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Multihomogeneous resultant formulae by means of complexes

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Abstract

The first step in the generalization of the classical theory of homogeneous equations to the case of arbitrary support is to consider algebraic systems with multihomogeneous structure. We propose constructive methods for resultant matrices in the entire spectrum of resultant formulae, ranging from pure Sylvester to pure Bézout types, and including matrices of hybrid type of these two. Our approach makes heavy use of the combinatorics of multihomogeneous systems, inspired by and generalizing certain joint results by Zelevinsky, and Sturmfels or Weyman (J. Algebra, 163 (1994) 115; J. Algebraic Geom., 3 (1994) 569). One contribution is to provide conditions and algorithmic tools so as to classify and construct the smallest possible determinantal formulae for multihomogeneous resultants. Whenever such formulae exist, we specify the underlying complexes so as to make the resultant matrix explicit. We also examine the smallest Sylvester-type matrices, generically of full rank, which yield a multiple of the resultant. The last contribution is to characterize the systems that admit a purely Bézout-type matrix and show a bijection of such matrices with the permutations of the variable groups. Interestingly, it is the same class of systems admitting an optimal Sylvester-type formula. We conclude with examples showing the kinds of matrices that may be encountered, and illustrations of our MAPLE implementation.

Keywords: Sparse resultant; Multihomogeneous system; Determinantal formula; Sylvester and Beźout type matrix; Degree vector

1. Introduction

Resultants provide efficient ways for studying and solving polynomial systems by means of their matrices. This paper considers the sparse (or toric) resultant, which exploits

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a priori knowledge on the support of the equations. We concentrate on unmixed (i.e. with identical supports) systems where the variables can be partitioned into groups so that every polynomial is homogeneous in each group. Such polynomials, and the resulting systems, are called *multihomogeneous*. Multihomogeneous structure is a first step away from the classical theory of homogeneous systems towards fully exploiting arbitrary sparse structure. Multihomogeneous systems are encountered in several areas including geometric modeling (e.g. Chionh et al., 1998; Saxena, 1997; Zhang, 2000), game theory and computational economics (McKelvey and McLennan, 1997).

Known sparse resultant matrices are of different types. An the one end of the spectrum are the *pure Sylvester-type* matrices, where the polynomial coefficients fill in the nonzero entries of the matrix; such is the coefficient matrix of linear systems, Sylvester's matrix for univariate polynomials, and Macaulay's matrix for homogeneous systems. An the other end are the *pure Bézout-type* matrices, i.e. matrices where the coefficients of the *Bezoutian* associated to the input polynomials fill in the nonzero entries of the matrix, whereas hybrid matrices, such as Dixon's, contain blocks of both pure types. The examples in Section 7.2 show the intricacy of such matrices. Hence the interest to describe them in advance in terms of combinatorial data, which allows for a structured matrix representation, based on quasi-Toeplitz or quasi-Hankel structure (Emiris and Pan, 2002; Mourrain and Pan, 2000).

Our work builds on Weyman and Zelevinsky (1994) and their study of multihomogeneous systems through the determinant of a resultant complex. First, we give precise *degree vectors* together with algorithmic methods for identifying and constructing *determinantal formulae* for the sparse resultant, i.e. matrices whose determinant equals the sparse resultant. The underlying resultant complex is made explicit and computational tools are derived in order to produce the smallest such formula. Second, we describe and construct the smallest possible pure Sylvester matrices, thus generalizing the results of Sturmfels and Zelevinsky (1994) and Gelfand et al. (1994, Section 13.2, Proposition 2.2), already present in the interesting paper by McCoy (1933) pointed out by one of the referees. The corresponding systems include all systems for which exact Sylvester-type matrices are known. We consider more general Sylvester-type matrices, and show that in the search of small formulae, these more general matrices are not crucial. The third contribution of this paper is to offer sufficient and necessary conditions for systems to admit purely Bézout determinantal formulae, thus generalizing a result from Chtcherba and Kapur (2000). It turns out that these are precisely the same systems admitting optimal Sylvester-type formulae, and this is nothing but a special case of complexes with only two nonvanishing cohomologies. We also show a bijection of such matrices with the permutations on $\{1, \dots, r\}$, where r stands for the number of the variable groups. While constructing explicit Bézout-type formulae, we derive a precise description of the support of the Bezoutian polynomial.

The complex with terms $K_\nu(m)$ described in the next section is known as the *Weyman complex*. For any choice of dimensions, of degrees of the input equations and of an integer vector m, the multihomogeneous resultant equals the determinant of the Weyman complex (for the corresponding monomial basis at each of the terms), which can be expressed as a quotient of products of subdeterminants extracted from the differentials in the complex. This way of defining the resultant was introduced by Cayley (Gelfand et al., 1994, Appendix A; Weyman, 1994; Weyman and Zelevinsky, 1994). In the particular case in which the complex has just two terms, its determinant is nothing but the determinant of the only nonzero differential, which is therefore equal to the resultant. In this case, we say that there is a *determinantal* formula for the resultant and the corresponding degree vector m is called *determinantal*. In Weyman and Zelevinsky (1994), the multihomogeneous systems for which a determinantal formula exists were classified; see also Gelfand et al. (1994, Section 13.2). Their work, though, does not identify completely the corresponding morphisms nor the determinantal vectors m, a question we partially undertake. We follow the results in D'Andrea and Dickenstein (2001), which concerned the homogeneous case, inspired also by Jouanolou (1997).

The main result of Sturmfels and Zelevinsky (1994) was to prove that a determinantal formula of Sylvester type exists exactly when all defects are zero. In Sturmfels and Zelevinsky (1994, Theorem 2) (recalled in Gelfand et al., 1994, Section 13.2, Proposition 2.2; see also McCoy, 1933, Theorem 4) all such formulae are characterized by showing a bijection with the permutations of $\{1, \ldots, r\}$ and defined the corresponding degree vector m as in Definition 5.2 below. This includes all known Sylvester-type formulae, in particular, linear systems, systems of two univariate polynomials and bihomogeneous systems of three polynomials whose resultant is, respectively, the coefficient determinant, the Sylvester resultant and the Dixon resultant. In fact, Sturmfels and Zelevinsky characterized all determinantal Cayley–Koszul complexes, which are instances of the Weyman complexes when all the higher cohomologies vanish.

The incremental algorithm for sparse resultant matrices (Emiris and Canny, 1995) relies on the determination of a degree vector m. When $\delta = 0$, it produces optimal Sylvester matrices by Sturmfels and Zelevinsky (1994). For other multihomogeneous systems, Emiris and Canny (1995) heuristically produces small matrices, yet with no guarantee. For instance, on the system of Example 5.5, it finds a 1120×1200 matrix. The present paper explains the behavior of the algorithm, since the latter uses degree vectors following Definition 5.2 defined by random permutations. Our results provide immediately the smallest possible matrix. More importantly, the same software constructs all Sylvester-type formulae described here.

Pure Bézout-type formulae were studied in Chtcherba and Kapur (2000) for unmixed systems whose support is the direct sum of what they call basis simplices, i.e. the convex hull of the origin and another l_k points, each lying on a coordinate axis. This includes the case of multihomogeneous systems. They showed that in this case a sparse resultant matrix can be constructed from the Bezoutian polynomial, to be defined in Section 6, though the corresponding matrix formula is not always determinantal. Their Corollary 4.2.1 states that for multihomogeneous systems with null defect vector, the Bézout formula becomes determinantal; Saxena had proved the special case of all $l_k = 1$ (Saxena, 1997). In Chtcherba and Kapur (2000, Section 4.2) they indicate there are $r!$ such formulae and in Section 5 they study bivariate systems ($n = 2$) showing that then, these are the only determinantal formulae.

Section 6 proves these results in a different manner and characterizes the determinantal cases for multihomogeneous systems, showing that a null defect vector is a sufficient but also *necessary* condition for a determinantal formula of pure Bézout type for any n. Thus, there is an optimal Sylvester-type formula for the resultant if and only if there is an optimal pure Bézout-type formula (cf. Definition 6.1). This had been proven for arbitrary systems only in the bivariate case (Chionh et al., 1998). In particular, we explicitly exhibit a choice of the differential in the Weyman complex in this case (cf. Theorem 6.13), thus partially answering the "challenge to make these maps explicit" of Weyman and Zelevinsky (1994, Section 5.1).

Studies exist (Chionh et al., 1998; D'Andrea and Dickenstein, 2001; Zhang, 2000) for dealing with hybrid formulae including Bézout-type blocks or pure Bézout matrices, and concentrate on the computation of such matrices. In particular Zhang (2000) elaborates on the relation of Sylvester and Bézout-type matrices (called Cayley-type there) and the transformations that link them. The theoretical setting together with Pfaffian formulae for resultants is addressed in Eisenbud and Schreyer (2003). This is made explicit for any toric surface in Khetan (2002). In the recent preprint (Awane et al., 2002), not also the multihomogeneous resultant but the whole ideal of inertia forms is studied, extending results of Jouanolou (1980) in the homogeneous case.

This paper is organized as follows. The next section provides some technical facts useful later. Section 3 offers bounds in searching for the smallest possible determinantal (hybrid) formulae. Section 4 makes explicit one degree vector attached to any determinantal data of dimensions of projective spaces, and discusses further techniques for obtaining determinantal formulae. Sections 5 and 6 characterize matrices of pure Sylvester and pure Bézout type respectively. In Section 7 we fully describe the formulae of a system of three bilinear polynomials. Then, we provide an explicit example of a hybrid resultant matrix for a multidegree for which neither pure Sylvester nor pure Bézout determinantal formulae exist; this example illustrates the possible morphisms that may be encountered with multihomogeneous systems. Our MAPLE implementation is described in Section 8.

A preliminary version of certain results in this paper has appeared in Dickenstein and Emiris (2002).

2. Preliminary observations

We consider the r-fold product $X := \mathbb{P}^{l_1} \times \cdots \times \mathbb{P}^{l_r}$ of projective spaces of respective dimensions $l_1, \ldots, l_r$ over an algebraically closed field of characteristic zero, for some natural number r. We denote by $n = \sum_{k=1}^{r} l_k$ the dimension of X, i.e. the number of affine variables.

Definition 2.1. Consider $d = (d_1, \ldots, d_r) \in \mathbb{N}^r_{>0}$ and multihomogeneous polynomials $f_0, \ldots, f_n$ of degree d. The multihomogeneous resultant is an irreducible polynomial $R(f_0, \ldots, f_n) = R_{(l_1,\ldots,l_r),d}(f_0, \ldots, f_n)$ in the coefficients of $f_0, \ldots, f_n$ which vanishes iff the polynomials have a common root in X.

This is an instance of the sparse resultant (Gelfand et al., 1994). It may be chosen with integer coefficients, and it is uniquely defined up to sign by the requirement that it has relatively prime coefficients. The resultant polynomial is itself homogeneous in the coefficients of each f_i, with degree given by the multihomogeneous Bézout bound $\binom{n}{l_1,\ldots,l_r} d_1^{l_1} \cdots d_r^{l_r}$ (Gelfand et al., 1994, Proposition 13.2.1). This number is also called the m-homogeneous bound (Wampler, 1992).

Let V be the space of $(n+1)$ tuples $f = (f_0, \ldots, f_n)$ of multihomogeneous forms of degree d over X. Given a *degree vector* $m \in \mathbb{Z}^r$ there exists a finite complex $K_\cdot = K_\cdot(m)$ of free modules over the ring of polynomial functions on V (Weyman and Zelevinsky, 1994), whose terms depend only on $(l_1, \ldots, l_r)$, d and m and whose differentials are polynomials on V satisfying:

(i) For every given f we can specialize the differentials in $K_\cdot$ by evaluating at f to get a complex of finite-dimensional vector spaces.
(ii) This complex is exact iff $R(f_0, \ldots, f_n) \neq 0$.

In order to describe the terms in these complexes some facts from cohomology theory are necessary; see Hartshorne (1977) for details. Given a *degree vector* $m \in \mathbb{Z}^r$, define, for $\nu \in \{-n, \ldots, n+1\}$,

$$K_\nu(m) = \bigoplus_{p \in \{0,\ldots,n+1\}} H^{p-\nu}(X, m - pd)^{\binom{n+1}{p}}, \tag{1}$$

where for an integer r-tuple m', $H^q(X, m')$ denotes the qth cohomology of X with coefficients in the sheaf $\mathcal{O}(m')$ such that its global sections $H^0(X, m')$ are identified with multihomogeneous polynomials of (multi)degree m'. By the Künneth formula, we have

$$H^q(X, m - pd) = \bigoplus_{j_1+\cdots+j_r=q}^{j_k \in \{0,l_k\}} \bigotimes_{k=1}^{r} H^{j_k}(\mathbb{P}^{l_k}, m_k - pd_k),$$

where $q = p - \nu$ and the second sum runs over all integer sums $j_1 + \cdots + j_r = q$, $j_k \in \{0, l_k\}$. In particular, $H^0(\mathbb{P}^{l_k}, \alpha_k)$ is the space of all homogeneous polynomials in $l_k + 1$ variables with total degree α_k. By Serre's duality, for any $\alpha \in \mathbb{Z}^r$, we also know that

$$H^q(X, \alpha) \simeq H^{n-q}(X, (-l_1 - 1, \ldots, -l_r - 1) - \alpha)^*, \tag{2}$$

where * denotes dual. We recall Bott's formulae for these cohomologies.

Proposition 2.2. *For any $m \in \mathbb{Z}^r$, $H^{l_k}(\mathbb{P}^{l_k}, m_k - pd_k) = 0 \Leftrightarrow m_k - pd_k \geq -l_k$, $H^0(\mathbb{P}^{l_k}, m_k - pd_k) = 0 \Leftrightarrow m_k - pd_k < 0$, for $k \in \{1, \ldots, r\}$. Moreover,*

$$H^j(\mathbb{P}^{l_k}, m_k - pd_k) = 0, \qquad \forall j \neq 0, l_k,$$
$$\dim H^{l_k}(\mathbb{P}^{l_k}, m_k - pd_k) = \binom{-m_k + pd_k - 1}{l_k},$$
$$\dim H^0(\mathbb{P}^{l_k}, m_k - pd_k) = \binom{m_k - pd_k + l_k}{l_k}.$$

Consequently,

$$\dim H^q(X, m - pd) = \sum_{j_1+\cdots+j_r=q}^{j_k \in \{0,l_k\}} \prod_{k=1}^{r} \dim H^{j_k}(\mathbb{P}^{l_k}, m_k - pd_k),$$

and

$$\dim K_\nu(m) = \sum_{p \in [0,n+1]} \binom{n+1}{p} \dim H^{p-\nu}(X, m - pd).$$

Definition 2.3. Given r and $(l_1, \ldots, l_r), (d_1, \ldots, d_r) \in \mathbb{N}^r$, define the *defect* vector $\delta \in \mathbb{Z}^r$ (just as in Sturmfels and Zelevinsky, 1994; Weyman and Zelevinsky, 1994) by $\delta_k := l_k - \lceil \frac{l_k}{d_k} \rceil$. Clearly, this is a nonnegative vector. We also define the *critical degree* vector $\rho \in \mathbb{N}^r$ by $\rho_k := (n+1)d_k - l_k - 1$, for all $k = 1, \ldots, r$.

Lemma 2.4 (Weyman and Zelevinsky, 1994). *For any* $i \in [r] := \{1, \ldots, r\}$, $d_i l_i < d_i + l_i \Leftrightarrow \delta_i = 0 \Leftrightarrow \min\{l_i, d_i\} = 1$.

Let us establish a general technical lemma.

Lemma 2.5. *For any* $k \in \{1, \ldots, r\}$, $0 \leq (l_k - \delta_k)d_k - l_k \leq d_k - 1$.

Proof. By definition, $\delta_k = l_k - \lceil l_k/d_k \rceil \Leftrightarrow \lceil l_k/d_k \rceil = l_k - \delta_k = (l_k + t_k)/d_k$ for some integer t_k such that $0 \leq t_k \leq d_k - 1$. □

We detail now the main results in Weyman and Zelevinsky (1994). They show (Lemma 3.3(a)) that a vector $m \in \mathbb{Z}^r$ is determinantal iff $K_{-1}(m) = K_2(m) = 0$. They also prove in Theorem 3.1 that a determinantal vector m exists iff $\delta_k \leq 2$ for all $k \in [r]$. To describe a differential in the complex from $K_\nu(m)$ to $K_{\nu+1}(m)$, one needs to describe all the morphisms $\delta_{p,p'}$ from the summand corresponding to an integer p to the summand corresponding to another integer p', where both $p, p' \in \{0, \ldots, n+1\}$. Weyman and Zelevinsky (1994, Propositions 2.5, 2.6) proves this map is 0 when $p < p'$ and that, roughly speaking, it corresponds to a *Sylvester* map $(g_0, \ldots, g_n) \to \sum_{i=0}^n g_i f_i$ when $p = p' + 1$, thus having all nonzero entries in the corresponding matrix given by coefficients of $f_0, \ldots, f_n$. For $p > p' + 1$, the maps $\delta_{p,p'}$ are called higher-order differentials. By degree reasons, they cannot be given by Sylvester matrices. Theorem 2.10 also gives an explicit theoretical construction of the higher-order differentials in the pure Bézout case (cf. Definition 6.1).

3. Bounds for determinantal degree vectors

This section addresses the computational problem of enumerating all determinantal degree vectors $m \in \mathbb{Z}^r$. The "procedure" of Weyman and Zelevinsky (1994, Section 3) "is quite explicit but it seems that there is no nice way to parametrize these vectors", as stated in that paper. Instead, we bound the range of m to implement a computer search for them. In Section 4 we will give an explicit choice of degree vector m for each determinantal data $(l_1, \ldots, l_r; d_1, \ldots, d_r)$.

Given $k \in \{1, \ldots, r\}$ and a vector $m \in \mathbb{Z}^r$ define as in Weyman and Zelevinsky (1994):

$$P_k(m) = \left\{ p \in \mathbb{Z} : \frac{m_k}{d_k} < p \leq \frac{m_k + l_k}{d_k} \right\}.$$

Let $\tilde{P}_k(m)$ be the real interval $(\frac{m_k}{d_k}, \frac{m_k + l_k}{d_k}]$, so $P_k(m) = \tilde{P}_k(m) \cap \mathbb{Z}$. Using Lemma 3.3 in Weyman and Zelevinsky (1994), it is easy to give bounds for all determinantal vectors m for which all $P_k(m) \neq \emptyset$.

Lemma 3.1. *For a determinantal* $m \in \mathbb{Z}^r$ *and for all* $k \in \{1, \ldots, r\}$, $P_k(m) \neq \emptyset$ *implies,* $\max\{-d_k, -l_k\} \leq m_k \leq d_k(n+1) - 1 + \min\{d_k - l_k, 0\}$.

Proof. By Weyman and Zelevinsky (1994, Lemma 3.3(b)), $P_k(m) \subset [0, n+1] \Rightarrow m_k/d_k \geq -1$ and $(m_k + l_k)/d_k \geq 0$, which imply the lower bound. Also, $(m_k + l_k)/d_k < n+2 \Leftrightarrow m_k \leq (n+2)d_k - l_k - 1$ and $m_k/d_k < n+1 \Leftrightarrow m_k \leq (n+1)d_k - 1$ yield the upper bound. Notice that the possible values for m_k form a nonempty set, since the two bounds are negative and positive respectively. $\square$

Now, $p \in P_k(m)$ iff $H^{l_k}(\mathbb{P}^{l_k}, m_k - pd_k) = H^0(\mathbb{P}^{l_k}, m_k - pd_k) = 0$. Thus, a first guess could be that all determinantal vectors give $P_k(m) \neq \emptyset$. But this is not the case, as the following example shows:

Example 3.2. Set $l = (1,2), d = (2,3)$. We focus on degree vectors of the form $m = (2\mu_1, 3\mu_2)$, for $\mu_1, \mu_2 \in \mathbb{Z}$. Then, for all such m, the sets $P_k(m)$, for $k = 1, 2$, are empty. Nevertheless, there exist four determinantal vectors of this form, namely $m = (4,3), (0,6), (2,6)$ or $(6,3)$. Moreover, the vector $m = (6,3)$ gives a determinantal formula with a matrix of size 88, which is closer to the smallest possible one which has size 72. The largest determinantal formula is given by the determinant of a square matrix of size 180. Note that the degree of the multihomogeneous resultant is 216. More details on this example are provided in Section 8. $\square$

We wish now to get a bound for those determinantal vectors for which some $P_k(m)$ is empty. Let $[\cdot]_k \in \{0, 1, \ldots, d_k - 1\}$ denote the remainder after division by d_k.

Definition 3.3. Given $m \in \mathbb{Z}^r$ and $k \in \{1, \ldots, r\}$, define new vectors $m', m'' \in \mathbb{Z}^r$ whose jth coordinates equal those of m for all $j \neq k$ and such that $m'_k = m_k + d_k - [m_k]_k - 1 \geq m_k \geq m''_k = m_k - [m_k + l_k]_k$.

Lemma 3.4. *The vectors m', m'' differ from m at their kth coordinate if $P_k(m) = \emptyset$.*

Proof. Let us write $m_k = jd_k + [m_k]_k$ for $j \in \mathbb{Z}$. Then, $m_k/d_k = j + \frac{[m_k]_k}{d_k}$. If $P_k(m) = \emptyset, (m_k + l_k)/d_k < j+1 \Rightarrow jd_k + [m_k]_k + l_k < (j+1)d_k$, so $[m_k]_k \leq d_k - l_k - 1 \Rightarrow l_k \leq -[m_k]_k + d_k - 1$ and thus $1 \leq d_k - [m_k]_k - 1$. Also, $[m_k + l_k]_k \geq 1$ because $[m_k + l_k]_k = 0 \Rightarrow (m_k + l_k)/d_k \in P_k(m)$. $\square$

Lemma 3.5. *If $m \in \mathbb{Z}^r$ with $P_k(m) = \emptyset$ and $H^0(\mathbb{P}^{l_k}, m_k - pd_k) = 0$ (resp. $H^{l_k}(\mathbb{P}^{l_k}, m_k - pd_k) = 0$), then $P_k(m') \neq \emptyset$ and $H^0(\mathbb{P}^{l_k}, m'_k - pd_k) = 0$ (resp. $H^{l_k}(\mathbb{P}^{l_k}, m'_k - pd_k) = 0$), where $m'_k = m_k + d_k - [m_k]_k - 1$ as in Definition 3.3.*

Proof. Write $m_k = jd_k + [m_k]_k$ for some integer $j \in \mathbb{Z}$. To prove $P_k(m') \neq \emptyset$ we show $j+1 \in P_k(m')$, i.e. $m'_k/d_k < j+1 \leq m'_k + l_k/d_k \Leftrightarrow m'_k < (j+1)d_k \leq m'_k + l_k \Leftrightarrow jd_k + d_k - 1 < (j+1)d_k \leq jd_k + d_k - 1 + l_k$, which is clearly true since $1 \leq l_k$.

Now, $H^{l_k}(\mathbb{P}^{l_k}, m_k - pd_k) = 0 \Leftrightarrow m_k + l_k \geq pd_k$ hence $m'_k + l_k \geq pd_k$ because $m'_k \geq m_k$. $H^0(\mathbb{P}^{l_k}, m_k - pd_k) = 0 \Leftrightarrow m_k < pd_k$ so $j \leq p-1$. Hence, $m_k - [m_k]_k = jd_k \leq (p-1)d_k \Leftrightarrow m_k - [m_k]_k + d_k \leq pd_k$ which is the desired conclusion. By Weyman and Zelevinsky (1994), $P_k(m') \subset [0, n+1]$ from which $j \in \{-1, 0, \ldots, l\}$. $\square$

Lemma 3.6. *If $m \in \mathbb{Z}^r$ with $P_k(m) = \emptyset$ and $H^0(\mathbb{P}^{l_k}, m_k - pd_k) = 0$ (resp. $H^{l_k}(\mathbb{P}^{l_k}, m_k - pd_k) = 0$), then $P_k(m'') \neq \emptyset$ and $H^0(\mathbb{P}^{l_k}, m''_k - pd_k) = 0$ (resp. $H^{l_k}(\mathbb{P}^{l_k}, m''_k - pd_k) = 0$), where $m''_k = m_k - [m_k + l_k]_k$ as in Definition 3.3.*

Proof. Write $m_k + l_k = jd_k + [m_k + l_k]_k$ for some integer $j \geq 0$. To prove $P_k(m''_k) \neq \emptyset$ we show it contains j, i.e. $\frac{m_k - [m_k + l_k]_k}{d_k} < j \leq \frac{m_k - [m_k + l_k]_k + l_k}{d_k} \Leftrightarrow m_k - [m_k + l_k]_k < jd_k \leq m_k - [m_k + l_k]_k + l_k \Leftrightarrow jd_k - l_k < jd_k \leq jd_k$, which is clearly true since $0 < l_k$. $H^0(\mathbb{P}^{l_k}, m_k - pd_k) = 0 \Leftrightarrow m_k < pd_k \Rightarrow m''_k < pd_k$ because $m''_k \leq m_k$, hence $H^0(\mathbb{P}^{l_k}, m''_k - pd_k) = 0$. $H^{l_k}(\mathbb{P}^{l_k}, m_k - pd_k) = 0 \Leftrightarrow m_k + l_k \geq pd_k$, then $j \geq p$. Hence $m_k + l_k - [m_k + l_k]_k \geq pd_k$ which finishes the proof. By Weyman and Zelevinsky (1994), $P_k(m'') \subset [0, n+1]$ hence $j \in \{0, \ldots, n+1\}$.

Lemmas 3.5 and 3.6 imply

Theorem 3.7. *For any determinantal $m \in \mathbb{Z}^r$, define vectors $m', m'' \in \mathbb{Z}^r$ as in Definition 3.3 which differ from m only at the kth coordinates, $1 \leq k \leq r$, such that $P_k(m) = \emptyset$. Then $P_k(m') \neq \emptyset$, $P_k(m'') \neq \emptyset$ and both m', m'' are determinantal.*

Corollary 3.8. *For a determinantal $m \in \mathbb{Z}^r$ with $P_k(m) = \emptyset$ for some $k \in \{1, \ldots, r\}$, we have* $0 \leq m_k \leq d_k(n+1) - l_k - 1$.

Proof. Since m'_k, m''_k define $P_k(m') \neq \emptyset, P_k(m'') \neq \emptyset$, we can apply Lemma 3.1. We use the lower bound with m''_k because $m''_k < m_k < m'_k$. $P_k(m) = \emptyset \Rightarrow d_k > l_k$, so $m''_k = m_k - [m_k + l_k]_k \geq -l_k \Rightarrow m_k \geq [m_k + l_k]_k - l_k \geq 1 - l_k$, because $[m_k + l_k]_k \geq 1$ by the proof of Lemma 3.4. If $m_k < 0$, for $P_k(m)$ to be empty we need $m_k + l_k < 0 \Leftrightarrow m_k < -l_k$ which contradicts the derived lower bound; so $m_k \geq 0$. For the upper bound, $m'_k = m_k + d_k - [m_k]_k - 1 \leq d_k(n+1) - 1 \Rightarrow m_k \leq d_k n + [m_k]_k \leq d_k(n+1) - l_k - 1$; the latter follows from $[m_k]_k < d_k - l_k$ (Lemma 3.4). $(m_k + l_k)/d_k \leq n + 1 - (1/d_k) < n + 1$ implies the inclusion of the half-open interval in $(0, n+1)$. The possible values for m_k form a nonempty set, since the lower bound is zero and the upper bound is $d_k(n+1) - l_k - 1 \geq d_k - 1 > 0$ since $d_k > l_k \geq 1$. □

So, in fact, the real interval $\tilde{P}_k \subset (0, n+1)$.

Corollary 3.9. *For a determinantal $m \in \mathbb{Z}^r$ and $k \in [r]$,* $\max\{-d_k, -l_k\} \leq m_k \leq d_k(n+1) - 1 + \min\{d_k - l_k, 0\}$.

This implies there is a finite number of vectors to be tested in order to enumerate all possible determinantal m. This could also be deduced from the fact that the dimension of $K_0(m)$ equals the degree of the resultant. Corollary 3.9 gives a precise bound for the box in which to search algorithmically for all determinantal m, including those that are "pure" in the terminology of Weyman and Zelevinsky (1994). Our MAPLE implementation, along with examples, is presented in Section 8.

If we take $r = 1 = l_1$ and $m = 2d_1 - 1$ we obtain the classical Sylvester formula and the upper bound given by Corollary 3.8 is attained. At the lower bound $m = -1$, the corresponding complex $H^1(\mathbb{P}^1, -1 - 2d) = H^0(\mathbb{P}^1, 2d_1 - 1)^* \to H^1(\mathbb{P}^1, d_1 - 1)^2 = (H^0(\mathbb{P}^1, d_1 - 1)^*)^2$ yields the same matrix transposed. In addition, the system of three bilinear polynomials in Section 7.1 admits a pure Sylvester formula with $m = (2, -1)$, which attains both lower and upper bounds of the corollary. The bounds in Lemma 3.1 can also be attained (see Example 5.5 continued in Section 8) hence Corollary 3.9 is tight. It is possible that some combination of the coordinates of m restricts the search space.

4. Explicit determinantal degree vectors

We focus on the case of $\delta_k \le 2$ for all k, which is a necessary and sufficient condition for the existence of a determinantal complex (Weyman and Zelevinsky, 1994). We shall describe specific degree vectors m that yield determinantal complexes.

Theorem 4.1. *Suppose that $0 \le \delta_k \le 2$ for all $k = 1, \dots, r$ and $\pi : [r] \to [r]$ is any permutation. Then, the degree vector $m^\pi \in \mathbb{Z}^r$ with*

$$m_k^\pi = \left(1 - \delta_k + \sum_{\pi(j) \ge \pi(k)} l_j\right) d_k - l_k, \tag{3}$$

for $k = 1, \dots, r$ defines a determinantal complex.

Proof. First, $K_1 \ne 0$ and $K_0 \ne 0$ because they each contain at least one nonzero direct summand, namely $H^0(X, m^\pi - pd)$ for $p = 1, 0$ respectively. To see this, it suffices to prove $m_k^\pi - d_k \ge 0, k = 1, \dots, r$, which follows from $(1 - \delta_k + l_k)d_k - l_k - d_k \ge 0 \Leftrightarrow (-\delta_k + l_k)d_k - l_k \ge 0$. This holds by Lemma 2.5.

To demonstrate that $K_2 = 0$ we shall see that every one of its direct summands $H^q(X, m^\pi - (q + 2)d)$ vanishes for $q = \sum_{\pi(j) \in J} l_j$, where J is any proper subset of $[r]$. Recall that the case $J = [r]$ is irrelevant by the definitions of Section 2. It is enough to show $H^0(\mathbb{P}^{l_k}, m_k^\pi - (q + 2)d_k) = 0$, for some k with $\pi(k) \notin J$. Let k be the maximum of the indices verifying $\pi(k) \notin J$; such a $k \in [r]$ always exists because $J \ne [r]$. Since $(1 - \delta_k + l_k)d_k - l_k < 2d_k$ by Lemma 2.5, it easily follows that

$$m_k^\pi < \left(2 + \sum_{\pi(j) \in J} l_j\right) d_k. \tag{4}$$

It now suffices to establish $K_{-1}(m) = 0$. Since this module has no zero cohomology summand, let $q = \sum_{\pi(j) \in J} l_j$, with $J \ne \emptyset$. Let $k \in J$ such that $\pi(j) \ge \pi(k)$ for all $j \in J$. Then, $\delta_k \le 2$ implies $(2 - \delta_k)d_k \ge 0$, from which $m_k^\pi - (q - 1)d_k \ge -l_k$, and so $H^q(X, m^\pi - (q - 1)d) = 0$ for any direct summand of $K_{-1}(m)$. □

When the defects are at most 1, we can give another explicit choice of determinantal degree vector for each permutation of $[r]$.

Theorem 4.2. *Suppose that $0 \le \delta_k \le 1$ for all $k = 1, \dots, r$ and $\pi : [r] \to [r]$ is any permutation. Then, the degree vector $m^\pi \in \mathbb{Z}^r$ with*

$$m_k^\pi = \left(-\delta_k + \sum_{\pi(j) \ge \pi(k)} l_j\right) d_k - l_k, \tag{5}$$

for $k = 1, \dots, r$ defines a determinantal complex.

Proof. Let us modify the previous proof. First, $K_0 \ne 0$ because it contains $H^0(X, m^\pi - pd) \ne 0$ for $p = 0$. This follows from $(-\delta_k + l_k)d_k - l_k \ge 0 \Leftrightarrow (-\delta_k + l_k)d_k - l_k \ge 0$ for all k. This holds by Lemma 2.5.

Next, $K_1 \neq 0$ because for $p = n+1$, $H^{l_k}(\mathbb{P}^{l_k}, m_k^\pi - pd_k) \neq 0$ for all k. This follows from $(-\delta_k + n)d_k - l_k < (n+1)d_k - l_k \Leftrightarrow -\delta_k < 1$.

To demonstrate that $K_2 = 0$ we repeat the corresponding argument in the proof of Theorem 4.1. Then, it suffices to show that $(-\delta_k + l_k)d_k - l_k < 2d_k$, which is weaker than the inequality before (4) above. Similarly, to establish $K_{-1}(m) = 0$ the steps in the previous proof lead us to showing $m_k^\pi - (q-1)d_k \geq -l_k \Leftarrow (1-\delta_k)d_k \geq 0$ which holds for $\delta_k \leq 1$. □

One can easily verify that the vectors m^π in both theorems above satisfy the bounds of Corollary 3.9.

A natural question is how to give an explicit degree vector yielding a determinantal formula of smallest size. When $r = 1$, this is the content of Lemma 5.3 in D'Andrea and Dickenstein (2001). Even for $r = 2$, this seems to be a difficult task in general.

Example 4.3. Set $l = (2,2), d = (3,2)$. Let $\pi_1 : [2] \to [2]$ denote the identity and $\pi_2 : [2] \to [2]$ the permutation which interchanges 1 and 2. The two determinantal vectors defined according to (3) are $m_{\pi_1} = (10,2)$ and $m_{\pi_2} = (4,6)$, which yield determinantal matrices of respective sizes 396 and 420. The two determinantal vectors defined according to (5) are $m_{\pi_1} = (7,0)$ and $m_{\pi_2} = (1,4)$, which yield determinantal matrices of respective sizes 756 and 780. But as we will see in Section 6, these latter formulae give instead the vectors providing the smallest determinantal formulae when all defects are 0. These vectors can be computed by the function `comp_m` of our implementation, discussed in Section 8.

Also, when $r = 1$ it is shown in D'Andrea and Dickenstein (2001) that the smallest formula is attained for "central" determinantal degree vectors. In this example, all degree vectors m satisfy that either $1 \leq m_1 \leq 5$ and $4 \leq m_2 \leq 7$ or $7 \leq m_1 \leq 11$ and $0 \leq m_2 \leq 4$; these bounds are computed by the routines described in Section 8. The smallest resultant matrix has size 340×340, and corresponds to the degree vectors $(3,6)$ or $(9,1)$, which are in a "central" position among determinantal degree vectors, in other words, their coordinates lie in a "central" position between the respective coordinates of other determinantal vectors. However, the vectors $(k,4)$ are determinantal for k from 1 to 5, but the size of a matrix corresponding to the vector $(3,4)$ (both of whose coordinates lie between the coordinates of the vectors $(1,4)$ and $(5,4)$), equals 580. So, it is bigger than the size of a resultant matrix associated to the vector $(5,4)$, which is of dimension 540. □

In the above example, the two degree vectors giving the smallest resultant matrices satisfy $(9,1) + (3,6) = (12,7)$, which is the critical vector from Definition 2.3. It is clear that for any determinantal m, the vector $(12,7)$—m is also determinantal yielding the same matrix dimension. This is a consequence of Serre's duality recalled in (2). The general statement is summarized in the next proposition.

Proposition 4.4. *Assume $m, m' \in \mathbb{Z}^r$ satisfy $m + m' = \rho$, the latter being the critical degree vector of Definition 2.3. Then, $K_\nu(m)$ is dual to $K_{1-\nu}(m')$ for all $\nu \in \mathbb{Z}$. In particular, m is determinantal if and only if m' is determinantal, yielding matrices of the same size, namely* $\dim(K_0(m)) = \dim(K_1(m'))$.

Proof. Based on the equality $m + m' = \rho$ we deduce that for all $p = 0, \ldots, n+1$, it holds that $(m' - pd) = (-l_1 - 1, \ldots, -l_r - 1) - (m - (n+1-p)d)$. Therefore, for all $q = 0, \ldots, n$, Serre's duality (2) implies that $H^q(X, m' - pd)$ and $H^{n-q}(X, m - (n+1-p)d)$ are dual. Since $(n+1-p)-(n-q) = 1-(p-q)$, we deduce that $K_\nu(m)$ is dual to $K_{1-\nu}(m')$ for all $\nu \in \mathbb{Z}$, as desired. Observe that in particular $K_{-1}(m) \simeq K_2(m')^*$ and $K_0(m) \simeq K_1(m')^*$, the latter giving the matrix dimension in the case of determinantal formulae. □

We end this section making explicit a consequence of Proposition 3.7 in Weyman and Zelevinsky (1994) (and giving an independent proof), which gives a generalization of the characterization of determinantal complexes in Sturmfels and Zelevinsky (1994) and Weyman and Zelevinsky (1994). This result will allow us to give in Section 6 explicit expressions for all degree vectors yielding a determinantal formula of smallest size when all defects vanish.

Theorem 4.5. *There exists a determinantal vector m such that the Weyman complex is reduced to only one nonzero cohomology group on each of $K_0(m)$, $K_1(m)$ if and only if all defects vanish, i.e. $\delta_k = 0$ for all $k = 1, \dots, r$.*

Proof. Recall that for each $p \in \{0, \dots, n+1\}$ there exists at most one integer j such that $H^j(X, m - pd) \neq 0$ in $K_\nu(m)$ where $p = j - \nu$ (Weyman and Zelevinsky, 1994, Proposition 2.4). In fact, let $A(p) := \{k : m_k - pd_k < -l_k\}$ and $B(p) := \{k : m_k - pd_k \geq 0\}$. Denote $j(p) := \sum_{k \in A(p)} l_k$. Then $H^{j'}(X, m - pd) = 0$ for all $j' \neq j(p)$ and $H^{j(p)}(X, m - pd) \neq 0$ iff $A(p) \cup B(p) = \{1, \dots, r\}$.

The assumption of the theorem means that there exist exactly two integers $p_1, p_2 \in \{0, \dots, n+1\}$ for which $H^{j(p_i)}(X, m - p_i d) \neq 0, i = 1, 2$; cf. also Weyman and Zelevinsky (1994, Lemmma 3.3(a)). Then, for any $p \in \{0, \dots, n+1\} \backslash \{p_1, p_2\}$, there exists k such that $k \notin A(p) \cup B(p)$, i.e. $p \in P_k(m)$. The latter set is defined at the beginning of Section 3. Then, $\{0, \dots, n+1\} \backslash \{p_1, p_2\} \subseteq \cup_{k=1}^r P_k(m)$ and so

$$l_1 + \cdots + l_r = n \leq \sum_{k=1}^{r} \# P_k(m) \leq \sum_{k=1}^{r} \left\lceil \frac{l_k}{d_k} \right\rceil,$$

where the first inequality uses the fact that $\# \cup_{k=1}^r P_k(m) \leq \sum_{k=1}^r \# P_k(m)$ and the second follows from the definition of $P_k(m)$. Since $l_k \geq \lceil \frac{l_k}{d_k} \rceil$ for all k, we deduce that $l_k = \lceil \frac{l_k}{d_k} \rceil$, and this can only happen iff $l_k = 1$ or $d_k = 1$. □

5. Pure Sylvester-type formulae

This section constructs rectangular matrices of pure Sylvester-type that have at least one maximal minor which is a nontrivial multiple of the sparse resultant, coming from a complex of the form:

$$\cdots \to K_2(m) \to K_1(m) \to K_0(m) \to K_{-1}(m) = 0,$$

where $K_1(m) = H^j(X, m - d)^{n+1}$, and $K_0(m) = H^0(X, m)$ for a nonnegative vector $m \in \mathbb{Z}^r_{\geq 0}$.

We assume that $H^0(X, m - pd) \neq 0$ for $\nu = 0, 1$ and $p - \nu = 0$. This implies $H^0(\mathbb{P}^{l_k}, m_k - \nu d_k) \neq 0$ for all $k \in \{1, \dots, r\}$. Moreover, we must have $H^{p-\nu}$

$(X, m - pd) = 0$ for $p - \nu = \sum_{j \in J} l_j$ where J is any subset satisfying $\emptyset \neq J \subset \{1, \ldots, r\}$. Note that we do not require in general that $K_2(m) \neq 0$.

Lemma 5.1. *If $m \neq m' \in \mathbb{Z}^r_{\geq 0}$ yield a Sylvester-type matrix and $m'_k \geq m_k$ for all $k \in \{1, \ldots, r\}$ then, the Sylvester matrix associated to m' is strictly larger than the Sylvester matrix associated to m.*

Proof. We must show $\dim K_0(m') \geq \dim K_0(m)$, i.e. $\dim H^0(\mathbb{P}^{l_k}, m'_k - pd_k) \geq \dim H^0(\mathbb{P}^{l_k}, m_k - pd_k)$ for $k \in \{1, \ldots, r\}$, i.e. $\binom{m'_k - pd_k + l_k}{l_k} \geq \binom{m_k - pd_k + l_k}{l_k}$. The cohomology is nonzero, thus $m'_k - pd_k \geq m_k - pd_k \geq 0$ and this implies the desired inequality because $\binom{s+l_k}{l_k} = (s + l_k) \cdots (s + 1)/l_k!$. The inequality is strict since there exists an index k such that $m'_k > m_k$. $\square$

Definition 5.2. For each choice of a permutation $\pi : \{1, \ldots, r\} \to \{1, \ldots, r\}$, consider the degree vector m^π defined by

$$m^\pi_k := \left(1 + \sum_{\pi(j) \geq \pi(k)} l_j\right) d_k - l_k, \qquad k = 1, \ldots, r.$$

When all defects are zero, these are the vectors defined in Sturmfels and Zelevinsky (1994) yielding determinantal Sylvester formulae and they also coincide with those defined in (3) in Theorem 4.1.

Lemma 5.3. *If $m \in \mathbb{Z}^r_{\geq 0}$ yields a Sylvester-type matrix, it is possible to define a permutation $\pi : [r] \to [r]$ such that for $i = \pi^{-1}(1)$ it holds that $m_i \geq m^\pi_i$. Moreover $H^0(\mathbb{P}^{l_i}, m_i - pd_i) \neq 0$ where $p \leq 1 + \sum_{\pi(j) \in J} l_j$, for any subset J such that $\emptyset \neq J \subset \{2, \ldots, r\}$.*

Proof. For $p = n + 1$, $\nu = 1$, a necessary condition is that $H^n(X, m - (n + 1)d) = 0$. Hence, there exists $i \in [r] : H^{l_i}(\mathbb{P}^{l_i}, m_i - (n+1)d_i) = 0 \Leftrightarrow m_i - (n+1)d_i \geq -l_i \Leftrightarrow m_i \geq m^\pi_i$ by choosing $\pi(i) = 1$. For any p as in the statement, $H^0(\mathbb{P}^{l_i}, m_i - pd_i) \neq 0 \Leftrightarrow m_i \geq pd_i$. Since $m_i - (n+1)d_i \geq -l_i$ it suffices to prove $(n+1)d_i - l_i \geq d_i(1 + \sum_{j \neq i} l_j) \geq d_i p$. The latter inequality is obvious for all p, whereas the former reduces to $l_i d_i \geq l_i$ which holds since $d_i \geq 1$. $\square$

Theorem 5.4. *A degree vector $m \in \mathbb{Z}^r_{\geq 0}$ gives a Sylvester-type matrix iff there exists a permutation π such that $m_j \geq m^\pi_j$ for $j = 1, \ldots, r$. Moreover, the smallest Sylvester matrix is attained among the vectors m^π.*

Proof. We prove the forward direction by induction on $k = 1, \ldots, r$. Assume m gives a Sylvester-type complex and consider the necessary condition $K_1(m) = H^0(X, m - d)$. The base case $k = 1$ was proven in Lemma 5.3. The inductive hypothesis for $k \in \{1, \ldots, r - 1\}$ specifies which cohomologies vanish and which not, where $m_u \geq m^\pi_u$, $\pi(u) \leq k$. In particular, for all subsets J such that $\emptyset \neq J \subset \{1, \ldots, r\} \backslash \{1, \ldots, k\}$, $p = 1 + \sum_{\pi(j) \in J} l_j$, $p_0 = p + l_v$, for some v such that $\pi(v) \leq k$, we assume:

$$H^{l_u}(\mathbb{P}^{l_u}, m_u - p_0 d_u) = 0, \qquad H^0(\mathbb{P}^{l_u}, m_u - pd_u) \neq 0. \tag{6}$$

For the inductive step, we exploit the necessary condition that $H^{p-1}(X, m - pd) = 0$ for $p = 1 + \sum_{\pi(j)>k} l_j$. By (6), there exists i such that $H^{l_i}\left(\mathbb{P}^{l_i}, m_i - pd_i\right) = 0$. Then, $m_i \geq pd_i - l_i = m_i^\pi$ where we define $\pi(i) = k + 1$. To complete the step, we show $H^0\left(\mathbb{P}^{l_u}, m_u - pd_u\right) \neq 0$ where $\pi(u) \leq k + 1$, $p = 1 + \sum_{\pi(j)\in J} l_j$, and any subset J such that $\emptyset \neq J \subset \{k + 2, \ldots, r\}$. The nonvanishing of the cohomology is equivalent to $m_u \geq pd_u$. It suffices to prove $m_i^\pi \geq d_i(1 + \sum_{\pi(j)>k+1} l_j)$. By definition, this reduces to $-l_i + d_i l_i \geq 0 \Leftrightarrow d_i \geq 1$. The converse direction follows from analogous arguments as above. The claim on minimality follows from Lemma 5.1. □

This gives an algorithm for finding the minimal Sylvester formulae by testing at most $r!$ vectors m^π, which is implemented in MAPLE (Section 8). To actually obtain the square submatrix whose determinant is divisible by the sparse resultant, it suffices to execute a rank test. These matrices exhibit quasi-Toeplitz structure, implying that asymptotic complexity is quasi-quadratic in the matrix dimension (Emiris and Pan, 2002). Observe that $P_k(m^\pi) \neq \emptyset$ because there exists $p \in \mathbb{Z}$ such that $p = 1 + \sum_{\pi(j)\leq\pi(k)} l_j$ such that $m_k^\pi < d_k p = m_k^\pi + l_k$ for all k.

Example 5.5. Let $l = (2, 1, 1)$, $d = (2, 2, 2)$; the degree of the resultant is 960. Let $\sigma = \pi^{-1}$ be the permutation inverse to π; then the corresponding degree vector can be written as $m^\pi_{\sigma(k)} := (1 + \sum_{j\geq k} l_{\sigma(j)})d_{\sigma(k)} - l_{\sigma(k)}$. Here is a list of the $6 = 3!$ degree vectors m^π, among which we find the smallest Sylvester matrix of row dimension 1080, whereas the sparse resultant's degree is 960. Also shown are the permutations σ and the corresponding matrix dimensions. The symmetry between the last two polynomials makes certain dimensions appear twice.

$$
\begin{array}{llll}
m^\pi = & (8,5,3) & \sigma = & (1,2,3) \quad 1080 \times 1120 \\
& (8,3,5) & & (1,3,2) \quad 1080 \times 1120 \\
& (6,9,3) & & (2,1,3) \quad 1120 \times 1200 \\
& (4,9,7) & & (2,3,1) \quad 1200 \times 1440 \\
& (6,3,9) & & (3,1,2) \quad 1120 \times 1200 \\
& (4,7,9) & & (3,2,1) \quad 1200 \times 1440.
\end{array}
$$

Our MAPLE program, discussed in Section 8, enumerates 81 purely rectangular Sylvester matrices (none of which is determinantal). All Sylvester matrices not shown here have dimensions 1260×1400 or larger. □

The map $K_1(m) \to K_0(m)$ is surjective, i.e. the matrix has at least as many columns as rows. In searching for a minimal formula, we should reduce $\dim K_0(m)$, i.e. the number of rows, since this defines the degree of the extraneous factor in the determinant. It is an open question whether $\dim K_0(m)$ reduces iff $\dim K_1(m)$ reduces. In certain system solving applications, the extraneous factor simply leads to a superset of the common isolated roots, so it poses no limitation. Even if it vanishes identically, perturbation techniques yield a nontrivial projection operator (D'Andrea and Emiris, 2001).

It is possible to obtain a pure Sylvester-matrix whose determinant equals the multihomogeneous resultant when the complex has as only nonzero terms $K_1(m) = H^j(X, m - (j + 1)d)^{\binom{n+1}{j+1}}$, and $K_0(m) = H^j(X, m - jd)^{\binom{n+1}{j}}$ for any $j = 0, \ldots, n$. But in this case we deduce from Theorem 4.5 that all defects vanish. So (cf. Sturmfels and Zelevinsky, 1994) there exists a pure Sylvester-type determinantal formula associated to a nonnegative degree vector (i.e. $m_k \geq 0$, $k = 1, \ldots, r$), or equivalently, for $j = 0$. Thus, if a determinantal data $(l_1, \ldots, l_r; d_1, \ldots, d_r)$ admits a degree vector yielding a two term Sylvester complex for some j, it admits such a formula for $j = 0$ as well. Hence, concentrating on $j = 0$ is not restrictive in the case of determinantal complexes. The Sylvester-type matrices for positive j correspond to degree vectors with some negative entries, unlike the assumption in Sturmfels and Zelevinsky (1994, p. 118). We show such an example in the bilinear case in Section 7. Thus, the first part of conjecture (Sturmfels and Zelevinsky, 1994, Conjecture 3) can be true only for nonnegative degree vectors.

6. Pure Bézout-type formulae

In this section, we will study the following complexes:

Definition 6.1. A Weyman complex is of pure Bézout type if $K_{-1}(m) = 0$, $K_1(m) = H^{l_1+\cdots+l_r}(X, m-(n+1)d)$ and $K_0(m) = H^0(m)$.

Weyman complexes of pure Bézout type correspond to generically surjective maps

$$H^{l_1+\cdots+l_r}(X, m-(n+1)d) \to H^0(m) \to 0 \tag{7}$$

such that any maximal minor is a nontrivial multiple of the multihomogeneous resultant. In fact, we shall show that the only possible such formulae are determinantal (i.e. $K_2(m) = 0$). We shall exhibit the corresponding differential in terms of the Bezoutian and characterize the possible degree vectors. We show that there exists a pure Bézout-type formula iff there exists a pure Sylvester formula. We remark that the dimension of the matrix with pure Bézout coefficients equals the dimension of the Sylvester matrix divided by $n+1$. Now we can generalize results in Chtcherba and Kapur (2000) and Saxena (1997) (cf. Section 1).

Theorem 6.2. *There exists a determinantal formula of pure Bézout type iff for all k either $l_k = 1$ or $d_k = 1$, i.e. all defects vanish.*

Proof. This is just a special instance of Theorem 4.5, where the only nonzero cohomologies in the complex correspond to $p_1 = 0$, $p_2 = n+1$. □

Let us study degree vectors yielding pure Bézout formulae, which will then provide the smallest determinantal formulae in case all defects vanish.

Definition 6.3. For each choice of a permutation $\pi : \{1, \ldots, r\} \to \{1, \ldots, r\}$, let us define a degree vector

$$m_k^\pi := -l_k + d_k \sum_{\pi(j) \geq \pi(k)} l_j, \qquad k = 1, \ldots, r.$$

When all $\delta_k = 0$ these are precisely the vectors defined in (5) in Theorem 4.2. Note that our assumptions in (7) imply $H^{l_j}(\mathbb{P}^{l_j}, m_j - (n+1)d_j) \neq 0$, $H^0(\mathbb{P}^{l_j}, m_j) \neq 0$.

Lemma 6.4. *The existence of any pure Bézout formula implies $0 \leq m_j < (n+1)d_j - l_j$, for all j.*

In fact, the m^π of Definition 6.3 satisfy these constraints for all permutations π.

Proof. The nonnegativity of all m_j is deduced from the fact that $H^0(X, m) \neq 0$. On the other side, the nonvanishing of $H^n(X, m - (n+1)d)$ implies the other inequality. $\square$

Lemma 6.5. *If $m \in \mathbb{Z}^r$ yields a pure Bézout-type complex, then there exists a permutation $\pi : [r] \to [r]$ such that $i = \pi^{-1}(1)$ verifies $m_i \geq m_i^\pi$ and*

$$H^{l_i}(\mathbb{P}^{l_i}, m_i - (q + l_i + \nu)d_i) = 0, \qquad \nu = 0, -1,$$
$$H^0(\mathbb{P}^{l_i}, m_i - qd_i) \neq 0, \qquad q = \sum_{j \in J} l_j,$$

for any $J \subset \{1, \ldots, r\} \backslash \{i\}$, $J \neq \emptyset$.

Proof. Since $H^n(X, m - nd) = 0$, there exists an index $i \in \{1, \ldots, r\}$ such that $m_i - nd_i \geq -l_i$. It is enough to define $\pi(i) = 1$. $\square$

Theorem 6.6. *If $m \in \mathbb{Z}^r$ yields a pure Bézout-type complex, it is possible to find a permutation π such that the degree vector m verifies $m_i \geq m_i^\pi$ for all $i = 1, \ldots, r$.*

Proof. We use induction; the base case follows from Lemma 6.5. The inductive hypothesis, for $k \in \{1, \ldots, r - 1\}$, is: there exists a subset $U \subset \{1, \ldots, r\}$, $|U| = k$, such that $\pi(u) \leq k$, $m_u \geq m_u^\pi$ for all $u \in U$ and

$$H^{l_u}(\mathbb{P}^{l_u}, m_u - (q + l_u + \nu)d_u) = 0, \qquad \nu = 0, -1,$$
$$H^0(\mathbb{P}^{l_u}, m_u - qd_u) \neq 0, \qquad q = \sum_{j \in J} l_j, \tag{8}$$

for all $J \subset \{1, \ldots, r\} \backslash U$, $J \neq \emptyset$. Now the inductive step: The hypothesis on K_0 implies $H^p(X, m - pd) = 0$ for $p = \sum_{j \notin U} l_j$. Considering the inequality in (8) for $q = p$, there exists $i \in [r] \backslash U$ such that $H^{l_i}(\mathbb{P}^{l_i}, m_i - pd_i) = 0 \Leftrightarrow m_i + l_i \geq pd_i$ i.e. $m_i \geq m_i^\pi$ for $\pi(i) = k + 1$ because $j \notin U \Leftrightarrow \pi(j) \geq k + 1$. It suffices now to extend (8) for $q' = \sum_{j \in J'} l_j$ where $\emptyset \neq J' \subset [r] \backslash (U \cup \{i\})$. First, $m_i + l_i \geq pd_i \geq (q' + l_i)d_i$ implies the equations below. Second, $m_i \geq -l_i + pd_i = (p - l_i)d_i + l_i(d_i - 1) \geq q'd_i$ yields the inequality, so

$$H^{l_i}(\mathbb{P}^{l_i}, m_i - (q' + l_i + \nu)d_i) = 0, \qquad \nu = 0, -1.$$
$$H^0(\mathbb{P}^{l_i}, m_i - q'd_i) \neq 0. \tag{9}$$

Then, m satisfies the hypothesis $K_{-1}(m) = 0$ for $\nu = -1$ because every summand in $K_{-1}(m)$ contains some cohomology as in (9). Since $p \geq 0 \Rightarrow q = p - \nu \geq 1$ no summand has only zero cohomologies. By Lemma 6.4 and (9) for $\nu = 0$, m gives $K_0(m) = H^0(X, m)$ because $H^{l_i}(\mathbb{P}^{l_i}, m_i - pd_i) = 0$ for any p, i. $\square$

Theorem 6.7. *A pure Bézout and generically surjective formula exists for some vector m iff it equals m^π of Definition 6.3, for some permutation π, and all defects are zero.*

Proof. It suffices to consider $K_1(m) = H^n(X, m-(n+1)d)$; it is nonzero by Lemma 6.4. We prove by induction that $m = m^\pi$ by using the fact that all other summands in (1) for $K_1(m)$ vanish. For $H^0(X, m-d)$ to vanish, there must exist $i \in [r]$ such that $H^0(\mathbb{P}^{l_i}, m_i - d_i) = 0 \Leftrightarrow m_i < d_i$. Hence we need to define $\pi(i) = r$ because $\pi(i) < r \Rightarrow m_i^\pi \geq -l_i + d_i(l_i+1) = d_i + l_i(d_i - 1) \geq d_i$. Moreover, $m_i^\pi = -l_i + d_i l_i < d_i \Leftrightarrow \delta_i = 0$ by Lemma 2.4.

There is a unique integer in $[m_i^\pi, d_i)$ because $m_i^\pi + 1 \geq d_i \Leftrightarrow -l_i + d_i l_i + 1 \geq d_i \Leftrightarrow (l_i - 1)(d_i - 1) \geq 0$. Hence $m_i = d_i - 1 < d_i(q+1)$ for any $q \geq 0$, therefore $H^0(\mathbb{P}^{l_i}, m_i - (q+1)d_i) = 0$. Furthermore, for $q \geq l_i$, $H^{l_i}(\mathbb{P}^{l_i}, m_i - (1+q)d_i) \neq 0 \Leftrightarrow m_i + l_i < (q+1)d_i \Leftrightarrow l_i - 1 < qd_i$ which holds. This proves the inductive basis. The inductive hypothesis is: for all $u \in U \subset [r]$, where $|U| = k$, $\pi(u) > r - k$, then $\delta_u = 0$, $m_u = m_u^\pi$ and

$$H^0\left(\mathbb{P}^{l_u}, m_u - \left(1 + \sum_{\pi(j)<\pi(u)} l_j\right) d_u\right) = 0 \neq H^{l_u}\left(\mathbb{P}^{l_u}, m_u - \left(1 + \sum_{j\in J} l_j\right) d_u\right), \tag{10}$$

for all J such that $U \subset J \subset [r]$. For the inductive step, consider that $H^q(X, m - (1+q)d)$ must vanish for $q = \sum_{j\in U} l_j$. None of its summand cohomologies $H^{l_u}\left(\mathbb{P}^{l_u}, m_u - (1+q)d_u\right)$ vanish due to the last inequality. So there exists i such that $H^0(\mathbb{P}^{l_i}, m_i - (1+q)d_i) = 0 \Leftrightarrow m_i < (1+q)d_i$.

Hence $\pi(i) = r - k$ so that $m_i = m_i^\pi = -l_i + d_i \sum_{\pi(j)\geq r-k} l_j < (1+q)d_i \Leftrightarrow -l_i + d_i l_i < d_i \Leftrightarrow \delta_i = 0$ by Lemma 2.4. No larger m_i works because m_i^π is the maximum integer strictly smaller than $(1+q)d_i$. And $\pi(i) < r - k$ would make m_i too large. Now extend the inequality (10) to J' where $(U \cup \{i\}) \subset J'$ and observe $m_i^\pi < d_i \sum_{\pi(j)\geq r-k} l_j < d_i(1 + \sum_{j\in J'} l_j)$.

The hypothesis is proven for all $U \subset [r]$, including the case $|U| = r$. For the converse, assume there exists a permutation π such that $m = m^\pi$ and all defects vanish. Then $K_0(m), K_1(m)$ satisfy all conditions for a pure Bézout formula. Furthermore, $K_{-1}(m) = 0$, hence the formula is generically surjective. □

The condition $K_2(m) = 0$, which yields a square matrix, is obtained by the hypothesis of a pure Bézout and generically surjective formula; i.e. there is no rectangular surjective pure Bézout formula.

Corollary 6.8. *If a generically surjective formula is of pure Bézout type, then it is determinantal. Furthermore, for any permutation π, the matrix is of the same dimension, i.e.* $\dim K_0(m) = \deg R/(n+1)$.

6.1. Explicit Bézout-type formulae

We start by defining a dual permutation π' to any permutation π.

Definition 6.9. For any permutation $\pi : [r] \to [r]$, define a new permutation $\pi' : [r] \to [r]$ by $\pi'(i) = r + 1 - \pi(i)$.

Lemma 6.10. *Assuming all defects are zero, $m^{\pi} + m^{\pi'} = \rho$ for any permutation $\pi : [r] \to [r]$, where $\rho \in \mathbb{N}^r$ is the critical vector of Definition 2.3.*

Proof. $m_i^{\pi} + m_i^{\pi'} = d_i(n + l_i) - 2l_i$ for all i because the sum in the parentheses includes $\{l_j : \pi(j) \geq \pi(i)\} \cup \{l_j : \pi'(j) \geq \pi'(i)\}$, and the latter set is $\{l_j : \pi(j) \leq \pi(i)\}$. So $m_i^{\pi} + m_i^{\pi'} = d_i l_i - l_i + \rho_i - d_i + 1 = \rho_i + d_i(l_i - 1) - (l_i - 1) = \rho_i + (l_i - 1)(d_i - 1) = \rho_i$ because of the zero defects. $\square$

Denote by x_i (resp. x_{ij}) the ith variable group (respectively the jth variable in the group), $i \in [r]$, $j = 0, \ldots, l_i$. Introduce r new groups of variables y_i with the same cardinalities and denote by y_{ij} their variables.

Given a permutation π, let the associated *Bezoutian* be the polynomial $B^{\pi}(x, y)$ obtained as follows: first dehomogenize the polynomials by setting $x_{i0} = 1, i = 1, \ldots, r$; the obtained polynomials are denoted by $f_0, \ldots, f_n$. Second, construct the $(n+1) \times (n+1)$ matrix with jth column corresponding to polynomial f_j, $j = 0, \ldots, n$, and whose x_{ij} variables are gradually substituted, in successive rows, by each respective y_{ij} variable. This construction is named after Bézout or Dixon and is well-known in the literature, e.g. Cardinal and Mourrain (1996) and Emiris and Mourrain (1999). A general entry is of the form

$$\begin{aligned} &f_j(y_{\sigma(1)}, \ldots, y_{\sigma(k-1)}, y_{\sigma(k)1}, \ldots, y_{\sigma(k)t}, \\ &\quad x_{\sigma(k)(t+1)}, \ldots, x_{\sigma(k)l_{\sigma(k)}}, x_{\sigma(k+1)}, \ldots, x_{\sigma(r)}) \end{aligned} \tag{11}$$

where $\sigma := \pi^{-1}, k = 0, \ldots, r, t = 1, \ldots, l_k$. There is a single first row for $k = 0$, containing all the polynomials in the x_{ij} variables, whereas the last row has the same polynomials with all variables substituted by the y_{ij}. All intermediate rows contain the polynomials in a subset of the x_{ij} variables, the rest having been substituted by each corresponding y_{ij}. The number of rows is $1 + \sum_{j \in [r]} l_j = 1 + n$. Lastly, in order to obtain $B^{\pi}(x, y)$, we divide the matrix determinant by

$$\prod_{i=1}^{r} \prod_{j=1}^{l_i} (x_{ij} - y_{ij}). \tag{12}$$

Example 6.11. Let $l = (1, 2), d = (2, 1)$. If $\pi = (12), \pi' = (21)$, then $m^{\pi} = (5, 0), m^{\pi'} = (1, 1)$. For both degree vectors, the matrix dimension is 6. To obtain $B^{\pi}(x, y)$ we construct a 4×4 matrix whose jth column contains $f_j(x_1, x_{21}, x_{22})$, $f_j(y_1, x_{21}, x_{22})$, $f_j(y_1, y_{21}, x_{22})$, $f_j(y_1, y_{21}, y_{22})$, for $j = 0, \ldots, 3$. Here x_1 (and y_1) is a shorthand for x_{11} (and y_{11}). Then $B^{\pi}(x, y)$ contains the following monomials in the x_i and y_i variables respectively, 6 in each set of variables: $1, x_1, x_{21}, x_{22}, x_1x_{21}, x_1x_{22}, 1, y_1, y_1^2, y_1^3, y_1^4, y_1^5$. So the final matrix is indeed square of dimension 6. More details on this example are provided in Section 8. □

Lemma 6.12. *Let $B^\pi(x, y) = \sum b_{\alpha\beta} x^\alpha y^\beta$ where $\alpha = (\alpha_{ij}), \beta = (\beta_{ij}) \in \mathbb{Z}^n$, $i = 1, \ldots, r$, $j = 1, \ldots, l_i$. Set $\alpha_i = \sum_{j=1,\ldots,l_i} \alpha_{ij}$, $\beta_i = \sum_{j=1,\ldots,l_i} \beta_{ij}$, for all α, β. Then, $0 \le \alpha_i \le m_i^{\pi'}$, $0 \le \beta_i \le m_i^\pi$ and $0 \le \alpha_i + \beta_i \le \rho_i$, $i = 1, \ldots, r$.*

Proof. By Lemma 6.10 it suffices to bound α_i, β_i. But α_i is the degree of the x_i in the determinant decreased by l_i in order to account for the division by (12). The former equals the product of d_i with the number of rows where an x_{ij} variable appears for any $j \in [1, l_i]$. These are the first row, the rows where y_j are introduced for $j \in \{\sigma(1), \ldots, \sigma(k-1)\}$ such that $\sigma(k) = i$, and another $l_i - 1$ rows when $\sigma(k) = i$. The condition on $j : \pi(j) < \pi(i)$ is equivalent to $r + 1 - \pi(j) > r + 1 - \pi(i)$, hence $\alpha_i \le -l_i + d_i \sum_{\pi'(j) \ge \pi'(i)} l_j = m_i^{\pi'}$. Similarly, we prove the upper bound on β_i. The rows containing y_{ij} for some $j \in [1, l_i]$ are those where $j \in \{\sigma(k+1), \ldots, \sigma(r)\} : \sigma(k) = i$, another $l_i - 1$ rows when $\sigma(k) = i$, and the last row. Now, $\pi(j) \ge k + 1 > k = \pi(i)$, so $\beta_i \le -l_i + d_i \sum_{\pi(j) \ge \pi(i)} l_j = m_i^\pi$. Clearly $\alpha_i, \beta_i \ge 0$. □

For generic polynomials, the upper bounds of α_i, β_i are attained. The lemma thus gives tight bounds on the support of the Bezoutian.

Theorem 6.13. *Assume all defects are zero and $B^\pi(x, y)$ is defined as above. For any π, $(b_{\alpha\beta})$ is a square matrix of dimension*

$$\dim K_0(m) = \binom{l}{l_1, \ldots, l_r} d_1^{l_1} \cdots d_r^{l_r} = \frac{\deg R}{(n+1)}.$$

Furthermore, $\det(b_{\alpha\beta}) = R(f_0, \ldots, f_n)$.

Proof. First, we show that $(b_{\alpha\beta})$ is square of the desired size. The dimensions are given by the number of exponent vectors α, β bounded by Lemma 6.12 which are exactly $\dim K_0(m^{\pi'})$, $\dim K_0(m^\pi)$ respectively. Both $m^\pi, m^{\pi'}$ are determinantal, hence both of these numbers are equal to $\deg R/(n+1)$, by Theorem 6.7 and Corollary 6.8. $R(f_0, \ldots, f_n)$ divides every nonzero maximal minor of the matrix $(b_{\alpha\beta})$; cf. Cardinal and Mourrain (1996) and Emiris and Mourrain (1999, Theorem 3.13). Since any nonzero proper minor has degree $< \deg R$, the determinant of the matrix $(b_{\alpha\beta})$ is nonzero and equals the resultant. □

Note that there is not a unique choice of higher differentials in the Weyman complexes. We could chase the arrows in a resultant spectral sequence as in Gelfand et al. (1994, Chapter 2, Proposition 5.4) to show that the matrix we propose comes from the explicitization of one possible choice. We have followed instead the more direct route based on the above property of the Bezoutian in Cardinal and Mourrain (1996), which uses more elementary tools.

7. Two examples

7.1. The bilinear system

The generic system of three bilinear polynomials is

$$f_0 = a_0 + a_1x_1 + a_2x_2 + a_3x_1x_2,$$
$$f_1 = b_0 + b_1x_1 + b_2x_2 + b_3x_1x_2,$$
$$f_2 = c_0 + c_1x_1 + c_2x_2 + c_3x_1x_2,$$

and has type $(1, 1; 1, 1)$. The degree of the resultant in the coefficients is $3\binom{2}{1,1} = 6$. We shall enumerate all 14 possible determinantal formulae in the order of decreasing matrix dimension, from 6 to 2, and shall make the corresponding maps explicit. This study goes back to the pioneering work of Dixon (1908).

For $\pi = (1, 2)$, Definition 5.2 yields $m = (2, 1)$ and the complex is $0 \to K_1 = H^0(1, 0)^{\binom{3}{1}} \to K_0 = H^0(2, 1) \to 0$. The corresponding determinantal pure Sylvester matrix is, when transposed, equal to

$$\begin{bmatrix} a_0 & a_1 & a_2 & a_3 & 0 & 0 \\ b_0 & b_1 & b_2 & b_3 & 0 & 0 \\ c_0 & c_1 & c_2 & c_3 & 0 & 0 \\ 0 & a_0 & 0 & a_2 & a_1 & a_3 \\ 0 & b_0 & 0 & b_2 & b_1 & b_3 \\ 0 & c_0 & 0 & c_2 & c_1 & c_3 \end{bmatrix},$$

with rows corresponding to the input polynomials and the same set multiplied by x_1, whereas the columns are indexed by $1, x_1, x_2, x_1x_2, x_1^2, x_1^2x_2$. By symmetry, another formula is possible by interchanging the roles of x_1, x_2. Further formulae are obtained by taking the transpose of these two matrices, namely with $m = (-1, 0)$, where the complex is $H^2(-4, -3) = H^0(2, 1)^* \to (H^2(-3, -2))^3 = (H^0(1, 0)^*)^3$, and with $m = (0, -1)$. Recall the definition of duality from equation (2). Sylvester maps, as well as other types of maps, are further illustrated in Section 7.2.

There are additional determinantal Sylvester formulae corresponding to $m = (2, -1)$ and $m = (-1, 2)$. Their matrices contain the f_i and the f_i multiplied by x_1^{-1} or by x_2^{-1}. In the former case, the complex is $H^1(0, -3)^3 = (H^0(0) \otimes H^0(1)^*)^3 \to H^1(1, -2)^3 = (H^0(1) \otimes H^0(0)^*)^3$, the transposed matrix is

$$\begin{bmatrix} a_0 & a_1 & a_2 & a_3 & 0 & 0 \\ b_0 & b_1 & b_2 & b_3 & 0 & 0 \\ c_0 & c_1 & c_2 & c_3 & 0 & 0 \\ a_1 & 0 & a_3 & 0 & a_0 & a_2 \\ b_1 & 0 & b_3 & 0 & b_0 & b_2 \\ c_1 & 0 & c_3 & 0 & c_0 & c_2 \end{bmatrix},$$

and the columns are indexed by $1, x_1, x_2, x_1x_2, x_1^{-1}, x_1^{-1}x_2$. This construction can be verified by hand calculations.

For $m = (1, 1)$ the complex becomes $0 \to K_1 = H^0(0, 0)^{\binom{3}{1}} \oplus H^2(-2, -2) \to K_0 = H^0(1, 1) \to 0$. The matrix is square, of dimension equal to 4, and hybrid. We compute the two maps by hand; for a larger example see Section 7. The foundations for constructing such matrices can be found in Weyman and Zelevinsky (1994). The transposed 4×4 determinantal formula is written as follows, by using brackets:

$$\begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ c_0 & c_1 & c_2 & c_3 \\ [012] & [013] & [032] & -[123] \end{bmatrix}, \qquad \text{where } [ijk] = \det \begin{bmatrix} a_i & a_j & a_k \\ b_i & b_j & b_k \\ c_i & c_j & c_k \end{bmatrix}.$$

The matrix rows contain the f_i and a rational multiple of the affine toric Jacobian, whereas the columns are indexed by $1, x_1, x_2, x_1x_2$. This formula is obtained in Cattani et al. (1998) in a more general toric setting. An analogous 4×4 matrix corresponds to $m = (0, 0)$.

There are four "partial Bézout" determinantal formulae of dimension 3×3 for $m = (-1, 1), (1, -1)$ and for $m = (2, 0), (0, 2)$. We omit the details of the computation. In the first case, the complex is $H^2(-4, -2) = H^0(2, 0)^* \to H^1(-2, 0)^3 = (H^0(0)^* \otimes H^0(0))^3$, and a choice of the matrix is, in terms of brackets,

$$\begin{bmatrix} [-02] & [-03] + [-12] & [-13] \\ [0-2] & [0-3] + [1-2] & [1-3] \\ [02-] & [12-] + [03-] & [13-] \end{bmatrix}, \qquad \text{where } [ij-] = \det \begin{bmatrix} a_i & a_j \\ b_i & b_j \end{bmatrix},$$

and analogously for the 2×2 brackets $[i - k]$, $[-jk]$. The columns of this resultant matrix are indexed by $1, x_1, x_1^2$, which is the support of the three Bezoutian polynomials filling in the rows. In particular, these polynomials are defined for $\{i, j, k\} = \{0, 1, 2\}$ in the standard way:

$$B_k = \det \begin{bmatrix} f_i(x_1, x_2) & f_i(x_1, y_2) \\ f_j(x_1, x_2) & f_j(x_1, y_2) \end{bmatrix} \bigg/ (x_2 - y_2).$$

For $m = (1, 0)$ the complex becomes $0 \to K_1 = H^2(-2, -3)^{\binom{3}{3}} = H^0(0, 1)^* \to K_0 = H^0(1, 0) \to 0$. The corresponding determinantal pure Bézout-type formula is obtained from the Bezoutian polynomial

$$B = \det \begin{bmatrix} f_0(x_1, x_2) & f_0(y_1, x_2) & f_0(y_1, y_2) \\ f_1(x_1, x_2) & f_1(y_1, x_2) & f_1(y_1, y_2) \\ f_2(x_1, x_2) & f_2(y_1, x_2) & f_2(y_1, y_2) \end{bmatrix} \bigg/ (x_1 - y_1)(x_2 - y_2),$$

supported by $\{1, x_2\}, \{1, y_1\}$. The resultant matrix is given in terms of brackets as follows:

$$\begin{bmatrix} [123] & [023] \\ -[103] & [012] \end{bmatrix}.$$

7.2. A hybrid determinantal formula

Assume $l = (3, 2)$, $d = (2, 3)$. We present explicit formulae which can be extrapolated in general, giving an answer to the problem stated in Weyman and Zelevinsky (1994, p. 578). We plan to carry this extensively in a future work, but we include here the example without proofs as a hint for the interested reader. Our MAPLE program enumerates 30 determinantal vectors m, among which we find $m^{(2,1)} = (3, 13)$ according to Theorem 4.1.

The minimal matrix dimension is 1320 and is achieved at $m = (6, 3)$ and $(2, 12)$. In both cases, $P_2(m) = \emptyset$, whereas $P_1(6, 3) = \{4\}$ and $P_1(2, 12) = \{2\}$. This shows that the minimum matrix dimension may occur for some empty P_k, contrary to what one may think.

Moreover, the degree of the sparse resultant is $6\binom{5}{3,2}2^3 3^2 = 4320$. Since 1320 does not divide 4320, the minimal matrix is not of pure Bézout type; it is not of pure Sylvester type either. To specify the cohomologies and the linear maps that make the matrix formula explicit we compute, for the degree vector $m = (6, 3)$ and $p = 1, \ldots, 6$ the different values of $m - pd$: $(4, 0), (2, -3), (0, -6), (-2, -9), (-4, -12), (-6, -15)$. The complex becomes $K_2 = 0 \to K_1 \to K_0 \to K_{-1} = 0$, with nonzero part

$$\begin{aligned} &H^0(4, 0)^{\binom{6}{1}} \oplus H^2(0, -6)^{\binom{6}{3}} \oplus H^5(-6, -15)^{\binom{6}{6}} \\ &\quad \to H^0(6, 3)^{\binom{6}{0}} \oplus H^2(2, -3)^{\binom{6}{2}} \oplus H^5(-4, -12)^{\binom{6}{5}}, \end{aligned}$$

where we omitted the reference to the space $X = \mathbb{P}^3 \times \mathbb{P}^2$ in the notation of the cohomologies. Then $\dim K_1 = 210 + 200 + 910 = 1320 = 840 + 150 + 330 = \dim K_0$. By a slight abuse of notation, let $\delta_{\alpha,\beta}$ stand for the restriction of the above map to $H^\alpha \to H^\beta$. Then $\delta_{02} = \delta_{05} = \delta_{25} = 0$ by Weyman and Zelevinsky (1994, Proposition 2.5) and it suffices to study the maps below, of which the first three are of pure Sylvester type by Weyman and Zelevinsky (1994, Propostion 2.6) and the last three are of pure Bézout type as those of Section 6. These maps can be simplified using the dual cohomologies:

$$H^j(\mathbb{P}^{l_k}, m_k - pd_k) = H^{l_k - j}(\mathbb{P}^{l_k}, (\rho_k - m_k) - (n + 1 - p)d_k)^*,$$

where ρ is the critical vector of Definition 2.3. So, we have maps

$$\begin{aligned} \delta_{00} &: \ H^0(4, 0)^6 \to H^0(6, 3) \\ \delta_{22} &: \ (H^0(0) \otimes H^0(3)^*)^{\binom{6}{3}} \to (H^0(2) \otimes H^0(0)^*)^{\binom{6}{2}} \\ \delta_{55} &: \ H^0(2, 12)^* \to (H^0(0, 9)^*)^6 \\ \delta_{20} &: \ (H^0(0) \otimes H^0(3)^*)^{\binom{6}{3}} \to H^0(6, 3) \\ \delta_{50} &: \ H^0(2, 12)^* \to H^0(6, 3) \\ \delta_{52} &: \ H^0(2, 12)^* \to (H^0(2) \otimes H^0(0)^*)^{\binom{6}{2}}. \end{aligned}$$

The resultant matrix (of the previous map in the natural monomial bases) has the following aspect, indicated by the row and column dimensions:

$$\begin{array}{c c} & \begin{array}{ccc} 840 & 150 & 330 \end{array} \\ \begin{array}{c} 210 \\ 220 \\ 910 \end{array} & \begin{bmatrix} \delta_{00} & 0 & 0 \\ \delta_{20} & \delta_{22} & 0 \\ \delta_{50} & \delta_{52} & \delta_{55} \end{bmatrix} \end{array} = \begin{bmatrix} S_{00} & 0 & 0 \\ B_{20}^{x_2} & \delta_{22} & 0 \\ B_{50} & B_{52}^{x_1} & S_{55}^T \end{bmatrix}$$

where S_{ij}, B_{ij}, $B_{ij}^{x_k}$ stand for pure Sylvester and Bézout blocks, the latter coming from a Bezoutian with respect to variables x_k for $k = 1, 2$, and S_{55}^{T} represents a transposed Sylvester matrix, corresponding to the dual of the Sylvester map $H^0(0, 9)^6 \to H^0(2, 12)$.

Table 1
The main functionalities of our software

Routine	Function
`comp_m`	Compute the degree vector m by some specified formula
`allDetVecs`	Enumerate all determinantal formulae
`allsums`	Compute all possible sums of the l_i's adding to $q \in \{0, \ldots, \sum_{i=1}^{r} l_i\}$
`coHzero`	Test whether $H^q(X, m - pd)$ vanishes
`coHdim`	Compute the dimension of $H^q(X, m - pd)$
`dimKv`	Compute the dimension of K_ν i.e. of the corresponding matrix
`findBez`	Find all m-vectors yielding a pure Bézout-type formula
`findSyl`	Find all m-vectors yielding a pure Sylvester-type formula
`minSyl`	Find all m^π-vectors yielding a pure Sylvester-type formula
`hasdeterm`	Test whether a determinantal formula exists

Let us take a closer look at δ_{22}, which denotes both the map and the corresponding matrix. Let $\alpha \in \mathbb{N}^{l_1}, \beta \in \mathbb{N}^{l_2}$, be the degree vectors of the elements of $H^0(2), H^0(3)^*$ respectively, thus $|\alpha| \le 2, |\beta| \le 3$. Let $I, J \subset \{0, \ldots, 5\}, |I| = 3, |J| = 2$ express the chosen polynomials according to the cohomology exponents. Then the entries are given by

$$\delta_{22}(x_1^\alpha \otimes T_J, 1 \otimes S_I^\beta) = \begin{cases} 0, & \text{if } J \not\subset I, \\ \text{coef}(f_k) \text{ of } x_1^\alpha x_2^\beta, & \text{if } I \backslash J = \{k\}, \end{cases}$$

where $T_J \in H^0(0)^*, S_I^\beta$ are elements of the respective dual bases of monomials. We expect such a construction to be generalizable, but such a proof would be part of future work.

Now take the Bézout maps: the matrix entries are given in (11) for $\sigma = (2, 1)$: the entry $(i, j), i, j \in \{0, \ldots, 5\}$ contains $f_j(x^{(1)}, \ldots, x^{(5-i)}, y^{(6-i)}, \ldots, y^{(5)})$, where each $x^{(i)}$ is a leading subsequence of $x_{11}, x_{12}, x_{13}, x_{21}, x_{22}$; similarly with the new variables $y^{(i)}$. The degree of the determinant, i.e. the Bezoutian, is $6, 3, 2, 12$ in x_1, x_2, y_1, y_2 respectively and these coefficients fill in the matrix B_{50}. For the Bézout block $B_{52}^{x_1}$, consider "partial" Bezoutians defined from the six polynomials with the exception of those indexed in J, where J, I are as above. Only the x_1 variables are substituted by new ones, thus yielding a 4×4 matrix. For $B_{20}^{x_2}$, take all polynomials indexed in I and develop the Bezoutian with new variables y_2 from a 3×3 matrix. Hence the entries of the Bézout blocks have, respectively, degree $6, 4, 3$ in the coefficients of the f_i.

8. Implementation

We have implemented on MAPLE V routines for the above operations, including those in Table 1. They are illustrated below and are available in file `mhomo.mpl` through: http://www.di.uoa.gr/~emiris/index-eng.html.

Example 3.2 (Continued). Recall that $l = (1, 2), d = (2, 3)$ and let $m = (6, 3)$:

```
> Ns:=vector([1,2]): Ds:=vector([2,3]):
> summs:=allsums(Ns):
> hasdeterm(Ns,Ds,vector([6,3]),summs);
```

true

```
> dimKv(Ns,Ds,vector([6,3]),summs,1);
```

$$88$$

```
> dimKv(Ns,Ds,vector([6,3]),summs,0);
```

$$88 \quad \square$$

Example 5.5 (Continued). Recall that $l = (2, 1, 1)$, $d = (2, 2, 2)$, then $\delta = (1, 0, 0)$. The MAPLE session first computes all 81 pure Sylvester formulae by searching the appropriate range of 246 vectors. The smallest formulae are shown.

```
> Ns:=vector([2,1,1]):Ds:=vector([2,2,2]):
> minSyl(Ns,Ds):
```

list of minimal S-matrices: m-vector and K1, K0-dims

[[8, 5, 3, 1120, 1080], [8, 3, 5, 1120, 1080], [6, 9, 3, 1200, 1120],
[6, 3, 9, 1200, 1120], [4, 9, 7, 1440, 1200], [4, 7, 9, 1440, 1200]]

```
> allSyl:=findSyl(Ns,Ds):
Search of degree vecs from [4,3,3] to [8,9,9].
First array [4,7,9]: dimK1=1440, dimK0=1200,
dimK(-1)=0(should be 0).
```

#pure-Sylvester degree vectors = 81

```
tried 246, got 81 pure-Sylv formulae [m,dimK1,dimK0]:
> sort(convert(%,list),sort_fnc);
```

[[8, 5, 3, 1120, 1080], [8, 3, 5, 1120, 1080],
[6, 9, 3, 1200, 1120], [6, 3, 9, 1200, 1120],
[4, 9, 7, 1440, 1200], [4, 7, 9, 1440, 1200],
[8, 6, 3, 1400, 1260], [8, 3, 6, 1400, 1260], . . .]

```
> allDetVecs(Ns,Ds):
> allmsrtd := sort(convert(%,list),sort_fnc);
```

From, $[-4, -3, -3]$, *to*, $[9, 10, 10]$, *start at*, $[-4, -3, -3]$

```
Tested 1452 m-vectors: assuming Pk's nonempty.
Found 488 det'l m-vecs, listed with matrix dim:
```

allmsrtd := [[6, 3, 1, 224], [6, 1, 3, 224], [1, 7, 5, 224],
[1, 5, 7, 224], [3, 7, 1, 240], [4, 7, 1, 240], [3, 1, 7, 240],
[4, 1, 7, 240], [6, 3, 0, 262], [6, 0, 3, 262], [1, 8, 5, 262],
[1, 5, 8, 262], . . .]

```
> for i from 1 to nops( allmsrtd ) do print (
allmsrtd[i],Pksets(Ns,Ds,vector([allmsrtd[i][1],
allmsrtd[i][2],allmsrtd[i][3]]))): od:
```

$[6, 3, 1, 224], [[4, 4], [2, 2], [1, 1]]$
$[6, 1, 3, 224], [[4, 4], [1, 1], [2, 2]]$
$[1, 7, 5, 224], [[1, 1], [4, 4], [3, 3]]$
$[1, 5, 7, 224], [[1, 1], [3, 3], [4, 4]]$
$[3, 7, 1, 240], [[2, 2], [4, 4], [1, 1]]$
$[4, 7, 1, 240], [[3, 3], [4, 4], [1, 1]]$
$[3, 1, 7, 240], [[2, 2], [1, 1], [4, 4]]$
$[4, 1, 7, 240], [[3, 3], [1, 1], [4, 4]]$
$[6, 3, 0, 262], [[4, 4], [2, 2], [0, \mathit{NO\_INT}, 0]]$
$\dots$

The last two commands find all 488 determinantal vectors. The smallest formulae are indicated (the minimum dimension is 224) and for some we report the P_k's. No determinantal formulae is pure Sylvester. Notice that the assumption of empty sets P_k is used only in order to bound the search, but within the appropriate range empty P_k's are considered, so no valid degree vector is missed. This is illustrated by the last $P_3 = \emptyset$ marked *NO_INT*.

The vectors predicted by Theorem 4.1 are found among those produced above, including $m^{(123)} = (6, 5, 3), m^{(213)} = (4, 9, 3), m^{(312)} = (0, 9, 7)$. The corresponding matrix dimensions are 672, 600, and 800. □

Example 6.11 (Continued). Recall that $l = (1, 2), d = (2, 1)$. The only pure Bézout formulae are the two determinantal formulae of Example 6.11, for which we have $m^{\pi} = (5, 0), m^{\pi'} = (1, 1)$.

```
> Ns:=vector([1,2]):Ds:=vector([2,1]):
> summs:=allsums(Ns):
> findBez(Ns,Ds,true); #not only determinantal
```

$$\textit{low} - \textit{upper bounds},\ \textit{1st candidate} :, [0, 0], [6, 1], [0, 0]$$

```
Searched degree m-vecs for ANY pure Bezout formula.
Tested 15, found 2 pure-Bezout [m,dimK0,dimK1]:
```

$$\{[5, 0, 6, 6], [1, 1, 6, 6]\}$$

The search examined 15 degree vectors between the shown bounds. It is clear that both vectors are determinantal because the matrix dimensions are for both 6×6. □

9. Further work

Our results can be generalized to polynomials with scaled supports or with a different degree d per polynomial. One question is whether the vectors m', m'' of Definition 3.3 lead to smaller or larger matrices than m. Notice that certain cohomologies, which were nonzero for m, may vanish for m' or m''. We plan to complete the description of hybrid determinantal formulae. We would also like to answer in general the question stated in Section 4 of determining a priori the degree vectors yielding the smallest determinantal formulae in all possible cases. A problem related to the Sylvester formulae calls for identifying in advance the nonzero maximal minor in the matrix, which leads to finding a determinant with exact degree in some polynomial.

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