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Continuation and Surjectivity Theorems for Uniform Limits of A-proper Mappings with Applications

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INTRODUCTION

In [19] the authors established several continuation theorems and studied the approximation-solvability of operator equations involving multivalued A-proper mappings. In this paper we continue our study of the solvability of equations involving the uniform limits of multivalued A-proper mappings and their perturbations.

In Section 1 we state the basic definitions and some of the preliminaries to be used in subsequent sections. For more details the reader should consult [20].

In Section 2 we establish two continuation theorems and some surjectivity results for uniform limits of A-proper mappings under various growth conditions. The proofs of some theorems in this section are based on our results in [19]. All the results of this section except Theorem 2.4 and a version of Theorem 2.6 are new also in the single-valued case.

Section 3 deals with some special classes of uniform limits of A-proper mappings. In the first part of this section we introduce various classes of multivalued A-proper mappings involving quasi-K-monotone, generalized pseudo-K-monotone, and K-semibounded mappings. In the second part of this section we use the theorems of Section 2 to deduce various surjectivity results for the above mentioned mappings and their perturbations. As special cases we deduce from our results (for separable spaces) certain surjectivity results of Browder, Browder and Hess, Hess, Fitzpatrick, Petryshyn, Petryshyn and Fitzpatrick, Rockafellar, Wille, and others.

In Section 4 we apply some of our results (Theorems 2.4, 2.6, and 3.5) to the variational solvability of elliptic boundary value problems involving completely continuous perturbations of operators with semibounded variation. Our surjectivity results extend some of those obtained earlier by Browder, Dubinsky and others (see Sects. 3 and 4 for detailed historical comments). As our second

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application we treat a boundary value problem involving nonlinear ordinary differential operators of order $m > 1$ satisfying general homogeneous boundary conditions. The interesting feature of this example is that it involves mappings acting between two different spaces and thus surjectivity results of the other authors are not directly applicable in this case. Furthermore our growth conditions are weaker than those used by other authors for operators acting from a Banach space to its dual space.

1. Basic Definitions and Preliminaries

Let $\{E_n\}$ and $\{F_n\}$ be two sequences of oriented finite-dimensional spaces and let $\{V_n\}$ and $\{W_n\}$ be two sequences of continuous linear mappings with V_n mapping E_n into X and W_n mapping Y onto F_n, where X and Y are real normed linear spaces.

Remark 1.1. For the sake of notational simplicity we use the same symbol $\| \|$ to denote the norm in the respective space X, Y, E_n, and F_n, and from the context it will be clear which norm is meant. We also use the symbols "$\rightarrow$" and"$\rightharpoonup$" to denote *strong* and *weak* convergence, respectively.

Definition 1.1. A quadruple of sequences $\Gamma = \{E_n, V_n; F_n, W_n\}$ is said to be an ***admissible scheme*** for (X, Y) if $\dim E_n = \dim F_n$ for each n, V_n is injective, $\operatorname{dist}(x, V_nE_n) \rightarrow 0$ as $n \rightarrow \infty$ for each x in X, and $\{W_n\}$ is uniformly bounded.

Note that in Definition 1.1 we do not require that E_n and F_n be subspaces of X and Y, respectively, nor that V_n and W_n be linear projections. The following examples of admissible shcemes (for others see [21]), which we subscript for further references, will be used in this paper. For the present we shall assume that $\{X_n\}$ is a sequence of oriented finite-dimensional subspaces of X such that $\operatorname{dist}(x, X_n) \rightarrow 0$ as $n \rightarrow \infty$ for each x in X, and let V_n be an inclusion map of X_n into X.

(a) Let $\{Y_n\}$ be a sequence of finite-dimensional oriented subspaces of Y such that $\dim Y_n = \dim X_n$ and let Q_n be a continuous linear map of Y onto Y_n such that $\| Q_n \| \leqslant M$ for all n and some $M > 0$. Then $\Gamma_a = \{X_n, V_n; Y_n, Q_n\}$ is admissible for (X, Y).

(b) If $Y = X$, $Y_n = X_n$ and $W_n = P_n$, where P_n is a projection of X onto X_n such that $P_n(x) \rightarrow x$ for each x in X and $\| P_n \| \leqslant M$ for all n, then $\Gamma_b = \{X_n, V_n; X_n, P_n\}$ is an admissible projection scheme for (X, X). Note that when X is complete, then the assumption that $\| P_n \| \leqslant M$ is superfluous.

(c) If $Y = X^*$, $Y_n = X^*_n$ and $W_n = V^*_n$, then $\Gamma_c = \{X_n, V_n; X^*_n, V^*_n\}$ is an admissible injective shceme for (X, X^*).

We add that Γ_c always exists when X is separable. Let D be a given set in X, $D_n \equiv V_n^{-1}(D)$, $T: D \to 2^Y$ and $T_n \equiv W_n T V_n |_{D_n}: D_n \to 2^{F_n}$. For what follows we shall need the following basic definition.

DEFINITION 1.2. A multivalued mapping $T: D \subset X \to 2^Y$ is said to be *A-proper* w.r.t. $\Gamma = \{E_n, V_n; F_n, W_n\}$ if $T_n: D_n \to 2^{F_n}$ is upper semicontinuous for each n and if for any sequence $\{u_{n_j} \mid u_{n_j} \in D_{n_j}\}$ such that $\{V_{n_j}(u_{n_j})\}$ is bounded in X and $\| W_{n_j}(y_{n_j}) - W_{n_j}(y)\| \to 0$ as $j \to \infty$ for some $y_{n_j} \in TV_{n_j}(u_{n_j})$ and $y \in Y$, there exist a subsequence $\{u_{n_{j(k)}}\}$ and $x_0 \in D$ such that $V_{n_{j(k)}}(u_{n_{j(k)}}) \to x_0$ and $y \in T(x_0)$.

Remark 1.2. The class of single-valued A-proper mappings, whose study was initiated by Petryshyn, has been investigated by many authors (see [20] for the survey). The theory of A-proper mappings proved to be useful in the constructive solvability for abstract and differential equations. It provided the unification and the extension of various results from the theory of operators of monotone, condensing, and P-compact type. Multivalued A-proper mappings have been first studied extensively by Milojević [17, 18] and subsequently by Milojević and Petryshyn [19]. For various examples of A-proper single-valued and multivalued A-proper mappings see [20, 17, 19].

The purpose of this paper is to continue the study initiated by the authors in [19]. Our particular interest is the study of the solvability of equations involving operators which are uniform limits of A-proper mappings. The theory presented here unifies and extends a number of results obtained by other authors for various special classes of mappings such as mappings of monotone type, condensing, P-compact and others.

In what follows $K(X)$, $BK(X)$, and $CK(X)$ will denote the families of nonempty closed and convex, nonempty bounded closed and convex, and nonempty compact and convex subsets of X, respectively.

2. CONTINUATION AND SURJECTIVITY RESULTS

In this section we shall consider the solvability of a larger class of operator equations than that considered in the paper [19]. Namely, for a given $T_t: X \to 2^Y$, $t \in [0, 1]$, we shall consider the solvability of equations of type $f \in T_1(x)$, where T_t is not A-proper for $t \in [0, 1]$, but is the uniform limit of multivalued A-proper mappings, i.e., T_t is such that $T_{t,\mu} = T_t + \mu G$ is A-proper for each $\mu > 0$ and $t \in [0, 1]$ and some given multivalued mapping G. We then apply these results to the solvability of equations of the form $f \in T(x)$ with T being a uniform limit of A-proper mappings. Examples of such mappings will be given in Section 3. The solvability of the perturbed equations, i.e., of $f \in T(x) + A(x)$ is also considered. Unlike the results of [19], those obtained here will be only of the existence type.

In what follows we shall say that a mapping $T: Y \to 2^Y$ satisfies *condition* $(+)$ if $\{x_n\}$ is any sequence and $u_n \to g$ in Y for some $u_n \in T(x_n)$, then $\{x_n\}$ is bounded. T is said to satisfy *condition* $(++)$ if $\{x_n\} \subset X$ is any bounded sequence such that whenever $u_n \to g$ in Y for some $u_n \in T(x_n)$ then there exists $x \in X$ such that $g \in T(x)$.

THEOREM 2.1. *Let X and Y be normed spaces with an admissible scheme Γ and let $T_t(\cdot) \equiv T(t, \cdot)$ map $[0, 1] \times X \to 2^Y$ with $W_n T_t(x) \in CK(F_n)$. Suppose there exists a bounded mapping $G: X \to 2^Y$ with $W_n G(x) \in CK(F_n)$ for each $x \in X$ and that $T_t + \mu G$ is A-proper w.r.t. Γ for each $t \in [0, 1]$, $\mu > 0$. Suppose that for each $f \in Y$ there exists $r_f > 0$ such that the following hypotheses hold:*

(H1) *There exists $\gamma > 0$ such that $\| f - y \| > \gamma$ for all $y \in T_t(x)$ with $x \in \partial B(0, r_f)$ and $t \in [0, 1]$.*

(H2) *$\| sf - y \| > \gamma$ for all $y \in T_0(x)$ with $x \in \partial B(0, r_f)$ and $s \in [0, 1]$.*

(H3) *If for some $\mu > 0$, $t_n \in [0, 1]$ and $u_n \in \partial B_n(0, r_f) = \partial V_n^{-1}(B(0, r_f))$ we have $W_n(f) \in W_n T_{t_n} V_n(u_n) + \mu W_n G V_n(u_n)$, then there exists a subsequence $\{t_{n_k}\}$ converging to t such that $\| W_{n_k}(f) - W_{n_k}(y_{n_k}) - \mu W_{n_k}(z_{n_k}) \| \to 0$ as $k \to \infty$ for some $y_{n_k} \in T_t V_{n_k}(u_{n_k})$ and $z_{n_k} \in G V_{n_k}(u_{n_k})$.*

(H4) *There exists $n_0 \geqslant 1$ such that for each $n \geqslant n_0$, $\mu > 0$, $\deg(L_n W_n T_0 V_n + \mu L_n W_n G V_n, B_n(0, r_f), 0\} \neq 0$, where L_n is some linear isomorphism between F_n and E_n.*

Then, if T_1 satisfies condition $(++)$, the equation $f \in T_1(x)$ is solvable for each $f \in Y$.

Proof. Let $f \in Y$ be fixed. Since G is bounded, there exists $\mu_0 > 0$ such that for each $\mu \in (0, \mu_0)$ we have

$$\| f - y - \mu z \| > \gamma/2 \qquad \text{for all } y \in T_t(x), \quad z \in G(x)$$
with
$$x \in \partial B(0, r_f), \qquad t \in [0, 1], \tag{2.1}$$

and

$$\| sf - y - \mu z \| > \gamma/2 \qquad \text{for all } y \in T_0(x), \quad z \in G(x)$$
with
$$x \in \partial B(0, r_f) \qquad \text{and} \qquad s \in [0, 1]. \tag{2.2}$$

Let $\mu \in (0, \mu_0)$ be fixed. Then we claim now that there exists n_1 such that

$$W_n(f) \notin W_n T_t V_n(u) + \mu W_n G V_n(u) \qquad \text{for all } n \geqslant n_1, \quad t \in [0, 1] \quad \text{and} \quad u \in \partial B_n(0, r_f). \tag{2.3}$$

If not, then there exist $t_n \in [0, 1]$ and $u_n \in \partial B_n(0, r_f)$ such that $W_n(f) \in W_n T_{t_n} V_n(u_n) + \mu W_n G V_n(u_n)$ for infinitely many n. By hypothesis (H3), there exists $\{t_{n_k}\}$ converging to t with $\| W_{n_k}(f) - W_{n_k}(y_{n_k}) - \mu W_{n_k}(z_{n_k}) \| \to 0$ as $k \to \infty$ for some $y_{n_k} \in T_t V_{n_k}(u_{n_k})$ and $z_{n_k} \in GV_{n_k}(u_{n_k})$. By the A-properness of $T_t + \mu G$ w.r.t. Γ, there exists $\{u_{n_{k(i)}}\}$ such that $V_{n_{k(i)}}(u_{n_{k(i)}}) \to x \in X$ and $f \in T_t(x) + \mu G(x)$ with $\| x \| = r_f$, in contradiction to (2.1). Thus, there exists n_1 such that (2.3) holds for each $n \geqslant n_1$.

Next, there exists an $n_2 \geqslant 1$ such that for each $n \geqslant n_2$

$$sW_n(f) \notin W_n T_0 V_n(u) + \mu W_n G V_n(u) \qquad \text{for all } s \in [0, 1] \quad \text{and} \quad u \in \partial B_n(0, r_f). \tag{2.4}$$

If not, there exist $s_n \in [0, 1]$ and $u_n \in \partial B_n(0, r_f)$ such that $s_n W_n(f) \in W_n T_0 V_n(u_n) + \mu W_n G V_n(u_n)$ for infinitely many n. Let $y_n \in T_0 V_n(u_n)$ and $z_n \in GV_n(u_n)$ be such that $s_n W_n(f) = W_n(y_n) + \mu W_n(z_n)$. We may assume that $s_n \to s$. Then

$$\begin{aligned} &\| W_n(y_n) + \mu W_n(z_n) - W_n(sf) \| \\ &\quad = \| W_n(s_n f) - W_n(sf) \| \leqslant \| W_n \| \cdot \| f \| \cdot | s_n - s | \to 0 \end{aligned}$$

as $n \to \infty$ by the uniform boundedness of $\{W_n\}$. Consequently, by the A-properness of $T_0 + \mu G$, there exists $\{u_{n_{k(i)}}\}$ such that $V_{n_{k(i)}}(u_{n_{k(i)}}) \to x \in X$ with $sf \in T_0(x) + \mu G(x)$, with $\| x \| = r_f$, in contradiction to (2.2). Thus, (2.4) holds. Let $N_f = \max\{n_0, n_1, n_2\}$. Then, for each $n \geqslant N_f$ relations (2.3), (2.4), and hypothesis (H4) hold. Hence, by Theorem 1.2 in [19], for each $\mu \in (0, \mu_0)$ there exists $x_\mu \in B(0, r_f)$ such that $f \in T_1(x) + \mu G(x)$. Let $\mu_n \in (0, \mu_0)$ be such that $\mu_n \to 0$ as $n \to \infty$ and let $x_{\mu_n} \equiv x_n \in X$ be such that $f \in T_1(x_n) + \mu_n G(x_n)$, or $f = y_n + \mu_n z_n$ for some $y_n \in T_1(x_n)$ and $z_n \in G(x_n)$. By the boundedness of G, $y_n = f - \mu_n z_n \to f$ which by condition $(++)$ implies the existence of $x \in X$ such that $f \in T_1(x)$. Q.E.D.

The following lemma gives some conditions which imply (H1)–(H3).

LEMMA 2.1. *Suppose that $T_t(x)$ is bounded and closed for all $t \in [0, 1]$, $x \in X$ and that*

(1) $f \notin \overline{T_t(\partial B(0, r_f))}$ *for all* $t \in [0, 1]$,

(2) $sf \notin \overline{T_0(\partial B(0, r_f))}$ *for all* $s \in [0, 1]$.

Then, if $\alpha(T_{t_n}(x), T_t(x)) = \sup_{y \in T_{t_n}(x)} d(y, T_t(x)) \to 0$ uniformly with respect to x in bounded sets when $t_n \to t$, hypotheses (H1), (H2) *hold. If, additionally, $W_n T_t(x)$ is compact for all n, $t \in [0, 1]$ and $x \in X$, then hypothesis* (H3) *also holds.*

Proof. Let $t_0 \in [0, 1]$ be fixed. Then there exists $\gamma(t_0) > 0$ such that $\| f - y \| > \gamma(t_0)$ for all $y \in T_{t_0}(x)$, $x \in \partial B(0, r_f)$. Let $\Delta(t_0)$ be an interval around t_0 such that for $t \in \Delta(t_0)$, $\alpha(T_t(x), T_{t_0}(x)) \leqslant \gamma(t_0)/3$. For each $x \in \partial B(0, r_f)$ and $y \in T_t(x)$, $t \in \Delta(t_0)$, let $y_0 \in T_{t_0}(x)$ be such that $\| y - y_0 \| \leqslant 2\gamma(t_0)/3$. Then $\| f - y \| \geqslant \| f - y_0 \| - \| y - y_0 \| \geqslant \gamma(t_0)/3$. Since $[0, 1]$ can be covered by a finite number of such intervals, we have that there exists $\gamma_1 > 0$ such that $\| f - y \| \geqslant \gamma_1$ for all $y \in T_t(x)$ with $\| x \| = r_f$ and $t \in [0, 1]$.

Now define $A = \{sf \mid s \in [0, 1]\}$ and $B = \overline{T_0(\partial B(0, r_f))}$. Since A is compact, B closed and $A \cap B = \phi$, it follows that there exists $\gamma_2 > 0$ such that $\| sf - y \| > \gamma_2$ for all $s \in [0, 1]$ and $y \in T_0(\partial B(0, r_f))$. Then $\gamma = \min\{\gamma_1, \gamma_2\}$ is such that hypotheses (H1) and (H2) hold.

Suppose now that for some $\mu > 0$, $t_n \in [0, 1]$ and $u_n \in \partial B_n(0, r_f)$, $W_n(f) \in W_n T_{t_n} V_n(u_n) + \mu W_n G V_n(u_n)$. Then for some subsequence we have $t_{n_k} \to t$. By the compactness of $W_n T_t(x)$, for each $z_{n_k} \in T_{t_{n_k}} V_{n_k}(u_{n_k})$, there exists $y_{n_k} \in T_t V_{n_k}(u_{n_k})$ such that

$$\| W_{n_k}(z_{n_k}) - W_{n_k}(y_{n_k}) \| = d(W_{n_k}(z_{n_k}), W_{n_k} T_t V_{n_k}(u_{n_k})).$$

This implies that

$$\begin{aligned} &\| W_{n_k}(z_{n_k}) - W_{n_k}(y_{n_k}) \| \\ &\quad \leqslant \alpha(W_{n_k} T_{t_{n_k}} V_{n_k}(u_{n_k}), W_{n_k} T_t V_{n_k}(u_{n_k})) \\ &\quad \leqslant \| W_{n_k} \| \alpha(T_{t_{n_k}} V_{n_k}(u_{n_k}), T_t V_{n_k}(u_{n_k})) \to 0 \qquad \text{as} \qquad k \to \infty. \end{aligned}$$

Since $W_{n_k}(f) = W_{n_k}(z_{n_k}) + \mu W_{n_k}(\bar{z}_{n_k})$ for some $z_{n_k} \in T_{t_{n_k}} V_{n_k}(u_{n_k})$ and $\bar{z}_{n_k} \in G V_{n_k}(u_{n_k})$, it follows for $\{y_{n_k}\}$ above chosen with respect to $\{z_{n_k}\}$ that

$$\| W_{n_k}(f) - W_{n_k}(y_{n_k}) - \mu W_{n_k}(\bar{z}_{n_k}) \| = \| W_n(z_{n_k}) - W_{n_k}(y_{n_k}) \| \to 0 \qquad \text{as} \qquad k \to \infty.$$

Thus hypothesis (H3) holds. Q.E.D.

Remark 2.1. It is easy to see that hypotheses (H1) and (H2) hold if T_t satisfies condition $(+)$ uniformly with respect to $t \in [0, 1]$.

Analyzing the proof of Theorem 2.1 we see that all we need for the solvability of $f \in T_1(x)$ is that relations (2.3) and (2.4) and hypothesis (H4) hold. So we have the following more general form of Theorem 2.1.

THEOREM 2.2. *Let X, Y be normed spaces with an admissible scheme Γ, $T: [0, 1] \times X \to 2^Y$ with $T_n: [0, 1] \times E_n \to CK(F_n)$ u.s.c. and $G: X \to 2^Y$ a bounded mapping with $W_n T_t(x)$, $W_n G(x) \in CK(F_n)$ and that for $\mu > 0$, $T_1 + \mu G$ is A-proper w.r.t. Γ. Suppose that for each $f \in Y$ there exist $r_f > 0$ and $n_0 \geqslant 1$ such that for each $n \geqslant n_0$ and $\mu \in (0, \mu_0)$ for some $\mu_0 > 0$ we have*

(H1′) $W_n(f) \notin W_n T_t V_n(u) + \mu W_n G V_n(u)$ *for* $t \in [0, 1]$, $u \in \partial B_n(0, r_f)$

(H2′) $s W_n(f) \notin W_n T_0 V_n(u) + \mu W_n G V_n(u)$ *for* $s \in [0, 1]$, $u \in \partial B_n(0, r_f)$.

and hypothesis (H4) *holds.*

Then, if T_1 satisfies condition $(++)$, *the equation* $f \in T_1(x)$ *is solvable for each* $f \in Y$.

For the solvability of the equation $f \in T_1(x) + A(x)$ we have the following:

THEOREM 2.3. *Let X and Y be normed spaces with an admissible scheme Γ and $T_t(\cdot) = T(t, \cdot)$ map $[0, 1] \times X \to 2^Y$ with $W_n T_t(x) \in CK(F_n)$ for all $(t, x) \in [0, 1] \times X$. Suppose that $T_t(x)$ is α-continuous in t uniformly with respect to x in bounded subsets of X. Suppose that A, G: $X \to 2^Y$ with $W_n A(x)$, $W_n G(x) \in CK(F_n)$, G is bounded, and that for each $\mu > 0$, $T_t + A + \mu G$ is A-proper w.r.t. Γ for each $t \in [0, 1]$. Suppose also that either for each $f \in Y$ there exists $r_f > 0$ such that:*

(1) $f \notin \overline{(T_t + A)(\partial B(0, r_f))}$ *for all* $t \in [0, 1]$

(2) $sf \notin \overline{(T_0 + A)(\partial B(0, r_f))}$ *for* $s \in [0, 1]$

or that T_t satisfies condition $(+)$ *uniformly* w.r.t. $t \in [0, 1]$.

(3) *There exists $n_0 \geq 1$ such that for each $n \geq n_0$, $\mu > 0$,* $\deg(L_n W_n T_0 V_n + L_n W_n A V_n + \mu L_n W_n G V_n, B_n(0, r_f), 0) \neq 0$, *where L_n is some linear isomorphism between F_n and E_n.*

Then, if $T_1 + A$ satisfies condition $(++)$, *the equation* $f \in T_1(x) + A(x)$ *is solvable for each* $f \in Y$.

Proof. Let $f \in Y$ be fixed. Since $\alpha(T_t(x) + A(x), T_{t_0}(x) + A(x)) \leq \alpha(T_t(x), T_{t_0}(x))$, it follows that $T_t(x) + A(x)$ is α-continuous in t uniformly with respect to x in bounded subsets of X, i.e., whenever $t_n \to t$, then $\alpha(T_{t_n}(x) + A(x), T_t(x) + A(x)) \to 0$ as $n \to \infty$ uniformly for x in bounded subsets of X. By Lemma 2.1, we see that hypotheses (H1), (H2), and (H3) of Theorem 2.1 hold for $T_t + A$, $T_0 + A$, and $T_t + A + \mu G$, respectively. In view of this and our assumption (3), there exists $\mu_0 > 0$ such that for each $\mu \in (0, \mu_0)$, the equation, $f \in T_1(x) + A(x) + \mu G(x)$ is solvable in $\bar{B}(0, r_f)$. Since $T_1 + A$ satisfies condition $(++)$ and G is bounded, we have that, as in Theorem 2.1, the equation $f \in T_1(x) + A(x)$ is solvable for each $f \in Y$. Q.E.D.

Let us now discuss conditions that imply hypothesis (H4). It is clear that if T_0 and G and T_0, A and G in Theorems 2.1 and 2.3, respectively, are odd, then hypothesis (H4) and condition (3) hold, respectively. Moreover, if K, K_n, and M_n satisfy conditions of Proposition 1.2 in [19], G and K, and T_0 and K satisfy condition (C3) of that proposition, then one can show that hypothesis (H4) holds. Similarly, one imposes conditions on T_0, A and G in Theorem 2.3 which imply the validity of its condition (3).

Remark 2.2 We add in passing that condition $(+)$ is implied by any one of the following conditions which have been used by many authors (see [19]) in their study of the equation $f \in T(x)$ with T either single-valued or multivalued mapping of monotone, ball-condensing and A-proper type:

Condition (1+). $(y, v)/\|v\| \to \infty$ as $\|x\| \to \infty$ for all $y \in T(x)$ and $v \in K(x)$ (i.e., T is *K-coercive*).

Condition (2+). For each unbounded sequence $\{x_n\} \subset X$, $\|y_n\| \to \infty$ as $n \to \infty$ for all $y_n \in T(x_n)$ (i.e., T is *norm-coercive*).

Condition (3+). For each $y_n \in T(x_n)$ and $v_n \in K(x_n)$, $\|y_n\| + ((y_n, v_n)/\|v_n\|) \to \infty$ as $\|x_n\| \to \infty$.

Condition (4+). $0 \notin \overline{T(\partial B(0, r))}$ and $T(tx) = t^\alpha T(x)$ for all $\|x\| \geqslant r$, $t > 1$ and some $\alpha > 0$.

The following result on the solvability of $f \in T(x)$, where T is a uniform limit of A-proper mappings, i.e., T is such that $T + \mu G$ is A-proper for each $\mu > 0$ and some mapping G, can be obtained either from Theorem 2.3 or Theorems 6.4 and 7.2 in [17]. We shall prove it by applying Theorems 6.4 and 7.2 in [17] since the argument is shorter.

THEOREM 2.4. *Let X, Y be normed spaces with an admissible scheme Γ. Suppose that $T: X \to 2^Y$ satisfies conditions (+) and (++) with $W_n T V_n: E_n \to CK(F_n)$ u.s.c. for each n and that $G: X \to 2^Y$ is bounded, $W_n G(x) \in CK(F_n)$ and $T_\mu = T + \mu G$ is A-proper w.r.t. Γ for each $\mu > 0$.*

Suppose further that either one of the following conditions holds:

(i) *There is $r_0 > 0$ such that T and G are odd on $X \backslash B(0, r_0)$.*

(ii) *There exist $K: X \to 2^{Y^*}$, $K_n: E_n \to 2^{F_n^*}$, and a linear isomorphism M_n of E_n onto F_n such that $0 \in K(x)$ implies $x = 0$ and that*

(C1) *for each n and $u \in E_n$, $v \in KV_n(u)$ there exists $w \in K_n(u)$ such that $(g, v) = (W_n g, w)$ for all $g \in Y$;*

(C2) *for each n, $(M_n u, w) > 0$ for all $w \in K_n(u)$ and $u \neq 0$ in E_n;*

(C3) *there exists $r_0 > 0$ such that $(u, v) \geqslant 0$ and $(w, v) \geqslant 0$ for all $u \in T(x)$, $w \in G(x)$ and $v \in K(x)$ with $\|x\| \geqslant r_0$.*

Then the equation $f \in T(x)$ is solvable for each $f \in Y$.

Proof. Let $f \in Y$ be fixed. We have noted in Remark 2.1 that condition (+) implies the existence of $r \geqslant r_0$ and $\gamma > 0$ such that $\|u - tf\| \geqslant \gamma$ for all $t \in [0, 1]$ and $u \in T(x)$ with $\|x\| = r$. This and the boundedness of G imply that there exists $\mu_0 > 0$ such that $\|v - tf\| \geqslant \gamma/2$ for $t \in [0, 1]$, $v \in T_\mu(x)$ with $\|x\| = r$ and $\mu \in (0, \mu_0)$. In view of Theorems 6.4 and 7.2 in [17], the equation $f \in T_\mu(x) = T(x) + \mu G(x)$ is solvable in $B(0, r)$ for each $\mu \in (0, \mu_0)$. Now, let $\mu_k \in (0, \mu_0)$ be such that $\mu_k \to 0$ as $k \to \infty$ and let $x_k \in B(0, r)$ be the corresponding solution of $f \in T_{\mu_k}(x)$. Since $\{G(x_k)\}$ is bounded, we have that $u_k =$

$f - \mu_k v_k \to f$ as $k \to \infty$ for some $u_k \in T(x_k)$ and $v_k \in G(x_k)$. This and condition $(++)$ imply that there exists an $x \in X$ such that $f \in T(x)$. Q.E.D.

Remarks 2.3. Theorem 2.4 remains valid if condition $(+)$ is replaced by any one of the conditions of Remark 2.2. We also add that condition $(++)$ holds if $T(\bar{B}(0, r))$ is closed in Y for each $r > 0$.

Remark 2.4. When all mappings involved are single-valued, Theorem 2.4 and Theorems 6.4 and 7.2 in [17] were proven by Petryshyn [21].

Remark 2.5. Instead of condition $(+)$ in Theorem 2.4 it is enough to require that for each $f \in Y$ there exists $r_f \geqslant r_0$ such that $\| u - tf \| \geqslant \gamma$ for all $t \in [0, 1]$ and $u \in T(x)$ with $\| x \| = r_f$ and some $\gamma > 0$.

Remark 2.6. The following are few choices of K, M_n, G and K_n in Theorem 2.4. If $Y = X^*$ with X separable and reflexive, we choose $\Gamma = \Gamma_c = \{X_n, V_n; X^*_n, V^*_n\}$, $K = I$, $K_n = V_n$, $G = J$ a duality mapping and define $M_n: X_n \to X^*_n$ by $M_n(x) = \sum_{j=1}^{n} (f_j, x) f_j$, where $\{\psi_1, \dots, \psi_n\}$ is a basis in X_n and $\{f_1, \dots, f_n\}$ is the corresponding biorthogonal basis in X^*_n. If $Y = X$ and $\Gamma = \Gamma_b = \{X_n, V_n; X_n, P_n\}$ is a projectionally complete scheme for (X, X), we choose $K = J$, $K_n = P^*_n J$, $M_n = I_n$ and $G = I$, where I_n and I are the identity mappings in X_n and X, respectively.

When condition $(+)$ is replaced by condition (C4) below, we have the following useful result.

THEOREM 2.5. *Let X, Y, K, K_n and M_n be as in Theorem 2.4. Suppose that $T: X \to 2^Y$ satisfies condition $(++)$ with $W_n T V_n: E_n \to CK(F_n)$ u.s.c. for each n. Suppose also that there exists a bounded mapping $G: X \to 2^Y$ such that $T_\mu = T + \mu G$ is A-proper w.r.t. Γ for each $\mu > 0$ and $W_n G(x) \in CK(F_n)$ for all $x \in V_n(E_n)$.*

Moreover, suppose that for each $f \in Y$ there exists an $r_f > 0$ such that

(C4) *$(u - f, v) \geqslant 0$ and $(w, v) \geqslant 0$ for all $u \in T(x)$, $w \in G(x)$ and $v \in K(x)$ with $\| x \| = r_f$.*

Then T is surjective, i.e., $T(X) = Y$.

Proof. Let $f \in Y$ be fixed. Then, for each $\mu > 0$ we have $(u + \mu w - f, v) = (u - f, v) + \mu(w, v) \geqslant 0$ for all $u \in T(x)$, $w \in G(x)$, $v \in K(x)$ with $\| x \| = r_f$. It follows from Theorem 6.4 in [17] that the equation $f \in T_\mu(x) = T(x) + \mu G(x)$ is solvable in $B(0, r_f)$ for each $\mu > 0$. Proceeding as in Theorem 2.4, we obtain that the equation $f \in T(x)$ is solvable. Thus, $T(X) = Y$. Q.E.D.

Finally, the following new result will prove to be useful.

THEOREM 2.6. *Let X, Y, K, K_n, and M_n be as in Theorem 2.4. Let $T: X \to 2^Y$ satisfy condition $(++)$ with $W_n T V_n: E_n \to CK(F_n)$ u.s.c. for each n and let*

$G\colon X \to 2^Y$ be bounded with $W_nGV_n(x) \in CK(F_n)$ and $T_\mu = T + \mu G$ be A-proper w.r.t. Γ for each $\mu > 0$. Suppose that the following conditions hold:

(1) $(u, v) \geqslant \| x \|^\alpha \| v \|$ *for all* $u \in G(x)$, $v \in K(x)$ *with* $x \in X$ *and some* $\alpha > 0$.

(2) *There exists* $c > 0$ *such that* $(u, v) \geqslant -c \| v \|$ *for all* $u \in T(x)$, $v \in K(x)$ *with* $x \in X$.

(3) *To each* f *in* Y *there corresponds* $c_f > 0$ *such that if* $f \in T(x_k) + \mu_k G(x_k)$ *for some* $x_k \in X$ *with* $\mu_k \to 0$, *then* $\| x_k \| \leqslant c_f$ *for all* k.

Then the equation $f \in T(x)$ *is solvable for each* f *in* Y.

Proof. Let $\mu_k > 0$ with $\mu_k \to 0$. By (1) and (2) we get that for each k and x in X

$$(u + \mu_k w, v) \geqslant (\mu_k \| x \|^\alpha - c) \| v \| \qquad \text{for all } u \in T(x), \quad w \in G(x), \quad v \in K(x).$$

This means that T_{μ_k} is K-coercive and A-proper and consequently, by Theorem 6.4 [17], the equation $f \in T(x) + \mu_k G(x)$ is solvable for each f in Y and each k. Moreover, for each f in Y the set of solutions is bounded by (3) and thus, by condition $(++)$, there exists x in X such that $f \in T(x)$. Q.E.D.

Remark 2.7. (a) Condition (3) in Theorem 2.6 is satisfied if T satisfies condition $(2+)$ (i.e., *if* $\| x_n \| \to \infty$ *then* $\| u_n \| \to \infty$ *for each* $u_n \in T(x_n)$) provided (1) and (2) hold and G is α-positively homogeneous (i.e., $G(tx) = t^\alpha G(x)$ for $x \in X$ and $t > 0$). Indeed, let $f \in T(x_k) + \mu G(x_k)$ with $\mu_k \to 0$. Then $f = u_k + \mu_k w_k$ for some $u_k \in T(x_k)$ and $w_k \in G(x_k)$ and for $v_k \in K(x_k)$, $(f, v_k) = (u_k, v_k) + \mu_k(w_k, v_k) \geqslant (\mu_k \| x_k \|^\alpha - c) \| v_k \|$. For $v_k \neq 0$ we have by the Schwartz–Buniakovsky inequality that $\mu_k \| x_k \|^\alpha = c + (f, v_k) \cdot \| v_k \|^{-1} \leqslant c + \| f \|$. Since G is bounded and $\mu_k G(x_k) = G(\mu_k^{1/\alpha} x_k)$, we have that $\{u_k\} = \{f - \mu_k w_k\}$ is bounded which, by $(2+)$, implies that $\{x_k\}$ is bounded.

The second author used the conditions and the method described in (a) to obtain a surjectivity theorem for the more general class of uniform limits of pseudo-A-proper maps (e.g. see Theorem 5.3A in [20]).

(b) Condition (3) of Theorem 2.6 holds if T satisfies condition $(3+)$, i.e.,

$(3+)$ $\| u_n \| + (u_n, v_n)/\| v_n \| \to \infty$ as $\| x_n \| \to \infty$ for each $u_n \in T(x_n)$, $v_n \in K(x_n)$, 1 and $(w, v) = \| w \| \cdot \| v \|$ for all $w \in G(x)$, $v \in K(x)$ and $x \in X$.

To show this, let $f \in T(x_k) + \mu_k G(x_k)$ with $\mu_k \to 0$. Then $f = u_k + \mu_k w_k$ for some $u_k \in T(x_k)$ and $w_k \in G(x_k)$ and for $v_k \in K(x_k)$,

$$\begin{aligned} \| u_k \| + \frac{(u_k, v_k)}{\| v_k \|} &= \| f - \mu_k w_k \| + \frac{(f - \mu_k w_k, v_k)}{\| v_k \|} \leqslant 2 \| f \| + \mu_k \| w_k \| - \mu_k \frac{(w_k, v_k)}{\| v_k \|} \\ &= 2 \| f \| \qquad \text{for each } k. \end{aligned}$$

Hence, by $(3+)$, $\{x_k\}$ is bounded.

(c) Analyzing the proof of Theorem 2.6, we see that instead of (1) and (2) it is enough to assume that for all $\mu \in (0, \mu_0)$ ($\mu_0 > 0$) and some $r_0 > 0$, $(u + \mu w, v) \geqslant 0$ for all $u \in T(x)$, $w \in G(x)$, $v \in K(x)$ and $\|x\| \geqslant r_0$ and $T + \mu G$ satisfies condition (+) or that T and G are odd on $X \backslash B(0, r_0)$ and $T + \mu G$ satisfies (t).

Remark 2.8. It is easy to see that condition (3+) implies condition (2+). However, if condition (2) of Theorem 2.6 holds, then conditions (2+) and (3+) are equivalent.

3. Surjectivity Results for Multivalued Mappings of Monotone and Ball-Contractive Type and Their Perturbations

In the first part of this section we introduce several new classes of multivalued A-proper mappings involving quasi-K-monotone, generalized pseudo-K-monotone and K-semibounded type of mappings. In the second part of this section we use the results of Section 2 in establishing various surjectivity type results for these classes of mappings and their perturbations. Ceratin surjectivity results for some perturbations of k-ball-contractive mappings are also proved. As special cases of our results, we deduce some results of Browder, Browder and Hess, Hess, Fitzpatrick, Petryshyn, Petryshyn and Fitzpatrick, Rockafellar, Wille, and others.

We begin this section by introducing some classes of multivalued mappings to be studied in the sequel.

Definition 3.1. (1) Let $K: X \to 2^{Y^*}$. Then a mapping $T: X \to 2^Y$ is said to be *quasi-K-monotone* if

(a) The set $T(x)$ is nonempty, bounded, closed and convex in Y for each $x \in X$;

(b) T is upper semicontinuous from each finite dimensional subspace F of X to the weak topology on Y;

(c) $x_n \rightharpoonup x$ in X implies that for each $u_n \in T(x_n)$ and $f_n \in K(x_n - x)$, $\limsup(u_n, f_n) \geqslant 0$.

(2) T is said to be of *type* (KS) if (a) and (b) of (1) hold and if $x_n \rightharpoonup x$ in X and $(u_n, f_n) \to 0$ for some $u_n \in T(x_n)$ and $f_n \in K(x_n - x)$ imply that $x_n \to x$ in X.

(3) T is said to be of *type* (KS_+) if (a) and (b) of (1) hold and if $x_n \rightharpoonup x$ in X and $\limsup(u_n, f_n) \leqslant 0$ for some $u_n \in T(x_n)$ and $f_n \in K(x_n - x)$ imply that $x_n \to x$ in X.

(4) T is said to be *pseudo-K-monotone* if conditions (a) and (b) of (1) hold and (d): if $x_n \rightharpoonup x$ in X and if $u_n \in T(x_n)$ and $f_n \in K(x_n - x)$ are such that

$\limsup(u_n, f_n) \leqslant 0$, then to each element $v \in X$ there exist $u(v) \in T(x)$, $g \in K(x - v)$, and $g_n \in K(x_n - v)$ such that

$$\liminf(u_n, g_n) \geqslant (u(v), g).$$

(5) T is said to be *generalized pseudo-K-monotone* if (a) and (b) of (1) hold and (e): if $x_n \rightharpoonup x$ in X and $u_n \in T(x_n)$, $f_n \in K(x_n - x)$ with $u_n \rightharpoonup u$ in Y and $\limsup(u_n, f_n) \leqslant 0$ imply that $u \in T(x)$ and $(u_n, f_n) \to 0$.

Let us add that the single-valued pseudomonotone mappings ($K = I$, $Y = X^*$) were introduced (in a somewhat different way) by Brézis [2] and that multivalued pseudomonotone and generalized pseudomonotone mappings were introduced by Browder and Hess [9]. Single-valued mappings of type (S) and (S_+) from X into X^* ($K = I$) were introduced and studied by Browder [5] and I-quasimonotone by Hess [13] and Calvert and Webb [10]. The single-valued mappings given by Definition 3.1 were studied by Petryshyn in a series of papers (see [20]).

To introduce our first class of multivalued A-proper mappings we need the following:

LEMMA 3.1. *Let X, Y be normed spaces, $T: X \to 2^Y$ a quasi-K-monotone mapping and $G: X \to 2^Y$ bounded and of type (KS_+). Then $T + \alpha G$ is of type (KS_+) for each $\alpha > 0$, except that in general $T(x) + \alpha G(x)$ may not be closed for all $x \in X$. In particular, if either T is pseudo-K-monotone with K being single-valued, and $K(0) = 0$ or T is a bounded generalized pseudo-K-monotone map with Y reflexive, then $T + \alpha G$ is of type (KS_+) for each $\alpha > 0$ except that in general $T(x) + \alpha G(x)$ may not be closed for each $x \in X$.*

Proof. Let $\alpha > 0$ be fixed and suppose that $x_n \rightharpoonup x_0$ in X and let $\limsup(u_n + \alpha v_n, f_n) \leqslant 0$ for some $u_n \in T(x_n)$, $v_n \in G(x_n)$, and $f_n \in K(x_n - x_0)$. Since T is quasi-K-monotone and G is bounded, it follows that $\limsup(v_n, f_n) \leqslant 0$. Hence, by the (KS_+) property of G, $x_n \to x$.

For the second part of the lemma we need only show that either a generalized pseudo-K-monotone or a pseudo-K-monotone mapping T is quasi-K-monotone. Suppose first that T is generalized pseudo-K-monotone. If T is not quasi-K-monotone, then for some $u_n \in T(x_n)$, $f_n \in K(x_n - x_0)$, where $x_n \rightharpoonup x_0$ in X, we have that $\limsup(u_n, f_n) < 0$. Since Y is reflexive and T is bounded, we may assume that $u_n \rightharpoonup u_0$ in Y. By the generalized pseudo-K-monotonicity of T, it follows that $\lim(u_n, f_n) = 0$, a contradiction. Thus T is quasi-K-monotone.

Assume now that T is pseudo-K-monotone. Again, supposing that it is not quasi-K-monotone, we obtain that $\limsup(u_n, K(x_n - x_0)) < 0$ for some $u_n \in T(x_n)$, with $x_n \rightharpoonup x_0$ in X. By the pseudo-K-monotonicity we have that for each $v \in X$ there exists $u(x) \in T(x_0)$ with

$$\liminf(u_n, K(x_n - v) \geqslant (u(v), K(x_0 - v)).$$

In particular, taking $v = x_0$ in the last inequality, we obtain

$$\liminf(u_n, K(x_n - x_0)) \geqslant 0 > \limsup(u_n, K(x_n - x_0)),$$

a contradiction. Thus, T is quasi-K-monotone. Q.E.D.

DEFINITION 3.2. We say that a mapping $T: D \subset X \to 2^Y$ is *K-quasi-bounded* if, for any bounded sequence $\{x_n\} \subset D$, $(y_n, f_n) \leqslant c \| x_n \|$ for each n and some $y_n \in T(x_n)$, $f_n \in K(x_n)$, $c > 0$, implies that $\{y_n\}$ is bounded in Y.

T is said to be *demiclosed* if $x_n \to x$ in X and $y_n \rightharpoonup y$ for some $y_n \in T(x_n)$, then $y \in T(x)$.

In view of Proposition 2.2 in [19] and Lemma 3.1 we have the following:

PROPOSITION 3.1. *Let X and Y be reflexive Banach spaces with an admissible scheme $\Gamma_a = \{X_n, V_n; Y_n, Q_n\}$ and $K: X \to Y^*$ a bounded mapping such that*

(a_1) *$K(x) = 0$ implies $x = 0$, K is α-positively homogeneous (i.e., $K(tx) = t^\alpha K(x)$ for each $t > 0$, x in X and some $\alpha > 0$), and the range of K is dense in Y^*.*

(a_2) *For each $x \in X_n$ and $g \in Y$, we have that $(Q_n(g), K(x)) = (g, K(x))$;*

(a_3) *K is weakly continuous at 0 and is uniformly continuous on closed balls in X.*

Let $T: X \to 2^Y$ be a K-quasi-bounded demiclosed and quasi-K-monotone mapping and $G: X \to 2^Y$ a bounded, demiclosed and of type (KS_+). Then $T + \alpha G$ is A-proper w.r.t. Γ_a for each $\alpha > 0$.

Remark 3.1. In the single-valued case Proposition 3.1 was established in Petryshyn [21], while the first part of Lemma 3.1 was proved by Hess [13] and Calvert and Webb [10] for $Y = X^*$, $K = I$ and $G = J$ a single-valued duality mapping and by Petryshyn [21] for general K and G. That a single-valued pseudomonotone mapping $T: X \to X^*$ is quasi-monotone ($K = I$) was proved by Fitzpatrick [12].

The following result will be needed for the next proposition and for establishing the surjectivity results of mappings involved. First we introduce the following:

DEFINITION 3.3. We say that a multivalued mapping $T: X \to 2^Y$ is *hemicontinuous* at $x \in X$, if $x \in X$, $t_n > 0$, $t_n \to 0$ and $y_n \in T(x + t_n v)$ imply that some subsequences $y_{n_k} \rightharpoonup y \in T(x)$. It is *demicontinuous* if $x_n \to x_0$ in X and $y_n \in T(x_n)$ imply that some subsequence $y_{n_k} \rightharpoonup y_0 \in T(x_0)$.

DEFINITION 3.4. Let X and Y be Banach spaces and $K: X \to 2^{Y^*}$. Then a mapping T from X into 2^Y is said to be *K-monotone*, if for all x, y in X there exists $f \in K(x - y)$ such that $(u - v, f) \geqslant 0$ for all $u \in T(x)$, $v \in T(y)$. T is *maximal K-monotone* if it is K-monotone and maximal in the sense of inclusions of graphs in the family of K-monotone mappings from X into 2^Y.

Monotone mappings ($K = I$, $Y = X^*$) were introduced independently by Vainberg, Kachuvorsky and Zarantonello and further studied by Minty, Browder, Kachuvorsky, Rockafellar, and others (see [5, 14] for references). The study of J-monotone mappings (J a duality mapping) was initiated by Browder, while those of K-monotone type were initiated by Kato and Petryshyn.

PROPOSITION 3.2. *Let X and Y be Banach spaces with an admissible scheme Γ_α and X reflexive. Let $K: X \to Y^*$ be weakly continuous, α-homogeneous with $\alpha \geq 1$ and $R(K) = Y^*$. Let D be an open convex subset of X and $T: \bar{D} \to 2^Y$ eihter hemicontinuous and K-monotone or pseudo-K-monotone. Then, if G is a bounded closed and convex subset of D, $T(G)$ is closed in Y.*

Proof. Let $u_k \in T(G)$ be such that $u_k \to u$ in Y as $k \to \infty$ and let $x_k \in G$ be such that $u_k \in T(x_k)$ for each k. By the reflexivity of X we may assume that $x_k \rightharpoonup x_0 \in G$.

Then, if T is K-monotone, $(u_k - y, K(x_k - v)) \geq 0$ for all $v \in D$, $y \in T(v)$ and all k. The passage to the limit in the latter inequality yields

$$(u, K(x_0 - v)) \geq (y, K(x_0 - v)) \qquad \text{for all } y \in T(v) \quad \text{and} \quad v \in D. \tag{3.1}$$

If T is pseudo-K-monotone, then

$$\limsup(u_k, K(x_k - x_0)) = \lim(u_k, K(x_k - x_0)) = 0$$

and consequently, for each $v \in D$ there exists $y(v) \in T(x_0)$ such that

$$\liminf(u_k, K(x_k - v)) \geq (y(v), K(x_0 - v)).$$

Passing to the limit, we obtain

$$(u, K(x_0 - v)) \geq (y(v), K(x_0 - v)) \qquad \text{for all } v \in D. \tag{3.2}$$

Now, each one of the inequalities (3.1), (3.2) implies that $u \in T(x_0)$. Consider (3.1) and suppose that $u \notin T(x_0)$. Since $T(x_0)$ is closed and convex and $R(K) = Y^*$, there exists $z_0 \in X$ such that

$$(u, K(z_0)) < \inf_{y \in T(x_0)} (y, K(z_0)). \tag{3.3}$$

Since D is open and $x_0 \in D$, for sufficiently small $t > 0$, we have that $v_t = x_0 - tz_0 \in D$. Choosing $x_0 - tz_0$ for v in (3.1), we obtain

$$(u, K(tz_0)) \geq (y(t), K(tz_0)), \qquad y(t) \in T(x_0 - tz_0)$$

or $(u, K(z_0)) \geq (y(t), K(z_0))$. Since T is hemicontinuous, passing to the limit when $t \to 0$, we get

$$(u, K(z_0)) \geq (y, K(z_0)) \qquad \text{for } y \in T(x_0),$$

in contradiction to (3.3).

Assuming now that the inequality (3.2) holds, we get in a similar fashion that for $t > 0$ small

$$(u, K(z_0)) \geqslant (z(t), K(z_0)) \qquad \text{with} \qquad z(t) \in T(x_0)$$

which contradicts (3.3). Q.E.D.

Now we are in a position to prove the following.

PROPOSITION 3.3. *Let X, Y, Γ_a, and K be as in Proposition 3.1 with K weakly continuous on X and $K(0) = 0$. If $T: X \to 2^Y$ is quasi-K-bounded, pseudo-K-monotone and either demiclosed or K is onto, and if $G: X \to 2^Y$ is a bounded demiclosed mapping of type (KS_+), then $T + \alpha G$ is A-proper w.r.t. Γ_a for each $\alpha > 0$.*

Proof. Let $\alpha > 0$ be fixed. Then by Lemma 3.1, $T + \alpha G$ is of type (KS_+). Since $T + \alpha G$ is K-quasi-bounded, by Proposition 2.2 in [19], it is sufficient to show that $T + \alpha G$ is demiclosed. So, let $x_n \to x$ in X and let $y_n = u_n + \alpha v_n \in T(x_n) + \alpha G(x_n)$ be such that $y_n \rightharpoonup y$ in Y. Since G is demiclosed and bounded, we have for some subsequence that $v_{n_j} \rightharpoonup v \in G(x)$, and consequently, $u_{n_j} \rightharpoonup y - \alpha v$. Hence, if T is demiclosed, we see that so is $T + \alpha G$. Next, let K be onto. By the continuity of K, $(u_{n_j}, K(x_{n_j} - x)) \to 0$. Now, continuing in the same fashion as in Proposition 3.2, we obtain that $y - \alpha v \in T(x)$, or $y \in T(x) + \alpha G(x)$. Q.E.D.

Remark 3.2. If K is linear, then it can be shown that a maximal K-monotone mapping T is pseudo-K-monotone and that, if K is also onto, pseudo-K-monotone mapping is generalized pseudo-K-monotone (see Browder and Hess [9] for the case $Y = X^*$ and $K = I$).

For K-quasi-bounded generalized pseudo-K-monotone mappings we have the following:

PROPOSITION 3.4. *Let X, Y, Γ_a and K be as in Proposition 3.1. If T is a K-quasibounded and generalized pseudo-K-monotone mapping of X into 2^Y and if G is a bounded demiclosed mapping of type (KS_+) from X into 2^Y, then $T_\mu = T + \mu G$ is A-proper w.r.t. Γ_a for each $\mu > 0$.*

Proof. Let $\mu > 0$ be fixed and let $\{x_{n_j} \mid x_{n_j} \in X_{n_j}\}$ be a bounded sequence such that for some $g \in Y$, $u_{n_j} \in T(x_{n_j})$, and $v_{n_j} \in G(x_{n_j})$, $Q_{n_j}(u_{n_j}) + \mu Q_{n_j}(v_{n_j}) - Q_{n_j}(g) \to 0$ as $j \to \infty$. Then, as in Proposition 2.2 of [19], we obtain that $(u_{n_j} + \mu v_{n_j}, K(x_{n_j} - x_0)) \to 0$ as $j \to \infty$ and $\{u_{n_j}\}$ is bounded, where $x_{n_j} \rightharpoonup x_0$. Now by the reflexivity of Y, we may assume that $u_{n_j} \rightharpoonup u_0$ and that $v_{n_j} \rightharpoonup v_0$ in Y. Moreover, we claim that $\limsup(\mu v_{n_j}, K(x_{n_j} - x_0)) \leqslant 0$. If not, then $\limsup(\mu v_{n_j}, K(x_{n_j} - x_0)) > 0$ and by passing to a subsequence, we may assume that $\lim(\mu v_{n_j}, K(x_{n_j} - x_0)) > 0$. Thus, $\limsup(u_{n_j}, K(x_{n_j} - x_0)) = \limsup(u_{n_j} + \mu v_{n_j}, K(x_{n_j} - x_0)) - \lim(\mu v_{n_j}, K(x_{n_j} - x_0)) < 0$. Since T is generalized pseudo-K-monotone, it follows that $u_0 \in T(x_0)$ and $(u_{n_j}, K(x_{n_j} - x_0)) \to 0$ as $j \to \infty$, in contradiction to $\lim(u_{n_j}, K(x_{n_j} - x_0)) < 0$.

Consequently, $\limsup(v_{n_j}, K(x_{n_j} - x_0)) \leqslant 0$ which, by the (KS_+) property of G, implies that $x_{n_j} \to x_0$.

Now, let $y \in X$ be arbitrary and let $y_n \in X_n$ be such that $y_n \to y$. Then, by (a_2), $(u_0 + \mu v_0 - g, Ky) = \lim(u_{n_j} + \mu v_{n_j} - g, K(y_{n_j})) = \lim(Q_{n_j}(u_{n_j}) + \mu Q_{n_j}(v_{n_j}) - Q_{n_j}(g), K(y_{n_j})) = (0, K(y)) = 0$. Hence, since $R(K)$ is dense in Y^* and $(u_0 + \mu v_0 - g, w) = 0$ for all $w \in R(K)$, it follows that $g = u_0 + \mu v_0$. Moreover, that $u_0 + \mu v_0 \in T(x_0) + \mu G(x_0)$ follows from the demiclosedness of G and the generalized pseudo-K-monotonicity of T since $(u_{n_j}, K(x_{n_j} - x_0)) \to 0$ as $j \to \infty$. Q.E.D.

In treating perturbation results, the following result will be useful.

LEMMA 3.2. *Let X and Y be reflexive Banach spaces, $K: X \to Y^*$ continuous and linear and T_1 and T_2 two generalized pseudo-K-monotone mappings from X into 2^Y. Suppose that T_1 is K-quasibounded and that there exists a continuous function $\psi: R^+ \to R$ such that for all $(x, y) \in G(T_2)$, $(y, K(x)) \geqslant -\psi(\|x\|)\|x\|$. Then $T_1 + T_2$ is generalized pseudo-K-monotone. In particular, if T_2 is maximal K-monotone with $0 \in T_2(0)$, then $T_1 + T_2$ is generalized pseudo-K-monotone.*

Proof. For $K = I$ and $Y = X^*$, Lemma 3.2 was proved by Browder and Hess [9]. Simple checking shows that the same arguments carry over to this more general setting. Q.E.D.

Finally, let us introduce another important class of multivalued mappings that contains K-monotone mappings as a proper subclass.

DEFINITION 3.5. Let $K: X \to Y^*$ and $T: X \to 2^Y$. Then T is called an operator with ***semibounded variation*** if $T(x)$ is nonempty, closed and convex for each x in X and for any x and y in X such that $\|x\| \leqslant R$, $\|y\| \leqslant R$ we have the inequality

$$(u - v, K(x - y)) \geqslant -c(R, \|x - y\|') \qquad \text{for all } u \in T(x), \quad v \in T(y),$$

where $\| \ \|'$ is a norm on X which is compact relative to the norm $\|\cdot\|$ and $c(R, \rho) \geqslant 0$ is a continuous function in R and ρ such that $c(R, t\rho)/t \to 0$ as $t \to 0$ for any fixed R and ρ.

Nonlinear operators $T: X \to X^*$ with semibounded variation have been studied by many authors (see, for example [11, 3, 25] and the literature cired there).

Using Lemma 1.1 of Fitzpatrick [12], we first prove the following result.

LEMMA 3.3. *Let $T: X \to 2^Y$ be an operator with semibounded variation and $K: X \to Y^*$ linear continuous and onto. Then*

(a) *T is locally bounded (i.e., if $x_n \to x$ in X and $u_n \in T(x_n)$, then $\{u_n\}$ is bounded in Y);*

(b) *if Y is reflexive and T is hemicontinuous, T is demicontinuous and strongly demiclosed (i.e., if $x_n \rightharpoonup x_0$ in X and $u_{n_k} \in T(x_{n_k})$ with $u_{n_k} \to f$ in Y, then $f \in T(x_0)$);*

(c) *T is K-quasi-bounded provided K is one-to-one.*

Proof. (a) Suppose that T is not locally bounded. Then there exists a sequence $\{x_n\} \subset X$ such that $x_n \to x_0$ and for some $u_n \in T(x_n)$, $\{u_n\}$ is unbounded. Since $y_n = K(x_n) \to K(x_0) = y_0$ and $\{u_n\} \subset Y \subsetneq Y^{**}$ with $\| u_n \| \to \infty$, by Lemma 1.1 in [12] we may choose a subsequence (if necessary) and $\tilde{z}_0$ in Y^* such that $\lim_n (u_n, y_n - y_0 - \tilde{z}_0) = -\infty$. Moreover, since $R(K) = Y^*$, there exists x_0 in X such that $K(\tilde{x}_0) = \tilde{z}_0$ and for each n and fixed $v_0 \in T(x_0 + \tilde{x}_0)$ we have

$$\begin{aligned}(u_n , K(x_n) &- K(x_0 + \tilde{x}_0)) \\ &\geqslant (v_0 , K(x_n) - K(x_0 + \tilde{x}_0)) - c(R, \| x_n - x_0 - \tilde{x}_0 \|'),\end{aligned} \tag{3.4}$$

where $R > 0$ is such that $\| x_0 + \tilde{x}_0 \| \leqslant R$ and $\| x_n \| \leqslant R$ for all n. Set $t_n = \| x_n - x_0 - \tilde{x}_0 \|$, $t_0 = \| -\tilde{x}_0 \|$ and note that $c(R, t_n) \to c(R, t_0)$ as $n \to \infty$. Hence, for a given $\epsilon > 0$, there exists $n_0 \geqslant 1$ such that $| c(R, t_n) - c(R, t_0)| < \epsilon$ for $n \geqslant n_0$. Consequently, $-c(R, t_n) > -c(R, t_0) - \epsilon$ for $n \geqslant n_0$, and therefore the right-hand side of (3.4) is bounded from below. This contradiction leads to the validity of (a).

(b) Let $x_n \to x_0$ in X and $u_n \in T(x_n)$. By (a), $\{u_n\}$ is bounded in Y and consequently there exists a subsequence $\{u_{n_j}\}$ such that $u_{n_j} \rightharpoonup u$ in Y. Let $r > 0$ be such that $\{x_n\} \subset B(x_0 , r) \subset X$ and $x \in B(x_0 , r)$. Then for each $v \in T(x)$ and some $R > 0$ we have

$$(u_{n_k} - v, K(x_{n_k} - x)) \geqslant -c(R, \| x_{n_k} - x \|').$$

Taking the limit in the above inequality as $n \to \infty$, we get $(u - v, K(x_0 - x)) \geqslant -c(R, \| x_0 - x \|')$ for each $v \in T(x)$, or

$$(u, K(x_0 - x)) \geqslant (v, K(x_0 - x)) - c(R, \| x_0 - x \|') \qquad \text{for all } v \in T(x),\ x \in B(x_0 , r). \tag{3.5}$$

This inequality implies that $u \in T(x_0)$. If not, then since $T(x_0)$ is closed and convex and $R(K) = Y^*$, there exists $z_0 \in X$ such that

$$(u, K(z_0)) < \inf_{y \in T(x_0)} (y, K(z_0)). \tag{3.6}$$

Let $t_n > 0$ be such that $t_n \to 0$ and $x_0 - t_n z_0 \in B(x_0 , r)$. Choosing $x_0 - t_n z_0$ as x in (3.5), we obtain $(u, K(t_n z_0)) \geqslant (v(t_n), K(t_n z_0)) - c(R, t_n \| z_0 \|')$, $v(t_n) \in$

$T(x_0 - t_n z_0)$, or $(u, K(z_0)) \geqslant (v(t_n), K(z_0)) - c(R, t_n \| z_0 \|')/t_n$. Since T is hemicontinuous, passing to the limit as $n \to \infty$ we obtain

$$(u, K(z_0)) \geqslant (v, K(z_0)) \qquad \text{for some } v \in T(x_0),$$

which contradicts (3.6). Thus, $u \in T(x_0)$.

The strong demiclosedness of T follows in the same fashion.

(c) Let $\{x_n\} \subset X$ be bounded and $(u_n, K(x_n)) \leqslant c_0 \| x_n \|$ for some $u_n \in T(x_n)$ and all n. Let $R > 0$ be such that $\{x_n\} \subset B(0, R)$ and observe that for each $x \in B(0, R)$ and $u \in T(x)$ we have

$$(u_n - u, K(x_n - x)) \geqslant -c(R, \| x_n - x \|').$$

Since T is locally bounded, there exists $\epsilon > 0$ such that $\| u \| \leqslant c_1$ for all $u \in T(x)$ with $\| x \| \leqslant \epsilon$. We may assume that $\epsilon < R$ and then for each $u \in T(x)$ with $\| x \| \leqslant \epsilon$ from the above inequality we obtain

$$|(u_n, K(x))| \leqslant |(u_n, K(x_n))| + |(u, K(x_n - x))| + c(R, \| x_n - x \|').$$

Now, since $(X, \| \cdot \|')$ is compactly embedded into $(X, \| \cdot \|)$, there exists $c_2 > 0$ such that $\| x \|' \leqslant c_2 \| x \|$ for all $x \in B(0, 2R)$ and consequently, $\| x_n - x \|' \leqslant c_2(R + \epsilon)$. By the continuity of $c(R, \rho)$ we have that $c(R, \| x_n - x \|') \leqslant c_3$ for all n and some $c_3 > 0$. Thus, for all $u \in T(x)$ with $\| x \| \leqslant \epsilon$, $|(u_n, K(x))| \leqslant c_0 + c_1 \| K(x_n - x) \| + c_3 \leqslant M$ for all n and some $M > 0$ and since K is injective and Y reflexive, we have for $0 < \epsilon_0 \leqslant \epsilon$ with $\epsilon_0 \| y \| \leqslant \epsilon$ for $y \in A = K^{-1}(B(0, 1))$ that

$$\epsilon_0 \| u_n \| = \epsilon_0 \sup_{y \in A} |(u_n, K(y))| = \sup_{y \in A} |(u_n, K(\epsilon_0 y))| \leqslant M,$$

i.e., $\| u_n \| \leqslant M/\epsilon_0$ for all n. Q.E.D.

The relationship of mappings with semibounded variation to pseudo K-monotone ones is given by the following:

PROPOSITION 3.5. *Let Y be reflexive, $K: X \to Y^*$ linear, continuous and surjective and $T: X \to K(Y)$ hemicontinuous with semibounded variation. Then T is pseudo-K-monotone.*

Proof. By Lemma 3.3(b), all we need show is part (d) of Definition 3.1. Suppose that $x_n \rightharpoonup x_0$ in X and $u_n \in T(x_n)$ with $\limsup(u_n, K(x_n - x_0)) \leqslant 0$. Then for a fixed $u_0 \in T(x_0)$ and some $R > 0$ we have $(u_0, K(x_n - x_0)) \leqslant (u_n, K(x_n - x_0)) + c(R, \| x_n - x_0 \|')$ and so $\liminf(u_n, K(x_n - x_0)) \geqslant 0$. Hence, $(u_n, K(x_n - x_0)) \to 0$ as $n \to \infty$. This implies that for each x in X, $\liminf(u_n, K(x_n - x)) = \lim(u_n, K(x_n - x_0)) + \liminf(u_n, K(x_0 - x)) = \liminf(u_n, K(x_0 - x))$. Now for $v \in T(x)$,

$$(v, K(x_n - x)) \leqslant (u_n, K(x_n - x)) + c(R, \| x_n - x \|')$$

and passing to the limit we obtain

$$(v, K(x_0 - x)) \leqslant \liminf(u_n, K(x_0 - x)) \qquad \text{for all } v \in T(x), \quad x \in X. \tag{3.7}$$

Let v_0 in X be fixed and define $x_t = x_0 + t(v_0 - x_0)$ for $t > 0$ with $t \to 0$. Then for any $v_t \in T(x_t)$ we get from (3.7) that

$$(v_t, K(x_0 - v_0)) \leqslant \liminf_n (u_n, K(x_0 - v_0)).$$

Since T is locally bounded and hemicontinuous, we may assume that $v_t \rightharpoonup v \in T(x_0)$ as $t \to 0$. Consequently, passing to the limit as $t \to 0$ in the last inequality, we get

$$(v, K(x_0 - v_0)) \leqslant \liminf_n (u_n, K(x_0 - v_0)) = \liminf_n (u_n, K(x_n - v_0)).$$

Q.E.D.

Let us now discuss some special cases of our results and their relationship to some results of other authors. Our first result in that direction is the following:

THEOREM 3.1. *Let X, Y, Γ_a and K be as in Proposition 3.1 with K linear and let, T: $[0, 1] \times X \to 2^Y$, A: $X \to BK(Y)$ and G: $X \to K(Y)$ be bounded, demi-closed and of type (KS_+). Suppose that*

(1) *$T_t = T(t, \cdot)$ is α-continuous in t uniformly for x in bounded subsets of X.*

(2) *For each f in Y there exists $r_f > 0$ such that $f \notin \overline{(T_t + A)(\partial B(0, r_f))}$ for all $t \in [0, 1]$.*

(3) *$sf \notin \overline{(T_0 + A)(\partial B(0, r_f))}$ for all $s \in [0, 1]$;*

(4) *There exists $n_f \geqslant 1$ such that for each $n \geqslant n_f$ and $\mu > 0$, $\deg(L_nQ_nT_0 + L_nQ_nA + \mu L_nQ_nG, B_n(0, r_f), 0) \neq 0$, where L_n is some linear isomorphism from Y_n onto X_n.*

Moreover, suppose that either T_t is quasi-K-monotone and K-quasi-bounded for each $t \in [0, 1]$ and A is maximal K-monotone, or T_t is generalized pseudo-K-monotone and K-quasi-bounded for each $t \in [0, 1]$ and A is generalized pseudo-K-monotone and K-quasi-bounded such that $(u, K(x)) \geqslant -\psi(\| x \|) \| x \|$ whenever $(x, u) \in G(A)$, where ψ: $R^+ \to R$ is a continuous function. Then the equation $f \in T_1(x) + A(x)$ is solvable for each f in Y provided T_1 satisfies condition $(++)$.

Proof. We first observe that for each $t \in [0, 1]$, $T_t + A$ is quasi-K-monotone in the first case, and generalized pseudo-K-monotone in the second case (Lemma 3.2, with possibly not closed images). Moreover, $T_t + A$ is K-quasi-bounded if A is maximal K-monotone. Next, suppose that A is generalized pseudo-K-monotone and that for some bounded $\{x_n\} \subset X$ we have that ($t \in [0, 1]$ fixed)

$(u_n + v_n, K(x_n)) \leqslant c \| x_n \|$ for some $u_n \in T_t(x_n)$, $v_n \in A(x_n)$ and some $c > 0$. Then

$$(u_n, K(x_n)) \leqslant c \| x_n \| + \psi(\| x_n \|) \| x_n \| \leqslant c_1 \| x_n \|$$

for some $c_1 > 0$. By the K-quasi-boundedness of T_t, $\{u_n\}$ is bounded. Since $(v_n, K(x_n)) \leqslant c \| x_n \| + |(u_n, K(x_n))| \leqslant c_2 \| x_n \|$, we have that $\{v_n\}$ is also bounded. Hence, $\{u_n + v_n\}$ is bounded, proving the K-quasi-boundedness of $T_t + A$. It remains to observe that since $T_t + A + \mu G$ is A-proper w.r.t. Γ_a for each $t \in [0, 1]$ and $\mu > 0$, the conclusion of our theorem follows from Theorem 2.3. Q.E.D.

Remark 3.3. If in Theorem 3.1 we assume that $(u, K(x)) \geqslant -c(t, \| x \|) \| x \|$ for all $u \in T_t(x)$, $t \in [0, 1]$, instead of K-quasi-boundedness of T_t, where $c: [0, 1] \times R^+ \to R$ is continuous, then one can easily see that the conclusion of Theorem 3.1 is still valid since $T_t + A$ is also K-quasi-bounded in this case.

Remark 3.4. When $Y = X^*$, $K = I$ and T_0 and A are odd on the boundary of a symmetric about 0 set D in X with A maximal monotone, T_1 pseudomonotone, T_t single-valued quasi-monotone for $t \in [0, 1]$, and $0 \notin \overline{(T_t + A)(\partial D)}$ instead of (2) and (3), the conclusion of Theorem 3.1 for $f = 0$ was proved by Hess [13].

As a consequence of our results of Section 2, we have the following surjectivity result for perturbations of quasi-K-monotone or generalized pseudo-K-monotone mappings.

THEOREM 3.2. *Let X, Y, Γ_a, and K be as in Proposition* 3.1, $T: X \to 2^Y$ *K-quasi-bounded and either demiclosed and quasi-K-monotone or generalized pseudo-K-monotone, and $G: X \to K(Y)$ bounded, demiclosed and of type* (KS_+). *Suppose that $C: X \to 2^Y$ is such that $T + C + \mu G$ is A-proper* w.r.t. Γ_a *for $\mu > 0$ and $Q_n C: X_n \to CK(Y_n)$ u.s.c. Moreover, suppose that either one of the following three conditions holds:*

(i) *$T + C$ satisfies condition $(+)$ and T, G, and C are odd on $X \backslash B(0, r)$ for some $r > 0$;*

(ii) *$T + C$ satisfies condition $(+)$, the mappings K, $K_n = Q^*{}_n K$ and M_n satisfy conditions* (a_1)–(a_3) *of Proposition* 3.1, *condition* (C2), *and $(u, K(x)) \geqslant 0$, $(v, Kx) \geqslant 0$, $(w, K(x)) \geqslant 0$ for all $u \in T(x)$, $v \in G(x)$, and $w \in C(x)$ with $\| x \| \geqslant r$;*

(iii) *K, K_n, and M_n are as in* (ii) *and for each f in Y there exists $r_f > 0$ such that $(u - f, K(x)) \geqslant 0$, $(v, K(x)) \geqslant 0$, and $(w, K(x)) \geqslant 0$ for all $u \in T(x)$, $v \in G(x)$, and $w \in C(x)$ with $\| x \| = r_f$.*

Then, if $T + C$ satisfies condition $(++)$, the equation $f \in T(x) + C(x)$ is solvable for each f in Y.

Remark 3.4. For $C = 0$ and T quasi-K-monotone, Theorem 3.2, parts (i) and (ii), was proved by Petryshyn [21] in the single-valued case and it extends some results of Browder [4], Calvert and Webb [10], Fitzpatrick [12], Rockafellar [24], and others (see [21] for more details) when $Y = X^*$, $K = I$ and T is quasi-monotone or maximal monotone. In the case when T is a maximal monotone mapping from X into 2^{X^*} and C is a single-valued completely continuous mapping, the conclusion of our theorem was proved by Fitzpatrick [12] under either (i) or (ii).

Remark 3.5. As C in our Theorem one can choose a completely continuous (multivalued) mapping or a K-quasi-bounded generalized pseudo-K-monotone mapping with K linear such that $(u, K(x)) \geqslant -\psi(\| x \|) \| x \|$ for all $u \in C(x)$ with $\psi: R^+ \to R$ a continuous function or a sum of two such mappings.

We say that $T: X \to 2^Y$ is ***completely continuous*** if $x_n \rightharpoonup x_0$ in X and $y_n \in T(x_n)$ imply that some subsequence $y_{n_k} \to y_0 \in T(x_0)$.

As an application of Theorem 2.6 and Remark 2.7(c) we have the following result:

THEOREM 3.3. *Let X be a reflexive separable Banach space, $T: X \to 2^{X^*}$ a quasi-bounded generalized pseudomonotone mapping and $C: X \to CK(X^*)$ completely continuous and such that*

$$\| u + v \| + \frac{(u + v, x)}{\| x \|} \to \infty \qquad \text{as} \qquad \| x \| \to \infty \tag{3^+}$$

for all $u \in T(x)$ and $v \in C(x)$ and condition (*) below holds for some ψ (which is so if, e.g., *for some $R > 0$, either T is bounded on $X \backslash B(0, R)$ or $(u, x) \geqslant -c \| x \|$ for $u \in T(x)$, $\| x \| \geqslant R$ and some c).*

Then the equation $f \in (T + C)(x)$ is solvable for each f in X^*.

Proof. Since for each f in X^*, $T - f$ and C satisfy the same conditions that T and C do, it is sufficient to show that $0 \in (T + C)(x)$ is solvable. Moreover, by the results of Asplund [1], we may assume that X and X^* are locally uniformly convex since the "monotonicity" and continuity properties of a mapping are not affected by renorming its domain and range space. Moreover, (3^+) holds in these renormed spaces.

Now by condition (3^+), if for some $r > 0$, $\| u + v \| + ((u + v, x)/\| x \|) \leqslant r$ for $u \in T(x)$ and $v \in C(x)$, then $\| x \| \leqslant R$ for some $R > 0$. Let $\psi: R^+ \to R^+$ be a continuous increasing function such that $\psi(t) = 0$ for $t \leqslant R$ and for $u \in T(x)$, $v \in C(x)$

$$\lim_{\|x\| \to \infty} \left(\psi(\| x \|) - \frac{|(u + v, x)|}{\| x \|} \right) = \infty. \tag{$*$}$$

Let J_0 be a duality mapping corresponding to this ψ. Then the mapping $T + C + J_0$ is coercise since for $u \in T(x)$, $v \in C(x)$ and $w \in J_0(x)$ we have that

$$\frac{(u + v + w, x)}{\| x \|} \geqslant \psi(\| x \|) - \frac{|(u + v, x)|}{\| x \|} \to \infty \qquad \text{as} \qquad \| x \| \to \infty.$$

Let J be the normalized duality mapping. Then, since $T + J_0$ is generalized pseudomonotone (see Lemma 3.2) and quasi-bounded and J is of type (S_+), the mapping $T + J_0 + \mu J$ is A-proper for each $\mu > 0$ w.r.t. the injective scheme $\Gamma_I = \{X_n, V_n; X^*_n, V^*_n\}$. Consequently, $T + J_0 + C + \mu J$ is A-proper w.r.t. Γ_I, $\mu > 0$.

Now, from the above discussion we see that the mappings $T + C + J_0$ and J satisfy all the assumptions of Theorem 2.6 (see Remark 2.7(b, c)). Hence, there exists $x_0 \in X$ such that $0 \in (T + C + J_0)(x_0)$. If $x_0 = 0$, then clearly $0 \in (T + C)(x_0)$. Suppose that $x_0 \neq 0$. Then $u + v + w = 0$ for some $u \in T(x_0)$, $v \in C(x_0)$ and $w \in J_0(x_0)$. This implies that $\psi(\| x_0 \|) = \| w \| = \| u + v \|$ and $\| x_0 \| \leqslant R$ since

$$0 = \frac{(u + v + w, x_0)}{\| x_0 \|} = \psi(\| x_0 \|) + \frac{(u + v, x_0)}{\| x_0 \|}.$$

By the definition of ψ, we have $\| u + v \| = \psi(\| x_0 \|) = 0$, i.e., $0 \in (T + C)(x_0)$.
Q.E.D.

Remark 3.6. If T is of semibounded variation in Theorem 3.3, then its conclusion is valid if T is bounded, since it is bounded and generalized pseudomonotone mapping (see Lemma 3.3 and Proposition 3.5). Moreover, if T is maximal monotone defined on all of X, then our condition (3^+) is equivalent to

$$\| u + v \| + \frac{(v, x)}{\| x \|} \to \infty \qquad \text{as} \qquad \| x \| \to \infty \quad \text{for all } u \in T(x), \quad v \in C(x).$$

Thus, our Theorem 3.3 includes a result of Wille [27] with T maximal monotone. For $C = 0$ and T single-valued bounded and generalized-pseudomonotone with $(Tx, x) \geqslant -k \| x \|$, see Browder [8].

In the next two theorems we extend the corresponding results of [19] to the case of noninjective and nonlinear perturbations of condensing like mappings as defined below.

For a bounded subset A of a Banach space X we define the ball-measure of noncompactness as $\chi(A) = \inf\{r > 0 \mid A$ can be covered by a finite number of balls of radius less than r with centers in $X\}$. A mapping $T\colon D \subset X \to CK(Y)$ is said to be ***k-ball-contractive*** if for each bounded subset $Q \subset D$, $\chi(T(Q)) \leqslant k\chi(Q)$; it is ***ball-condensing*** if for each $Q \subset D$ with $\chi(Q) \neq 0$, $\chi(T(Q)) < \chi(Q)$.

THEOREM 3.4. *Let X and Y be Banach spaces, $\Gamma_b = \{X_n, V_n; X_n, P_n\}$ a projectionally complete scheme for (X, X) with $\| P_n \| = 1$ and M a continuous linear mapping of X onto Y which satisfies the following condition:*

(1) *there exists a constant $c > 0$ such that $\chi(M(Q)) \geqslant c\chi(Q)$ for any bounded set $Q \subset X$.*

Suppose that $T: X \to CK(Y)$ *is bounded u.s.c. and such that*

(2) $\chi(T(Q)) \leqslant c\chi(Q)$ *for each bounded* $Q \subset X$.

Moreover, suppose that there exists $r > 0$ *such that* T *is odd on* $X \backslash B(0, r)$ *and that* $(T - M)^{-1}$ *maps relatively compact sets of* Y *into bounded sets of* X.

Then, if $T - M$ *satisfies condition* $(++)$, $T - M$ *maps* X *onto* Y.

Proof. Let $f_0 \in Y$ be fixed and define

$$\begin{aligned} F(t, y) &= M^{-1}(T(y) - tf_0) \\ &= \{x \in X \mid M(x) \in T(y) - tf_0\} \qquad \text{for all } (t, y) \in [0, 1] \times X. \end{aligned}$$

It is clear that the equation $f_0 \in T(y) - M(y)$ is equivalent to the equation $y \in F(1, y)$.

We shall prove now that $F(t, y)$ satisfies all the assumptions of Theorem 2.1. First we prove that $F(t, y)$ is u.s.c. on $[0, 1] \times X$. If F were not u.s.c. at some point (t_0, y_0), then there would exist a neighborhood U of $F(t_0, y_0)$ and sequences $y_n \to y_0$, $t_n \to t_0$ and $x_n \in F(t_n, y_n) \backslash U$ for all n. Then we would have $M(x_n) = u_n - t_n f_0$ for some $u_n \in T(y_n)$. By the u.s.c. of T we may assume that $u_n \to u_0 \in T(y_0)$, which implies that $\chi(\{M(x_n)\}) = 0$. By (1), $\chi(\{x_n\}) = 0$ and consequently, $x_{n_k} \to x_0$ for some subsequence $\{x_{n_k}\}$ with $x_0 \notin U$. Since M is continuous, we have that $M(x_0) \in T(y_0) - t_0 f_0$, i.e., $x_0 \in F(t_0, y_0) \subset U$, a contradiction. Using similar arguments, we obtain that M^{-1} is also u.s.c. on Y. Since M is continuous and linear with $R(M) = Y$, the Open Mapping Theorem implies that M maps open sets of X into open sets of Y and consequently, M^{-1} is lower semicontinuous.

Next we prove that for each $t \in [0, 1]$, $F_t = F(t, \cdot)$ is 1-ball-contractive. Indeed, for each bounded subset $Q \subset X$ we have

$$M(F([0, 1] \times Q)) \subset \{T(y) - tf_0 \mid y \in Q, t \in [0, 1]\}$$

and consequently, for each $t \in [0, 1]$, $\chi(M(F_t(Q))) \leqslant \chi(T(Q)) \leqslant c\chi(Q)$.

Assumption (1) implies that $F_t(Q)$ is bounded and $c\chi(F_t(Q)) \leqslant \chi(M(F_t(Q))) \leqslant c\chi(Q)$, i.e., $\chi(F_t(Q)) \leqslant \chi(Q)$ for each $t \in [0, 1]$. Then, as shown in [17], we have that $\mu I + I - F_t$ is A-proper with respect to Γ_b for each $t \in [0, 1]$ and $\mu > 0$.

Now we claim that $I - F_t$ satisfies condition $(+)$ uniformly for $t \in [0, 1]$. Let $g_n \equiv x_n - u_n \to g$ for some $u_n \in F(t_n, x_n)$. Then $M(u_n) = v_n - t_n f_0$ for some $v_n \in T(x_n)$ and $M(x_n) - v_n = M(g_n) + M(u_n) - v_n = M(g_n) - t_n f_0$, or $x_n \in (M - T)^{-1}(M(g_n) - t_n f_0)$. By the relative compactness of $\{M(g_n) - t_n f_0\}$ we obtain that $\{x_n\}$ is bounded. This implies that there exist $r_0 \geqslant r$ and $\gamma > 0$ such that $\| y - u \| \geqslant \gamma$ for all $u \in F_t(y)$ with $\| y \| = r_0$ and $t \in [0, 1]$. If not, then there would exist $\{t_n\} \subset [0, 1]$ and $\{y_n\} \subset X$ with $t_n \to t$ and $\| y_n \| \to \infty$ and for some $u_n \in F_{t_n}(y_n)$, $\| y_n - u_n \| \to 0$, in contradiction to condition $(+)$ for $I - F_t$. Thus such $r_0 > 0$ and $\gamma > 0$ exist.

Set $H_t = I - F_t$ for each $t \in [0, 1]$. Since T and M are odd on $X \setminus B(0, r_0)$, we have that $H_0(-y) = -H_0(y)$ for all $y \notin B(0, r_0)$. From this and our discussion above it follows that H_t satisfies all the hypotheses of Theorem 2.1 with $G = I$ and $f = 0$ except (H3) and condition $(++)$. The rest of the proof will be devoted to establishing the validity of (H3) for H_t. Thus, let for some $\mu > 0$, $t_n \in [0, 1]$, and $x_n \in \partial B_n(0, r_0)$, we have $0 \in P_n H(t_n, x_n) + \mu x_n = (\mu + 1) x_n - P_n M^{-1}(T(x_n) - t_n f_0)$ or $(\mu + 1) x_n = P_n(y_n)$ for some $y_n \in M^{-1}(T(x_n) - t_n f_0)$. Then $M(y_n) \in T(x_n) - t_n f_0$ and consequently, $c\chi(\{y_n\}) \leqslant \chi(\{M(y_n)\}) \leqslant \chi(\{T(x_n)\}) \leqslant c\chi(\{x_n\})$. Hence,

$$\chi(\{(\mu + 1) x_n\}) = \chi(\{P_n(y_n)\}) \leqslant \chi(\{y_n\}) \leqslant \chi(\{x_n\}),$$

a contradiction unless $\{x_n\}$ is precompact. It follows that we can choose subsequences $\{t_{n_k}\}$ and $\{x_{n_k}\}$ such that $t_{n_k} \to t_0$ and $x_{n_k} \to x_0$. Let $u_{n_k} \in T(x_{n_k}) - t_{n_k} f_0$ be such that $y_{n_k} \in M^{-1}(u_{n_k})$. By the u.s.c. of T, $u_m \to u_0 \in T(x_0) - t_0 f_0$ for some subsequence $\{u_m\} \subset \{u_{n_k}\}$. By the u.s.c. of M^{-1}, $y_{m_i} \to y_0 \in M^{-1}(u_0)$ for some subsequence $\{y_{m_i}\} \subset \{y_m\}$. Then, $P_{m_i}(y_{m_i}) \to y_0 = (1 + \mu) x_0$. Since M^{-1} is lower semicontinuous and $u_m = v_m - t_m f_0$ for some $v_m \in T(x_m)$ with $v_m \to v_0 \in T(x_0)$, we have that for $y_0 \in M^{-1}(u_0)$ there exists $z_{m_i} \in M^{-1}(T(x_{m_i}) - t_0 f_0)$ such that $z_{m_i} \to y_0$. Whence, $(1 + \mu) x_{m_i} - P_{m_i}(z_{m_i}) \to 0$ as $i \to \infty$, which shows that (H3) holds. Now, as in Theorem 2.1, we obtain an $\mu_0 > 0$ such that for each $\mu \in (0, \mu_0)$ there exists $y_\mu \in X$ with $0 \in \mu y_\mu + H_1(y_\mu) = (\mu + 1) y_\mu - M^{-1}(T(y_\mu) - f_0)$, or $f_0 \in (\mu + 1) M(y_\mu) - T(y_\mu)$. Let $\mu_k \in (0, \mu_0)$ with $\mu_k \to 0$. Then, by condition $(++)$ for $M - T$, there exists $y \in X$ such that $f_0 \in M(y) - T(y)$. Since $f_0 \in Y$ was arbitrary, we have that $M - T$ maps X onto Y. Q.E.D.

Remark 3.7. It is easy to check that if $K: X \to Y^*$ with $\| Kx \| \to \infty$ as $\| x \| \to \infty$ and if

$$(Tx, Kx) \geqslant c_1 \| Kx \|^2, \qquad (Mx, Kx) \leqslant c_2 \| Kx \|^2 + \phi(\| Kx \|)$$

with $c_1 - c_2 > 0$ and $\phi(r)/r \to 0$ as $r \to \infty$, then $T - M$ is K-coercive and, in particular, satisfies condition $(+)$.

Next we shall consider the case of nonlinear M. For such M we have the following:

THEOREM 3.5. *Let X and Y be Banach spaces, $\Gamma_b = \{Y_n, V_n; Y_n, Q_n\}$ a projectionally complete scheme for (Y, Y) with $\| Q_n \| = 1$, M a mapping of $D \subset X$ onto Y with M^{-1} u.s.c. and bounded, and $T: D \to CK(Y)$ u.s.c. and bounded. Suppose that $M - T$ satisfies conditions $(+)$ and $(++)$ and that TM^{-1}: $Y \to CK(Y)$ is 1-ball contractive.*

Moreover, suppose that either one of the following conditions holds:

(i) *M and T are odd on the symmetric* w.r.t. 0 *set D.*

(ii) *There exists* $R > 0$ *such that whenever* $0 \in tT(x) - M(x)$ *holds for any* $x \in D$ *and* $t \in (0, 1]$, *then* $\| x \| \leqslant R$.

Then $T - M$ *maps* D *onto* Y.

Proof. Let $f_0 \in Y$ be fixed and consider the equation $f_0 \in M(x) - T(x)$. Set $y = M(x)$. Then it is easy to see that the last equation is equivalent to the equation $f_0 \in y - TM^{-1}(y)$.

Let us now prove that $I - TM^{-1}$ satisfies conditions $(+)$ and $(++)$. Let $\{y_n\} \subset Y$ be such that $y_n - u_n \to g$ for some $u_n \in TM^{-1}(y_n)$. Then $u_n \in T(x_n)$ with $x_n \in M^{-1}(y_n)$ and consequently, $M(x_n) - u_n = y_n - u_n \to g$. Since $M - T$ satisfies condition $(+)$, $\{x_n\}$ is bounded and by the boundedness of T, $\| y_n \| = \| M(x_n) \| \leqslant \| M(x_n) - u_n \| + \| u_n \| \leqslant K$ for all n and some $K > 0$. Next, let $\{y_n\}$ be bounded and such that $y_n - u_n \to g$ for some $u_n \in TM^{-1}(y_n)$. Then for some $x_n \in M^{-1}(y_n)$, $u_n \in T(x_n)$ and $M(x_n) - u_n = y_n - u_n \to g$. Since $M - T$ satisfies condition $(++)$, there exists $x \in X$ such that $g \in M(x) - T(x)$ or $g \in y - TM^{-1}(y)$ with $y = M(x)$. Hence $I - TM^{-1}$ satisfies conditions $(+)$ and $(++)$.

Now we claim that there exist $\gamma > 0$ and $r_{f_0} > c$ ($c > 0$ given) such that for all $t \in [-1, 1]$ and $y \in \partial B(0, r_{f_0})$

$$\| y - u - tf_0 \| \geqslant \gamma \qquad \text{for all } u \in TM^{-1}(y). \tag{3.8}$$

If not, then there exist $\{t_n\} \subset [-1, 1]$ and $\{y_n\} \subset Y$ with $t_n \to t_0$ and $\| y_n \| \to \infty$ and $\| y_n - u_n - t_n f_0 \| \to 0$ for some $u_n \in TM^{-1}(y_n)$. Thus, $y_n - u_n \to t_0 f_0$ with $\{y_n\}$ unbounded, in contradiction to condition $(+)$ for $I - TM^{-1}$.

Case (1). Assume that T and M are odd. We claim that $H_t = I - TM^{-1} - tf_0$ satisfies hypothesis (H3) of Theorem 2.1 for $f = 0$ with $G = I$ i.e., if for some $\mu > 0$, $t_n \in [0, 1]$, and $y_n \in \partial B_n(0, r_{f_0})$ we have that

$$0 \in (1 + \mu) y_n - Q_n TM^{-1}(y_n) - t_n Q_n(f_0), \tag{3.9}$$

then there exists $\{t_{n_k}\}$ such that $t_{n_k} \to t_0$ and $t_0 Q_{n_k}(f_0) - (1 + \mu) y_{n_k} + Q_{n_k}(v_{n_k}) \to 0$ for some $v_{n_k} \in TM^{-1}(y_{n_k})$. Indeed, from (3.9) we get that for some $v_{n_k} \in TM^{-1}(y_{n_k})$, $0 = (1 + \mu) y_{n_k} - Q_{n_k}(v_{n_k}) - t_{n_k} Q_{n_k}(f_0)$, which implies that

$$t_0 Q_{n_k}(f_0) - (1 + \mu) y_{n_k} + Q_{n_k}(v_{n_k}) = (t_0 - t_{n_k}) Q_{n_k}(f_0) \to 0 \qquad \text{as} \qquad k \to \infty.$$

Now since T and M are odd, we have that $H_0 + \mu I = (1 + \mu) I - TM^{-1}$ is odd and hence hypothesis (H4) of Theorem 2.1 holds. Consequently, by Theorem 2.1 there exists $y \in Y$ such that $0 \in H_1(y) = y - TM^{-1}(y) - f_0$.

Case (2). Suppose that condition (ii) holds. We claim that there exists a constant C such that $\lambda y \notin TM^{-1}(y)$ for $\| y \| \geqslant C$ and $\lambda \geqslant 1$. Indeed, if such a C did not exist, then we could find sequences $\{y_n\} \subset Y$ and $\{\lambda_n \mid \lambda_n \geqslant 1\}$ with

$\|y_n\| \to \infty$ and $\lambda_n y_n \in TM^{-1}(y_n)$. Then $\lambda_n y_n = v_n$ for some $v_n \in T(u_n)$ and $u_n \in M^{-1}(y_n)$. Consequently, $y_n - (1/\lambda_n) v_n = M(u_n) - (1/\lambda_n) v_n = 0$, which by (ii) implies that $\{u_n\}$ is bounded. Since T is bounded and $\{1/\lambda_n\} \subset (0, 1]$, we have that the sequence $\{y_n\} = \{(1/\lambda_n) v_n\}$ is bounded, in contradiction to our assumption on $\{y_n\}$. It is clear now that for each $\mu \geqslant 0$, $\lambda(1 + \mu) y \notin TM^{-1}(y)$ for all $\|y\| \geqslant C$ and $\lambda \geqslant 1$. It follows from (3.8) that there exists $\mu_0 \in (0, 1)$ such that $\|(1 + \mu) y - u - tf_0\| \geqslant \gamma/2$ for $t \in [-1, 1]$, $\mu \in (0, \mu_0)$, $u \in TM^{-1}(y)$ with $\|y\| = r_{f_0}$. In view of this, a slight modification of Proposition 1.4 in [19] applied to TM^{-1} and $(1 + \mu) I$, $\mu \in (0, \mu_0)$, implies the existence of $y \in B(0, r_{f_0})$ such that $f_0 \in (1 + \mu) y_\mu - TM^{-1}(y_\mu)$ with $\mu \in (0, \mu_0)$. Let $\mu_k \in (0, \mu_0)$ be such that $\mu_k \to 0$ and $y_k \in B(0, r_{f_0})$ with $f_0 \in (1 + \mu_k) y_k - TM^{-1}(y_k)$. Let $u_k \in TM^{-1}(y_k)$ be such that $f_0 = (1 + \mu_k) y_k - u_k$. Then $y_k - u_k = -\mu_k y_k + f_0 \to f_0$ as $k \to \infty$ and consequently, since $I - TM^{-1}$ satisfies condition $(++)$, there exists $y \in Y$ such that $f_0 \in y - TM^{-1}(y)$, or $f_0 \in M(x) - T(x)$ for some $x \in M^{-1}(y)$. Since f_0 was arbitraty, $M - T$ maps D onto Y. Q.E.D.

When M is bijective and T single-valued, Theorem 3.5 was obtained by Petryshyn and Fitzpatrick [22]. We note that conditions (1) and (2) of Theorem 3.4 imply that M^{-1} is u.s.c. and bounded and TM^{-1} is u.s.c. and 1-ball-contractive. If T is a convex mapping (i.e., X is partially ordered by "$\leqslant$" and $T(\alpha x + (1 - \alpha) y) \leqslant \alpha T(x) + (1 - \alpha) Ty$ for all $x, y \in X$ and $\alpha \in (0, 1)$) and M is either linear or K-monotone with K linear, then $TM^{-1}(y)$ is convex for each y in Y. For M and T single-valued with $M^{-1}(y)$ convex for each y in Y and $M^{-1}T$ ball-condensing, the ontoness of $M - T$ was established by Webb [26] under either (i) or (ii).

4. Elliptic Boundary Value Problems Involving Semibounded Operators with Completely Continuous Perturbations

In the first part of this section we use Theorems 2.4 and 3.3 to establish the existence of weak solutions for boundary value problems for quasilinear elliptic operators in divergence form which involve completely continuous perturbations of operators with semibounded variation acting in $W_p{}^m(Q)$. The latter class of elliptic operator equations was first studied by Browder [3] and Dubinsky [11] and later by Pohodjayev [23], Skrypnik [25], and others (see [5, 25]). For the above-mentioned class of elliptic operator equations our existence results extend those of [3, 11].

In the second part of this section we apply Theorems 2.4 and 2.6 to establish the existence of strong solutions (i.e., functions lying in a closed subspace V of $W_2{}^m$) for nonlinear ordinary differential equations of order m involving completely continuous perturbations $B\colon V \to L_2$ of operators $A\colon V \to L_2$ with semibounded variation. Rather than obtaining the most general results, our aim is to show how one can choose the operators K, K_n, G, M_n and a suitable admissible scheme so that Theorems 2.4 and 2.6 are applicable.

Let Q be a bounded domain in R^n with a sufficiently smooth boundary ∂Q so that the Sobolev Imbedding Theorem holds on Q (see [7, 16]). For fixed $p \in (1, \infty)$, let $L_p \equiv L_p(Q)$ denote the real Banach space of functions $u(x)$ on Q with norm $\| u \|_p$. If $\alpha = (\alpha_1, ..., \alpha_n)$ is a multiindex of nonnegative integers, we denote by $D^\alpha = \partial^{\alpha_1}/\partial x_n^{\alpha_1} \cdots \partial^{\alpha_n}/\partial x_n^{\alpha_n}$ a differential operator of order $|\alpha| = \alpha_1 + \cdots + \alpha_n$. If m is a nonnegative integer, $W_p^m \equiv W_p^m(Q)$ denotes the real Sobolev space of all $u \in L_p$ whose generalized derivatives $D^\alpha u$, $|\alpha| \leqslant m$, also lie in L_p. W_p^m is a separable uniformly convex Banach space with respect to the norm $\| u \|_{m,p} = (\sum_{|\alpha| \leqslant m} \| D^\alpha u \|_p^p)^{1/p}$. In case $p = 2$, we get the Hilbert space W_2^m with the corresponding inner product $(\cdot, \cdot)_m$. Let $C_c^\infty(Q)$ be the family of infinitely differentiable functions with compact support in Q considered as a subset of W_p^m and let $\mathring{W}_p^m$ be the completion in W_p^m of $C_c^\infty(Q)$. Let $\langle u, v \rangle = \int_Q uv\, dx$ denote the natural pairing between $u \in L_p$ and $v \in L_q$ with $q = p(p-1)^{-1}$ and let R^{S_m} be the vector space whose elements are $\xi = \{\xi_\alpha \mid |\alpha| \leqslant m\}$.

Let V be a closed subspace of W_p^m such that $\mathring{W}_p^m \subseteq V \subseteq W_p^m$. For a given $f \in L_q$, the *BV problem corresponding to* V is the problem of finding a generalized solution $u \in V$ of the equation

$$\sum_{|\alpha| \leqslant m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, u, ..., D^m u) + \sum_{|\beta| \leqslant m-1} (-1)^{|\beta|} D^\beta B_\beta(x, u, ..., D^{m-1}u) = f(x) \tag{4.1}$$

such that for all v in V the following identity holds:

$$\sum_{|\alpha| \leqslant m} \langle A_\alpha(x, u, ..., D^m u), D^\alpha v \rangle + \sum_{|\beta| \leqslant m-1} \langle B_p(x, u, ..., D^{m-1}u), D^\beta v \rangle = \langle f, v \rangle. \tag{4.2}$$

The choices $V = \mathring{W}_p^m$ and $V = W_p^m$ lead to the generalized Dirichlet and Neumann BV problems for (4.1), respectively.

To formulate the BVP with respect to V as an equivalent operator equation involving mappings from V to V^* to which Theorems 2.4 and 3.3 are applicable, we impose the following conditions on the nonlinear functions $A_\alpha(x, \xi)$: $Q \times R^{S_m} \to R^1$ and $B_\beta(x, \eta): Q \times R^{S_{m-1}} \to R^1$.

(A1) *For each* $|\alpha| \leqslant m$, $A_\alpha(x, \xi)$ *satisfies the Carathéodory conditions, i.e., it is measurable in* $x \in Q$ *for fixed* $\xi \in R^{S_m}$ *and continuous in* $\xi \in R^{S_m}$ *for almost all* $x \in Q$, *and there exists a constant* $c_0 > 0$ *such that*

$$|A_\alpha(x, \xi)| \leqslant c_0 \left\{1 + \sum_{|\gamma| \leqslant m} |\xi_\alpha|^{p_{\gamma\alpha}}\right\}$$

with $p_{\gamma\alpha} \leqslant 1$ *for* $|\alpha| = |\gamma| = m$, *while in the lower order cases the exponents may have larger upper bounds* (see [8]).

(A2) *For any given ball* $B(0, R) \subset V$ *and* $u, v \in B(0, R)$ *we have*

$$\sum_{|\alpha| \leqslant m} \langle A_\alpha(x, u,..., D^m u) - A_\alpha(x, v,..., D^m v), D^\alpha(u - v)\rangle \geqslant -c(R, \| u - v \|_{m-1,p}),$$

where $c(R,\rho) \geqslant 0$ *is continuous in* R *and* ρ *and such that* $c(R, t\rho)/t \to 0$ *as* $t \to 0^+$ *for any fixed* R *and* ρ.

(B1) *For each* $|\beta| \leqslant m - 1$ *and each* u *in* W_p^{m-1} *the mapping* $u \to C_\beta(u) \equiv B_\beta(x, u, Du,..., D^{m-1}u)$ *yields a continuous and bounded mapping of* W_p^{m-1} *into* L_q, $q = p(p-1)^{-1}$.

Let us add in passing that it is not hard to show that condition (B1) holds if, for example, $B_\alpha: R^{S_{m-1}} \to R^1$ satisfies the Carathéodory conditions and there exist constants $b_\alpha \geqslant 0$ and a function $b(x) \in L_q$ such that for each $|\beta| \leqslant m - 1$

$$| B_\beta(x, \eta)| \leqslant \sum_{|\alpha| \leqslant m-1} b_\alpha | \eta_\alpha |^{p-1} + b(x) \qquad (\eta \in R^{S_{m-1}}, x \in \underset{\sim}{Q} \text{ (a.e.)}). \tag{4.3}$$

Now it follows from assumptions (A1) and (B1) and the standard results on Nemytsky operators (e.g., [16]) that the generalized forms

$$a(u, v) = \sum_{|\alpha| \leqslant m} \langle A_\alpha(x, u,..., D^m u), D^\alpha v\rangle \qquad (u, v \in W_p{}^m), \tag{4.4}$$

$$b(u, v) = \sum_{|\beta| \leqslant m-1} \langle B_\beta(x, u,..., D^{m-1}u), D^\alpha v\rangle \qquad (u, v \in W_p{}^m) \tag{4.5}$$

are well defined on $W_p{}^m$ and that for a given closed subspace V of $W_p{}^m$ with $\mathring{W}_p{}^m \subseteq V$ one can associate with $a(u, v)$ and $b(u, v)$ in a unique way bounded continuous mappings A and B of V into V^* such that

$$a(u, v) = (Au, v), \qquad b(u, v) = (Bu, v) \qquad (u, v \in V), \tag{4.6}$$

where (Au, v) and (Bu, v) denote the values of the functionals Au and Bu in V^* at v in V. Similarly, for each f in L_q there exists a unique w_f in V^* such that $\langle f, v\rangle = (w_f, v)$ for all v in V. Consequently, in view of (4.6), Eq. (4.2) is equivalent to the operator equation

$$Tu \equiv Au + Bu = w_f \qquad (u \in V) \tag{4.7}$$

for a given $w_f \in V^*$ and the mapping $T = A + B: V \to V^*$.

In order to apply Theorems 2.4 and 3.3 to Eq. (4.7) or Eq. (4.2) we must first select our admissible approximation shceme Γ for the pair (V, V^*). Since V is a separable uniformly convex Banach space, we may find a sequence of finite-dimensional subspaces $\{X_n\} \subset V$ such that $\operatorname{dist}(u, X_n) = \inf_{x \in X_n} \| u - x \| \to 0$ for each u in V as $n \to \infty$ and so the scheme $\Gamma = \{X_n, V_n; X^*_n, V^*_n\}$ is admissible for (V, V^*), where $V_n: X_n \to V$ is the inclusion map and $V^*_n: V^* \to X^*_n$.

We are now in a position to apply our theorems to obtain the existence of solutions of elliptic boundary value problems of form (4.1) involving operators with semibounded variation.

THEOREM 4.1. *Suppose $Q \subset R^n$ is a bounded domain for which the Sobolev Imbedding Theorem is valid, V is a closed subspace of W_p^m with $\mathring{W}_p^m \subseteq V$ for which the hypothesis* (A2) *holds, and assumptions* (A1) *and* (B1) *are satisfied. Then Eq.* (4.7) *is solvable for each w_f in V^* provided that any one of the following conditions holds:*

(D1) *A and C are odd on $V \backslash B(0, r)$ for some $r > 0$ and $T = A + B$ satisfies condition $(+)$ on V;*

(D2) *$a(u, u) + b(u, u) \geqslant 0$ on $V \backslash B(0, r)$ for some $r > 0$ and $T = A + B$ satisfies condition $(+)$ on V;*

(D3) *$\| Tu \| + \{a(u, u) + b(u, u)\}/\| u \|_{m,p} \to \infty$ as $\| u \|_{m,p} \to \infty$ for u in V.*

Proof. First, it follows from (A1) and (A2) that A: $V \to V^*$ is a bounded continuous mapping with semibounded variation. Hence, by Propositions 3.4 and 3.5 and Lemma 3.3 with $K = I$ and $G = J$, A is strongly demiclosed and $A + \mu J$ is A-proper with respect to Γ_I for each $\mu > 0$. Now it follows from assumption (B1) that B: $V \to V^*$ is completely continuous. Indeed, if $\{u_n\}$ is a sequence in $V (\subset W_p^m)$ such that $u_n \rightharpoonup u$ in V, then by the Sobolev Imbedding Theorem, $u_n \to u$ in W_p^{m-1} and thus, by (B1), $C_\beta(u_n) \to C_\beta(u)$ in L_q for each $|\beta| \leqslant m - 1$. Since

$$(Bu_n - Bu, v) = \sum_{|\beta| \leqslant m-1} \langle B_\beta(x, u_n, \ldots, D^{m-1}u_n) - B_\beta(x, u, \ldots, D^{m-1}u), D^\beta v \rangle$$

for each v in V, it follows by Hölder's inequality that

$$\| Bu_n - Bu \| = \sup\{|(Bu_n - Bu, v)| \mid \| v \|_{W_p^m} = 1\}$$
$$\leqslant \sum_{|\beta| \leqslant m-1} \left(\int | C_\beta(u_n) - C_\beta(u)|^q \, dx \right)^{1/q} \to 0 \quad \text{as} \quad n \to \infty.$$

Consequently, B is completely continuous. Now, since A is strongly demiclosed and B is completely continuous, $A + B$ satisfies condition $(++)$. Indeed, let $\{u_n\}$ be any bounded sequence such that $Au_n + Bu_n \to f$ in V^* for some f in V^*. Since V is reflexive, there exists a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that $u_{n_j} \rightharpoonup u_0$ for some u_0 in V as $j \to \infty$. Then $Bu_{nj} \to Bu_0$ and $Au_{nj} \to f - Bu_0$ in V^* as $j \to \infty$. This and the strong demiclosedness of A imply that $Au_0 + Bu_0 = f$, i.e., $A + B$ satisfies condition $(++)$. Finally, since V is reflexive and B is completely continuous, B is compact. Thus, $A + B + \mu J$: $V \to V^*$ is A-proper w.r.t. Γ_I for each $\mu > 0$.

Consequently, if condition (D1) or (D2) holds, then the assertion of Theorem

4.1 follows from Theorem 2.4, while if condition (D3) holds, then the assertion of Theorem 4.1 follows from Theorem 3.3 in view of Proposition 3.5.

Q.E.D.

Remark 4.1. In case $B \equiv 0$ and $T = A$ is coercive (in which case (D3) obviously holds), Theorem 4.1 was first proved by Browder [3] and Dubinsky [11]. Coercive elliptic operators of pseudomonotone type were first studied by Leray and Lions [15]. The study of elliptic equations involving odd operators was initiated by Pohodjayev [23] and continued by Browder [7, 8], Hess [13], and others (see [8, 16]). In [8] Browder studied the solvability of elliptic equations involving operators of pseudomonotone type satisfying the growth condition (D3) under the additional condition that $(Tu, u) \geq -k \| u \|$ for all $u \in X$ and some constant k.

As our second application we consider the following BVP for the nonlinear ordinary differential equation of order $m > 1$ of the form

$$\tilde{A}(x, u, u^{(1)},..., u^{(m)}) + \tilde{B}(x, u, u^{(1)},..., u^{(m-1)}) = h \qquad (h \in L_2), \tag{4.8}$$

$$W_i(u) \equiv \sum_{j=0}^{m-1} [\alpha_{ij} u^{(j)}(a) + \beta_{ij} u^{(j)}(b)] = 0 \qquad (0 \leqslant i \leqslant m), \tag{4.9}$$

where α_{ij}, β_{ij} are constants and the functions $\tilde{A}(x, \xi): [a, b] \times R^{m+1} \to R^1$ and $\tilde{B}(x, \eta): [a, b] \times R^m \to R^1$ satisfy the Caratheodory conditions and are such that

(a1) *For each* $u \in W_2^m \equiv W_2^m([a, b])$, *the map* $u \to A(u) \equiv \tilde{A}(x, u(x),..., u^m(x))$ *yields a bounded continuous mapping of* W_2^m *into* $L_2[a, b]$.

(a2) *For each* $u \in W_2^{m-1} \equiv W_2^{m-1}([a, b])$ *the map* $u \to B(u) \equiv \tilde{B}(x, u,..., u^{(m-1)})$ *yields a bounded continuous mapping of* W_2^{m-1} *into* L_2.

Let V *be a closed subspace of* W_2^m *given by*

$$V = \{u \in W_2^m \mid W_i(u) = 0 \text{ for } i = 0, 1,..., m\}.$$

Suppose further that $\tilde{A}(x, \xi)$ satisfies the following condition:

(a3) *For any given ball* $B(0, R) \subset V$ *and* u, v *in* $B(0, R)$ *the function* $\tilde{A}(x, \xi)$ *satisfies the condition:*

$$\int_a^b [\tilde{A}(x, u,..., D^m u) - \tilde{A}(x, v,..., D^m v)]\, D^m(u - v)\, dx \geq -c(R, \| u - v \|_{W_2^{m-1}}),$$

where $c(R, \rho) \geq 0$ *is continuous in* R *and* ρ *and such that* $c(R, t\rho)/t \to 0$ *as* $t \to 0^+$ *for fixed* R *and* ρ.

Suppose that the homogeneous equation $u^{(m)} = 0$ has only the trivial solution $u(x) \equiv 0$ satisfying (4.9). Then, as is well known, the linear mapping K defined on V by $Ku = u^{(m)}(x)$ is a homeomorphism of V onto L_2. Thus, if $\{X_n\} \subset V$ is a sequence of finite-dimensional subspaces such that $\operatorname{dist}(v, X_n) = \inf_{x \in X_n} \| v - x \|_{W_2^m} \to 0$ for each $v \in V$, the sequence $\{Y_n\} = \{KX_n\} \subset L_2$ has also the property that $\operatorname{dist}(f, Y_n) = \inf_{y \in Y_n} \| f - y \|_{L_2} \to 0$ as $n \to \infty$ for each f in L_2. Let $P_n: V \to X_n$ and $Q_n: L_2 \to Y_n$ be the orthogonal projections. Then $\Gamma_0 = \{X_n, P_n; Q_n, Y_n\}$ is projectionally complete for (V, L_2).

Now, it follows from conditions (a1), (a2), and (a3), that $A: V \to L_2$ and $B: V \to L_2$ are well-defined, continuous and bounded; moreover, A has a semibounded variation as a map of V into L_2 since (a3) implies that

$$(A(u) - A(v), K(u - v))_{L_2} \geqslant -c(R, \| u - v \|_{W_2^{m-1}}), \tag{4.10}$$

whenever $u, v \in B(0, R) \subset V$.

In view of the above discussion and Theorems 2.4 and 2.6 we have the following existence theorem for Eq. (4.8).

THEOREM 4.2. *Suppose that the functions $A(x, \xi)$ and $B(x, \eta)$ satisfy conditions* (a1), (a2), *and* (a3). *Then for each $h \in L_2$, the BV problem* (4.8)–(4.9) *has a solution $u \in V$ provided that any one of the following conditions holds:*

(d1) *$A(x, -\xi) = -A(x, \xi)$ for $\xi \in R^{m+1}$ and $B(x, -\eta) = -B(x, \eta)$ for $\eta \in R^m$ and $T \equiv A + B: V \to L_2$ satisfies condition $(+)$, i.e., if $\{u_n\} \subset V$ is a sequence that $A(u_n) + B(u_n) \to f$ for some f in L_2, then $\{u_n\}$ is bounded in V.*

(d2) *$(Tu, Ku) \geqslant 0$ for all u in V and $T = A + B: V \to L_2$ satisfies condition $(+)$.*

(d3) *$\| Tu \|_{L_2} + ((Tu, K(u)/\| Ku \|) \to \infty$ as $\| u \|_{W_2^m} \to \infty$ for $u \in V$, and $(Tu, Ku) \geqslant -c \| Ku \|$ for all $u \in V$ and some c.*

Proof. It follows from (4.10) that for each $\mu > 0$ and $u, v \in B(0, R)$ we have the inequality

$$\begin{aligned}((A + \mu K)\, u - (A + \mu K)\, v, K(u - v))\\ \geqslant \mu \| K(u - v)\|_{L_2}^2 - c(R, \| u - v \|_{W_2^{m-1}}).\end{aligned}$$

Since the imbedding of W_2^m into W_2^{m-1} is compact and the norm in W_2^m is equivalent to the norm given by $\| u \|_0 = \| Ku \|_{L_2}$ for $u \in V$, it follows from the above inequality that $A + \mu K: V \to L_2$ is A-proper with respect to Γ_0. Moreover, since $B: V \to L_2$ is completely continuous and thus compact because V and L_2 are reflexive, the operator $A + B + \mu K: V \to L_2$ is A-proper with respect to Γ_0.

It is obvious that if in Theorem 2.4 we take $X = V$, $Y = L_2$, $G = K$, and $K_n = M_n = K|_{X_n}: X_n \to Y_n$, then all the assumptions of Theorem 2.4 are satisfied. Thus, if either (d1) or (d2) holds, the conclusion of Theorem 4.2

follows from Theorem 2.4. If, on the other hand, condition (d3) holds, then, in view of Lemma 3.3, the conclusion of Theorem 4.2 follows from Theorem 2.6 and Remark 2.7(b). Q.E.D.

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