z) = \theta$ and $w(y_i, z) = \theta_i$. Then for $j = 1, \ldots, k$:

$$s_1^2 - s_2^2 \sum_{i \neq j} (\theta_i \theta_i^{-1} - 1) = (s_1 s_2 - 1)(s_1^2 - \theta^2)^2).$$

**Proof.** We may assume $j = 1$. We consider the matrix $N_{Y, Z}$ for

$$Y = \{x, z, y_2, \ldots, y_k\}, \quad Z = \{y_1, (x, y_1), (x, y_2), \ldots, (x, y_k)\}.$$ We fix an ordering of $Y$ and $Z$ as above and arrange the entries of $N_{Y, Z}$ according to this ordering. Since $|Y| = |Z| = k + 1 > k = m'(t_1)$, we have $\det N_{Y, Z} = 0$ by Lemma 2.1. We have $w(x, x) = w(y_i, y_j) = t_0$ and $w(x, y_i) = t_2$. Moreover, we have $w(y_i, y_j) = t_2$ for $i \neq j$, since $\Gamma$ has no 3-cycle. So the matrix $N_{Y, Z}$ is

$$N_{Y, Z} = \begin{pmatrix} \theta_1 \theta_1^{-1} & \theta_2 \theta_2^{-1} & \ldots & \theta_k \theta_k^{-1} \\ s_1 & s_1^{-1} & \ldots & s_k^{-1} \\ s_2 j & s_2^{-1} j & \ldots & s_k I + s_2^{-1} (J - I) \end{pmatrix},$$

where $\tau$ indicates the transpose, $j$ denotes the column vector of size $k - 1$ with all entries 1, $I$ denotes the unit matrix of size $k - 1$, and $J$ denotes the $(k - 1) \times (k - 1)$-matrix with all entries 1. The determinant of $N_{Y, Z}$ can be calculated as follows. Multiply the first row by $\theta^{-1}$, the second row by $\theta_1$, and the other rows by $s_2$. Then $N_{Y, Z}$ becomes

$$\begin{pmatrix} \theta_1 \theta_2^{-2} & \theta_1^{-1} & \theta_2^{-1} \ldots \theta_k^{-1} \\ s_1^2 \ldots \theta_1^{-1} \\ s_2^2 j j & s_1 s_2 I + (J - I) \end{pmatrix}.$$ Multiply the first column by $s_2^{-2}$, and subtract the second column from each of the other columns. Then we obtain

$$\begin{pmatrix} s_2^{-2} \theta_1 \theta_2^{-2} - \theta_1^{-1} & \theta_1^{-1} \theta_2^{-1} - \theta_1^{-1} \ldots \theta_k^{-1} - \theta_1^{-1} \\ s_1 s_2^{-2} - 1 & 1 & 0 \tau \\ 0 & j & (s_1 s_2 - 1) I \end{pmatrix},$$

where $0$ denotes the zero column vector of size $k - 1$. Note that $s_1 s_2 - 1 \neq 0$ by our assumption $t_0 \neq t_2$. Subtract $(s_1 s_2 - 1)^{-1} (\theta_i^{-1} - \theta_i^{-1})$ times the $(i+1)\text{th}$ row from the first row for $i = 2, \ldots, k$. Then the matrix becomes
\[
\begin{pmatrix}
(s_2^2 \theta_1 \theta_2^2 - \theta_1^{-1} \theta_1) - q & 0^t \\
(s_1 s_2^2 - 1) & 1 & 0^t \\
0 & j & (s_1 s_2 - 1) I
\end{pmatrix},
\]

where

\[q = (s_1 s_2 - 1)^{-1} \sum_{i=2}^{k} (\theta_i^{-1} - \theta_1^{-1}).\]

Thus \(\det NY, Z = 0\) implies

\[s_2^2 \theta_1 \theta_2^2 - \theta_1^{-1} - (s_2^2 s_2 - 1)(\theta_1^{-1} - q) = 0.\]

The result now follows from an easy computation. \(\square\)

**Corollary 3.2.** \((k - 1)(s_1^2 - s_2^2) = s_1^{-1} s_2 (1 - s_1^2).\)

**Proof.** In Lemma 3.1, take \(z = y_1\), so that we have \(\theta = \theta_1, \theta_1 = \theta_0,\) and \(\theta_i = \theta_2 (i = 2, ..., k).\) Then taking \(j = 1,\)

\[(s_1^2 - s_2^2) \sum_{i=2}^{k} (t_0 t_1^{-1} - 1) = (s_1 s_2 - 1)(s_1 t_1^{-2} t_2^2) .\]

Replacing \(t_1^{-1} t_2\) by \(s_1 (i = 1, 2),\) we get

\[(k - 1)(s_1^2 - s_2^2)(s_1^{-1} s_2^{-1} - 1) = (s_1 s_2 - 1)(s_1^2 s_2 - 1).\] \(\square\)

**Lemma 3.3.** Let \(x \in X,\) and let \(y_1, ..., y_k\) be the neighbours of \(x.\) Let \(z_1 z_2\) be an edge, and put \(w(x, z_1) = \theta, w(x, z_2) = \eta, w(y, z_1) = \theta, w(y, z_2) = \eta, (i = 1, ..., k).\) Then

\[\sum_{i=1}^{k} \theta \eta_i^{-1} = (s_1 s_2 + k - 1) s_1 s_2^{-1} \theta \eta^{-1} .\]

**Proof.** We consider the matrix \(NY, Z\) for

\[Y = \{x, y_1, y_2, ..., y_k\}, \quad Z = \{(z_1, z_2), (x, y_1), (x, y_2), ..., (x, y_k)\} .\]

Since \(|Y| = |Z| = k + 1 > k = m'(t_1)|,\) we have \(\det NY, Z = 0\) by Lemma 2.1. The matrix \(NY, Z\) is

\[NY, Z = \begin{pmatrix}
\theta \eta^{-1} & s_1^{-1} I \\
\theta \eta_1^{-1} & s_1 I + s_2^{-1} (J - I) \\
\theta \eta_2^{-1} & \end{pmatrix} .\]
where \( j, I, \) and \( J \) are of size \( k \). Multiply the first row by \( s_1 \), the other rows by \( s_2 \), and the first column by \( s_2^{-1} \). Then the above matrix becomes

\[
\begin{pmatrix}
\psi_1 s_2^{-1} \theta \eta^{-1} & j \\
\psi_1 \theta \eta^{-1} & \vdots \\
\theta \eta_k^{-1} & s_1 s_2 I + (J - I)
\end{pmatrix}.
\]

Subtract the first row from the other rows. Then we obtain

\[
\begin{pmatrix}
\psi_1 s_2^{-1} \theta \eta^{-1} & j \\
\psi_1 \theta \eta^{-1} - s_1 s_2^{-1} \theta \eta^{-1} & \vdots \\
\theta \eta_k^{-1} - s_1 s_2^{-1} \theta \eta^{-1}
\end{pmatrix}.
\]

Multiply the \( i \)th row by \((s_1 s_2 - 1)^{-1}\) for \( i = 2, \ldots, k + 1 \), and multiply the first column by \((s_1 s_2 - 1)\). Then the matrix becomes

\[
\begin{pmatrix}
(s_1 s_2 - 1) s_2^{-1} \theta \eta^{-1} & j \\
\psi_1 \theta \eta^{-1} - s_1 s_2^{-1} \theta \eta^{-1} & \vdots \\
\theta \eta_k^{-1} - s_1 s_2^{-1} \theta \eta^{-1}
\end{pmatrix}.
\]

Subtract the \( i \)th row from the first row for \( i = 2, \ldots, k + 1 \); then we obtain

\[
\begin{pmatrix}
q & \mathbf{0} \\
\psi_1 \theta \eta^{-1} - s_1 s_2^{-1} \theta \eta^{-1} & \vdots \\
\theta \eta_k^{-1} - s_1 s_2^{-1} \theta \eta^{-1}
\end{pmatrix}.
\]

where

\[
q = (s_1 s_2 - 1) s_2^{-1} \theta \eta^{-1} - \sum_{i=1}^k (\psi_1 \theta \eta_i^{-1} - s_1 s_2^{-1} \theta \eta^{-1}).
\]

Now \( q = \det N_{Y, \sigma} = 0 \).

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4. **Distance-Regularity: General Case**

Fix an integer \( r \) with \( 1 < r \leq d \), and pick two vertices \( x, z \) at distance \( r = \delta(x, z) \). Put \( c = c(z, x) = |\Gamma_{r-1}(z) \cap \Gamma_1(x)|, a = a(z, x) = |\Gamma_r(z) \cap \Gamma_1(x)| \) and
$b=b(z, x)=[\Gamma_{r+1}(z) \cap \Gamma_1(x)]$. Note that $c>0$. To simplify our arguments, we assign to $t_{r+1}$ a non-zero value different from $t_r$ and $t_{r'}$.

**Lemma 4.1.** (i) $(s_1^2 - s_2^2)(a(s_2 - s_1 - 1) + b(s_1 - s_2 + 1)) = (s_1 s_2 - 1) \times (s_1^2 - s_2^2)$,

(ii) $(s_1^2 - s_2^2)(a(s_2 - 1) + b(s_1 - s_2 + 1)) = (s_1^2 s_2 - 1)(s_2^2 - s_1^2)$.

**Proof.** Let $y_1, \ldots, y_k$ be the neighbors of $x$, where we choose the labels such that $y_1, \ldots, y_c \in \Gamma_1(z), y_{c+1}, \ldots, y_{c+a} \in \Gamma_{r+1}(z)$, and $y_{c+a+1}, \ldots, y_k \in \Gamma_r(z)$. Put $w(z, y_i) = \theta_i$; then we have $\theta_i = t_{r+1}$ for $1 \leq i \leq c$, $\theta_i = t_r$ for $c+1 \leq i \leq c+a$, and $\theta_i = t_{r+1}$ for $c+a+1 \leq i \leq k$. From Lemma 3.1 with $j=1$, we have

$(s_1^2 - s_2^2)((c-1)t_{r+1}t_{r-1} - 1) + a(t_{r-1}t_{r+1} - 1) + b(t_{r-1}t_{r+1} - 1))$

$=(s_1 s_2 - 1)(s_1^2 - s_2^2)$,

and this implies (i). Then (ii) is obtained from (i) by using the fact that $S=(X, w_\ast)$ is a spin model, where $w_\ast$ denotes the function defined by $w_\ast(x, y) = w(x, y)^{-1}$; see Eq. (2.10) in [10].

**Lemma 4.2.** If $b>0$, then

(i) $(s_1^2 - s_2^2)(c(s_1 s_2 - 1) + a(s_1 - 1)) = (s_1 s_2 - 1)(s_1^2 - s_2^2)$,

(ii) $(s_1^2 - s_2^2)(c(s_2 - 1) + a(s_2 - 1)) = (s_1^2 s_2 - 1)(s_2^2 - s_1^2)$.

**Proof.** In the same way as for Lemma 4.1. Use Lemma 3.1 with $j=c+a+1$ in this case.

**Lemma 4.3.** If $a>0$, then

(i) $(s_1^2 - s_2^2)(c(s_1 - 1) + a(s_2 - 1)) = (s_1 s_2 - 1)(s_1^2 - 1)$,

(ii) $(s_1^2 - s_2^2)(c(s_2 - 1) + a(s_2 - 1)) = (s_1^2 s_2 - 1)(s_2^2 - 1)$.

**Proof.** In the same way as for Lemma 4.1. Use Lemma 3.1 with $j=c+1$ in this case.

**Lemma 4.4.** (i) $b(s_1^2 - s_2^2)(s_1 s_2 - 1) = (s_1 s_2 - 1)(s_1^2 - s_2^2)$.

(ii) $a(s_1^2 - s_2^2)(s_1 s_2 - 1) = (s_1 s_2 - 1)(s_1^2 - s_2^2)$.

**Proof.** Eliminate $a$ (resp. $b$) from the two equations in Lemma 4.1 to get (i) (resp. to get (ii)).
**Lemma 4.5.** If $h > 0$, then

(i) $c(s_1^2 - s_2^2)(s_{r+1}^{-1} - 1)(s_1s_{r+1}^{-1} - 1) = (s_1s_2 - 1)(s_1^2 - s_2^2)(s_1s_2s_{r+1}^{-1} - 1)$,

(ii) $a(s_1^2 - s_2^2)(s_{r+1} - 1)(s_1 - 1) = (s_1s_2 - 1)(s_1^2 - s_2^2)(s_1s_2s_{r+1}^{-1} - 1)$.

**Proof.** These are obtained from Lemma 4.2.

**Lemma 4.6.** If $a > 0$, then

(i) $b(s_1^2 - s_2^2)(s_{r+1}^{-1} - 1)(s_1s_{r+1}^{-1} - s_r) = (s_1s_2 - 1)(s_1^2 - s_2^2)(s_1s_2s_{r+1}^{-1} - 1)$,

(ii) $c(s_1^2 - s_2^2)(s_{r+1}^{-1} - s_r^{-1}) = (s_1s_2 - 1)(s_1^2 - s_2^2)(s_1s_2s_{r+1}^{-1} - 1)$.

**Proof.** Obtained from Lemma 4.3.

**Lemma 4.7.** If $a > 0$, then $(s_1 - 1)(s_1^{-1}s_2 - s_2^{-1})(s_1^2 + s_2^{-1}) = 0$.

**Proof.** Obtained from Lemma 4.4(i) and Lemma 4.6(i).

**Lemma 4.8.** If $a > 0$ and $b > 0$, then $(s_1s_{r+1}^{-1})(s_1^r - s_2^{-1})(s_1^2 + s_{r+1}) = 0$.

**Proof.** Obtained from Lemma 4.5(i) and Lemma 4.6(ii).

**Proposition 4.9.** Assume $s_1^2 \neq s_2^2$ and $t_{r-1}, t_r, t_{r+1}$ are distinct. Then $c(z, x), a(z, x)$, and $b(z, x)$ depend only on the distance $d(z, x) = r$, rather than on the individual vertices $z, x$ with $d(z, x) = r$. Moreover, $a(z, x) = 0$ when $d(z, x) < d$.

**Proof.** We have $s_r \neq 1, s_{r+1} \neq 1, s_ts_{r+1} \neq 1$, since $t_{r-1}, t_r, t_{r+1}$ are distinct. Therefore by Lemma 4.4, $b = b(z, x)$ and $a = a(z, x)$ can be written in terms of $s_j$'s for all $z, x$ at distance $r$. Therefore $b, a$, and $c = c(z, x) = k = a - b$ are determined by the weights $t_0, ..., t_d$ and the valency $k = m'(t_i)$. This means $c, a$, and $b$ depend only on the distance $d(z, x) > 1$. The case $d(z, x) = 1$ is trivial since $G$ is triangle-free.

Now assume $r < d$, so that $b = b(z, x) > 0$ for $z, x \in X$ with $d(z, x) = r$.

The l.h.s. of Lemma 4.4(i) is not 0, since we have $b > 0, s_{r+1} \neq 1, s_ts_{r+1} \neq 1$. Hence we must have $s_1^{-1}s_2 - s_2^{-1} \neq 0$. By the same argument, we have $s_1^{-1}s_2 \neq s_{r+1}$ by Lemma 4.5(i). If $a > 0$, then Lemma 4.7 implies $s_2^{-1} + s_1 = 0$, and Lemma 4.8 implies $s_1^2 + s_{r+1} = 0$, so that $s_ts_{r+1} = s_1^{-1}s_1 = 1$, contradicting $t_{r-1} \neq t_{r+1}$. Hence we must have $a = 0$.

Now Theorem 1.1 follows from Proposition 4.9 when $s_1^2 \neq s_2^2$. We consider the case $s_1^2 = s_2^2$ in the next section.
5. Distance-Regularity: Case $s_1^2 = s_2^2$

**Proposition 5.1.** If $s_1 = 1$, then $\Gamma$ is a distance-regular graph with $d = 2$ and $c_2 = 2$, and one of the following two cases occurs:

(i) $k = 2$, $n = 4$, $t_0 = t_1 = \pm 1$, and $t_2 = -t_0$,

(ii) $k = 5$, $n = 16$, $t_0 = t_1 = \pm i$, and $t_2 = -t_0$, where $i^2 = -2$.

**Proof.** Corollary 3.2 with $s_1 = 1$ implies $s_2 = \pm 1$. If $s_2 = 1$, then $s_1 = s_2 = 1$ and, hence, $t_0 = t_2$, contradicting our assumption. So we have $s_2 = -1$. Lemma 4.1 implies $s_2^2 = 1$, so that $t_r = \pm t_{r-1}$ $(r = 2, \ldots, d)$; hence we have $-t_1 = t_2 = \cdots = t_d$ by our assumption $t_i \neq t_i$ $(i = 2, \ldots, d)$. Thus we have

$t_1 = t_0, \quad t_i = -t_0 \quad (i = 2, \ldots, d). \quad (2)$

Now we use (S2), given in the definition of a spin model. Let $u, v \in X$ with $u \neq v$, and put $D_j^r = \Gamma_j(u) \cap \Gamma_j(v)$. Then (S2) implies

$$\sum_{x \in X} w(u, x) w(v, x)^{-1} = 0. \quad (3)$$

First we consider Eq. (3) for $u, v$ with $\partial(u, v) = 1$. We partition $X$ as

$$X = \{u\} \cup \{v\} \cup D_2^1 \cup D_2^2 \cup \left( \bigcup_{i \geq 2, j \geq 2} D_i^j \right).$$

Taking the summation in (3) along with this partition, we obtain

$$t_0 t_1^{-1} + t_1 t_0^{-1} + (k - 1)(t_1 t_2^{-1} + t_2 t_1^{-1}) + \sum_{i=2}^d \sum_{j=2}^d t_i t_j^{-1} |D_j^i| = 0.$$

From Eq. (2), this implies

$$1 + 1 + (k - 1)((-1) + (-1)) + (n - 2 - 2(k - 1)) = 0,$$

so that

$$n = 4k - 4. \quad (4)$$

Next, we consider Eq. (3) for $u, v$ with $\partial(u, v) = 2$. We partition $X$ as

$$X = \{u\} \cup \{v\} \cup D_2^1 \cup D_2^2 \cup D_2^3 \cup D_2^4 \cup \left( \bigcup_{i \geq 2, j \geq 2} D_i^j \right).$$
So Eq. (3) implies
\[ t_0 t_1^{-1} + t_2 t_0^{-1} + |D_1^1| t_1 t_1^{-1} + |D_2^1| t_1 t_2^{-1} + |D_3^1| t_1 t_3^{-1} + |D_4^1| t_2 t_1^{-1} + \sum_{i=2}^{d} \sum_{j=2}^{d} |D_i^j| t_i t_j^{-1} = 0. \]

Let \(|D_i^j| = \mu|\), then
\[ |D_1^2 \cup D_3^2| = |D_1^2 \cup D_1^1| = k - \mu. \]

So we obtain
\[ (-1) + (-1) + \mu + (k - \mu)((-1) + (-1)) + (n - 2 - \mu - 2(k - \mu)) = 0, \]
so that \(\mu = 2\) by (4). This means that \(c(x, y) = 2\) holds for all \(x, y \in X\) at distance \(d(x, y) = 2\).

Now assume \(d \geq 3\), and consider Eq. (3) for \(u, v\) with \(\partial(u, v) = 3\). We partition \(X\) as
\[ X = \{u\} \cup \{v\} \cup D_1^2 \cup D_3^2 \cup D_1^1 \cup D_3^1 \cup \left( \bigcup_{i > j > 2} D_i^j \right). \]

Then we obtain
\[ (-1) + (-1) + k((-1) + (-1)) + (n - 2 - 2k) = 0, \]
so that \(n = 4k + 4\), contracting Eq. (4). Therefore we have \(d = 2\), and \(\Gamma\) is distance-regular with \(c_2 = 2\). Equivalently, \(\Gamma\) is strongly regular with parameters \(n = 4k - 4, k, \lambda = 0, \mu = 2\) (see, for instance, [5] for definitions).

We must have \(n = 1 + k + k(k - 1)/2\) by counting the number of vertices in \(\Gamma_i(u)\) \((i = 0, 1, 2)\). Comparing this with Eq. (4), we obtain
\[ 1 + k + \frac{k(k - 1)}{2} = 4k - 4, \]
and this becomes \((k - 2)(k - 5) = 0\), so that we have \(k = 2\) and \(n = 4\) or \(k = 5\) and \(n = 16\). We use (1) to determine the value of \(t_0\). When \(k = 2\), (1) becomes \(t_0 + 2t_1 + t_2 = \sqrt{4} t_0^{-1}\), and this implies \(t_0^{-1} = 1\), so \(t_0 = \pm 1\). When \(k = 5\), Eq. (1) becomes \(t_0 + 5t_1 + 10t_2 = \sqrt{16} t_0^{-1}\), and this implies \(t_0^{-1} = -1\).

**Remark 5.2.** The values of \(t_0, t_1, t_2\) given in Proposition 5.1 actually define a spin model \(S\). When \(k = 2\), \(\Gamma\) is a 4-cycle and \(S\) is a special case of “square spin model” (see [6, 7, 10]). When \(k = 5\), \(\Gamma\) is the antipodal...
quotient of the five-dimensional hypercube $H(5, 2)$, or, equivalently, the complement of the Clebsch graph, and the spin model $S$ appears in [6, 7].

**Proposition 5.3.** Assume $s_1^2 = s_2^2$, $s_1 \neq 1$, and $t_{r-2} \neq t_r$ (for $r = 2, \ldots, d$). Then $\Gamma$ is a bipartite distance-regular graph of diameter $d \leq 4$ with intersection numbers $c_r = r$ (for $r = 1, \ldots, d$), and the weights are given by $t_r = i_{r-1}$ (for $r = 1, \ldots, d$), where $i^2 = -1$.

**Proof.** Corollary 3.2 with $s_1^2 = s_2^2$ implies $s_1 = 1$. From our assumption, we have also $s_1 \neq 1$, $s_2 \neq 1$, and $s_1 s_2 \neq 1$. These imply $s_1^2 = s_2^2 = -1$. Moreover, Lemma 4.1(i) with $s_1^2 = s_2^2$ implies $s_r^2 = s_{r-2}^2$ (for $r = 2, \ldots, d$). So $s_r^2 = -1$, and hence $t_r = i_{r-1}$ (for $r = 1, \ldots, d$). It follows that $t_r = -t_{r-2}$ (for $r = 2, \ldots, d$). If $d \geq 5$, then $t_5 = -t_3 = t_1$, contradicting our assumption $t_1 \neq t_r$ (for $r = 2, \ldots, d$), so we must have $d \leq 4$, and (up to the exchange of $i$ and $-i$)

$$t_1 = i_{0}, \quad t_2 = -i_0, \quad t_3 = -i_{0}, \quad t_4 = i_0. \tag{5}$$

We claim that $\Gamma$ is bipartite. If there is an edge inside $\Gamma_i(x)$ for some $x \in X$ and for some $r$, then Lemma 4.3(i) implies $s_1^2 = 1$, a contradiction. This easily implies that $\Gamma$ is bipartite.

Now we use Lemma 3.3. Fix an edge $z_1 z_2$ and put $D_r = \Gamma_i(z_1) \cap \Gamma_i(z_2)$. Take $x \in D_{r+1}^r$ (for $0 < r < d$). Then $\Gamma_i(x)$ is partitioned as

$$\Gamma_i(x) = (\Gamma_i(x) \cap D_r^r) \cup (\Gamma_i(x) \cap D_r^{r+1}) \cup (\Gamma_i(x) \cap D_r^{r+2}).$$

Put $|\Gamma_i(x) \cap D_{r-1}^r| = c$ and $|\Gamma_i(x) \cap D_{r+1}^r| = c'$. Let $y_1, \ldots, y_k$ be the neighbors of $x$, where we choose the labels such that

$$\Gamma_i(x) \cap D_r^r = \{y_1, \ldots, y_c\},$$
$$\Gamma_i(x) \cap D_r^{r+1} = \{y_{c+1}, \ldots, y_c\},$$
$$\Gamma_i(x) \cap D_r^{r+2} = \{y_{c+2}, \ldots, y_k\}.$$

Then we have

$$\theta = t_r, \quad \theta_1 = \cdots = \theta_r = t_{r-1}, \quad \theta_{r+1} = \cdots = \theta_k = t_{r+1},$$
$$\eta = t_{r+1}, \quad \eta_1 = \cdots = \eta_c = t_r, \quad \eta_{c+1} = \cdots = \eta_k = t_{r+2}.$$

So Lemma 3.3 implies

$$ct_{r-1}^r + (c' - c) t_{r+1}^r + (k - c') t_{r+2}^r = (s_1 s_2 + k - 1) s_1 s_2^2 t_r^r.$$

Assigning the values given in (5), this implies $c' = c + 1$. When $r = 1$, we have $c = 1$, so that $c' = 2$. Therefore we obtain $|\Gamma_i(x) \cap \Gamma_{r-1}(z)| = r$ for all
$x, z$ at distance $d(x, z) = r$, by induction on $r$. So $\Gamma$ is a distance-regular graph with $c_r = r$ $(r = 1, \ldots, d)$.

**Remark 5.4.** It is known that a bipartite distance-regular graph $\Gamma$ with $c_r = r$ $(r = 1, \ldots, d)$ is isomorphic to the $d$-dimensional hypercube (see, for instance, 9.2.5 of [5]). As easily shown, for a suitably chosen value of $t_0$, the weights $t_r = t_{1,0}$ $(r = 0, 1, \ldots, d)$ actually give a spin model, which is a tensor product of $d$ Potts models of size 2 (see [6] for definitions).

**References**