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A Note on the Packing of Two Copies of Some Trees into Their Third Power

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Abstract—It is proved in [1] that if a tree T of order n is not a star, then there exists an edge-disjoint placement of two copies of this tree into its fourth power.

In this paper, we prove the packing of some trees into their third power.

Keywords—Embedding, Packing, Permutation, Placement, Power tree.

1. INTRODUCTION

Suppose $G_1, \dots, G_k$ are graphs of order n. We say that there is a *packing* of $G_1, \dots, G_k$ (into the complete graph K_n) if there exist injections $\alpha_i : V(G_i) \to V(K_n)$, $i = 1, \dots, k$, such that $\alpha_i^*(E(G_i)) \cap \alpha_j^*(E(G_j)) = \emptyset$ for $i \neq j$, where the map $\alpha_i^* : E(G_i) \to E(K_n)$ is the one induced by α_i.

A packing of k copies of a graph G will be called a *k-placement* of G. A packing of two copies of G (i.e., a two-placement) is an *embedding* of G (in its complement $\bar{G}$). So, an embedding of a graph G is a permutation σ on $V(G)$ such that if an edge xy belongs to $E(G)$ then $\sigma(x)\sigma(y)$ does not belong to $E(G)$. If there exists an embedding of G, we say that G is *embeddable.*

Let G be a graph. The simple graph G^p, with $p \geq 1$, is constructed from G by adding edge between any pair of vertices at distance at most p on G. Obviously, $G^1 = G$. We denote the distance between two vertices x, y on G by $\text{dist}_G(x, y)$.

The following theorem was proved, independently, in [2–4].

THEOREM 1. *Let $G = (V, E)$ be a graph of order n. If $|E(G)| \leq n - 2$, then G can be embedded in its complement.* ∎

This result has been improved in many ways. The main references of this paper and of other packing problems are to be found in the last chapter of Bollobás's book [2], the fourth chapter of Yap's book [5], and the survey paper [6].

In this paper, we shall consider the case in which G is a tree on n vertices. The example of the star S_n shows that Theorem 1 cannot be improved by raising the size of G even in the case when G is a tree. However, in that case, we have the following result.

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THEOREM 2. *Let T be a tree of order n, $T \neq S_n$. Then T is contained in its own complement.* ∎

Theorem 2 was first proved by Straight (unpublished, cf. [7]). Besides, this result has been improved in many ways. For instance, the packing of two trees was considered in [7] and the three-placement of a tree in [8].

Another possibility of improving Theorem 2 is to consider some additional information about embeddings, i.e., packing permutations.

An example of such a result is the following theorem contained as a lemma in [9] (cf. also [10]).

THEOREM 3. *Let T be a nonstar tree of order n, with $n > 3$. Then there exists a two-placement σ of T such that for every $x \in V(T)$, $\mathrm{dist}_T(x, \sigma(x)) \leq 3$.* ∎

This theorem immediately implies the following corollary.

COROLLARY 4. *Let T be a nonstar tree of order n, with $n > 3$. Then there exists an embedding σ of T such that $\sigma(T) \subset T^7$.*

Since T^7 is, in general, a proper subgraph of K_n, the last corollary can be considered as an improvement of Theorem 2.

Kheddouci *et al.* considered the problem of the two-placement of a tree T into T^p such that p is as small as possible. In [1], they proved the following result.

THEOREM 5. *Let T be a nonstar tree of order n, with $n > 3$. Then there exists a two-placement σ of T such that $\sigma(T) \subset T^4$.*

And they posed the following conjecture.

CONJECTURE 6. *Let T be a nonstar tree of order n, with $n > 3$. Then there exists a two-placement σ of T such that $\sigma(T) \subset T^3$.*

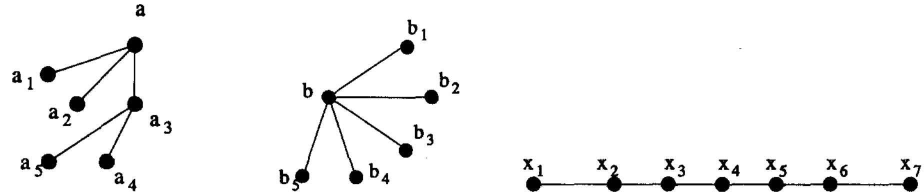
Note that the square of a path of order n contains only $(n-2)$ more edges. So, it is not possible in general to pack two copies of a tree T into T^2.

For the placement in T^3, the problem seems more difficult. It is due to some vertices which remain fixed by any possible permutation. Note that, for instance, the vertex x_4 of $P_7 = (x_1, x_2, \dots, x_7)$ remains always fixed for all possible permutations in P_7^3. It is clear that is not possible to make a placement of the tree T, gotten by two paths of order 7 joined by an edge between their fourth vertices, in T^3 by composition of two path permutations. In any case, the image of such an edge (that joins the fourth two vertices) remains itself.

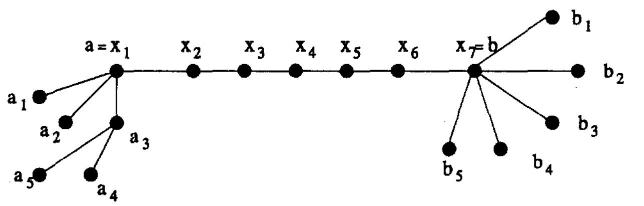
In this note, we prove Conjecture 6 for some families of tree for which the placement is not very difficult to do. In particular, we give the placement of a path, a star-path-star and a tree without vertices of degree two. During this modest study we noticed that some vertices of degree 2 tend to remain fixed and block the placement in the third power!

We shall need some additional definitions and notations in order to formulate our results.

Let x be a vertex of a tree T. The components of $T-x$ are called *neighbor trees* of x. If y is any neighbor of x in T, we denote by T_y the neighbor tree of x which contains y. Consequently, if we delete an edge $e = xy$ of T, we obtain two components of $T-e$, respectively, the neighbor tree T_x of y and the neighbor tree T_y of x. For each vertex $x \in V(T)$, if $d_T(x) \geq 2$ then x is an *internal* vertex. Consider two distinct rooted trees T' and T'', together with the path $(x_1, x_2, \dots, x_p)$. We denote $T = T' \cdot (x_1, x_2, \dots, x_p) \cdot T''$ the tree obtained by identifying x_1 with the root of the first tree, and x_p with the root of the second tree (see Figure 1). By doing this, the first tree becomes the neighbor tree T_{x_1} of x_2, and the second tree becomes the neighbor tree T_{x_p} of x_{p-1}. In the particular case where the first tree of this construction is reduced to a single vertex (thus, identified with x_1), we simplify the notation into $(x_1, x_2, \dots, x_p) \cdot T_b''$. A special case of this construction is exemplified by the star-path-star $S' \cdot (x_1, \dots, x_p) \cdot S''$, where S' and S'' are two stars with centers, respectively, x_1 and x_p. For $p = 2$ we call this tree a double-star.



(a) A tree T rooted on a. (b) A star S rooted on the center b. (c) A path P with end-vertices x_1 and x_7.



(d) A tree $T \cdot P \cdot S$ obtained by identifying a with x_1 and b with x_7.

Figure 1.

Let P be a path of end-vertices x and y. A permutation σ on $V(P)$ is said to be *P-nice* iff the following four conditions are satisfied:

(1) σ is a two-placement of P;
(2) $\sigma(P) \subset P^3$;
(3) $\mathrm{dist}_P(x, \sigma(x)) = 1$;
(4) $\mathrm{dist}_P(y, \sigma(y)) \in \{0, 1\}$.

The path P is said to be *nice* if there exists a P-nice permutation.

Let T be a tree of order at least three. Let $x \in V(T)$ be an internal vertex. A permutation σ on $V(T)$ is said to be (T,x)*-good* iff the following four conditions are satisfied:

(1) σ is a two-placement of T;
(2) $\sigma(T) \subset T^3$;
(3) $\mathrm{dist}_T(x, \sigma(x)) = 1$;
(4) for every end-vertex y of T, $\mathrm{dist}_T(y, \sigma(y)) = 2$.

The tree itself is said to be x*-good* if there exists a (T,x)-good permutation. Finally, T is called *good* if it is x-good for every internal vertex $x \in V(T)$.

By using this terminology, we shall prove the following propositions.

PROPOSITION 7. *All paths of order n, with $n > 3$, are nice.*

In the case of star-path-star we prove the following result.

PROPOSITION 8. *Let $T = S_1 \cdot P \cdot S_2$ be a star-path-star such that S_i, with $i = 1,2$, has at least one edge and P is at least of length 1. Then there exists a two-placement σ of T such that $\sigma(T) \subset T^3$.*

The main result of this paper is the following.

PROPOSITION 9. *Let T be a nonstar tree of order n, with $n > 5$, having no vertex of degree 2. Then T is good.*

PROPOSITION 10. *Let T be a nonstar tree of order n rooted on a vertex r or r', with $rr' \in E(T)$. If $d(r)$, $d(r') \geq 2$ and for each $x \in V(T) \setminus \{r, r'\}$, $d(x) \neq 2$, then T is r-good and r'-good.*

We call a k-ary tree ($k \geq 2$) a rooted tree on a vertex of degree k and all other vertices are of degree 1 or $k+1$. Proposition 10 implies the following corollary.

COROLLARY 11. *Let T be a nonstar k-ary tree, with $k \geq 2$, rooted on r. Then T is r-good.*

In the following result, we accept more vertices of degree 2.

PROPOSITION 12. *For $i = 1, 2$, let T_i be a rooted tree on an internal vertex such that the size of T_i is at least three and $d_{T_i}(x) \neq 2$ for each $x \in V(T_i)$. Let P be a path of order p, with $p \geq 2$. Then there exists a two-placement σ of $T = T_1 \cdot P \cdot T_2$, such that $\sigma(T) \subset T^3$.*

We start with a sequence of lemmas (Section 2) that we use in the main part of the proof given in Section 3.

2. LEMMAS

In this section, we show some basic lemmas needed to prove our propositions.

LEMMA 13. *The paths of order 4, 5, 6, and 7 are nice.*

PROOF. Let $n \in \{4, 5, 6, 7\}$. Let $P = (x_1, x_2, \dots, x_n)$ be a path of order n. A nice permutation σ_n associated to this path is given below (we use standard notation for its decomposition into cycles):

$$\begin{aligned}
\sigma_4 &= (x_1 \quad x_2 \quad x_4 \quad x_3);\\
\sigma_5 &= (x_1 \quad x_2 \quad x_4 \quad x_3)(x_5);\\
\sigma_6 &= (x_1 \quad x_2 \quad x_5 \quad x_4)(x_3)(x_6);\\
\sigma_7 &= (x_1 \quad x_2 \quad x_5)(x_3 \quad x_7 \quad x_6)(x_4);
\end{aligned}$$

Note that $\operatorname{dist}(x_1, \sigma_n(x_1)) = 1$ and $\operatorname{dist}(x_n, \sigma_n(x_n)) \in \{0, 1\}$, for each $4 \leq n \leq 7$. ∎

LEMMA 14. *Let S be a star at least of size two centered on x_1 and $P = (x_1, x_2, \dots, x_p)$, with $3 \leq p \leq 5$. Let $T = S \cdot P$. Then there exists a two-placement σ of T such that $\sigma(T) \subset T^3$ and $\operatorname{dist}(x_p, \sigma(x_p)) = 1$.*

PROOF. Let $\{z_1, z_2, \dots, z_t\}$ be the end-vertices of S. By hypothesis $t \geq 2$, since $p \geq 3$, then T is not a star. If $p = 3$, let $P' = (z_1, x_1, x_2, x_3)$. By Lemma 13, there exists a P'-nice permutation σ' such that $\operatorname{dist}(z_1, \sigma'(z_1)) = 1$, $\operatorname{dist}(x_1, \sigma'(x_1)) = 2$, and $\operatorname{dist}(x_3, \sigma'(x_3)) = 1$. We may extend this permutation to the required permutation σ' of $V(T)$ by putting $\sigma(z_i) = z_i$, for each $2 \leq i \leq t$. It is easy to see that $\operatorname{dist}(\sigma(z_i), \sigma(x_1)) \leq 3$, for each $1 \leq i \leq t$.

For $p = 4$, by Lemma 13, P is nice and there exists a P-nice permutation σ' such that $\operatorname{dist}(x_4, \sigma'(x_4)) = 1$. So the same corresponding permutation may be extended to T by the cycle $(z_1 z_2, \dots z_t)$.

In case $p = 5$, let $P' = (z_1, x_1, \dots, x_5)$. By Lemma 13, there exists a P'-nice permutation σ' such that $\operatorname{dist}(z_1, \sigma'(z_1)) = 0$ and $\operatorname{dist}(x_5, \sigma'(x_5)) = 1$. Therefore, $1 \leq \operatorname{dist}(x_1, \sigma'(x_1)) \leq 2$. We extend this permutation to T by putting $\sigma(z_i) = z_i$, for each $2 \leq i \leq t$. ∎

LEMMA 15. *Let T be a double-star. If the degree of each internal vertex is at least two, then T is good.*

PROOF. Let x and y be the two internal vertices T. Let $N(y) = \{x, y_1, y_2, \dots, y_p\}$ and $N(x) = \{y, x_1, x_2, \dots, x_{p'}\}$, with $p, p' \geq 2$ (see Figure 2). We give σ by the following cycle $(x\ x_1\ x_2 \dots x_{p'}\ y\ y_1\ y_2 \dots y_p)$. Thus, $\operatorname{dist}(x, \sigma(x)) = \operatorname{dist}(y, \sigma(y)) = 1$, $\operatorname{dist}(x_i, \sigma(x_i)) = 2$ for each $1 \leq i \leq p'$ and $\operatorname{dist}(y_i, \sigma(y_i)) = 2$ for each $1 \leq i \leq p$. Therefore, T is x-good and y-good, so T is good. ∎

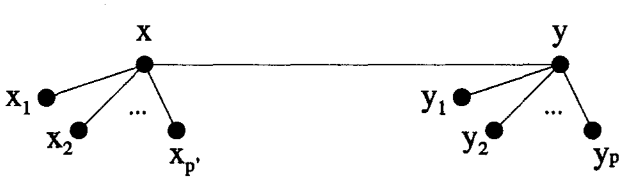


Figure 2. A double-star with centers x and y.

3. PROOFS OF PROPOSITIONS

In the following proofs, the verification for any two-placement σ of a tree T that $\text{dist}_T(\sigma(x), \sigma(y)) \leq 3$, for any $xy \in E(T)$, is easy and left to the reader to do.

3.1. Proof of Proposition 7

Let $P = (x_1, x_2, \ldots, x_n)$ be a path of order n. The proof is by induction on n. For $n = 4, 5, 6$, and 7, by Lemma 13, P is nice.

Suppose that the proposition holds for all $n' < n$ and $n \geq 8$. Let $x_j x_{j+1}$ be an edge of P such that $4 \leq j \leq n-4$. The neighbor paths P_{x_j} and $P_{x_{j+1}}$ of $P - x_j x_{j+1}$ are, respectively, of order j and $n-j$. By the induction hypothesis, P_{x_j} and $P_{x_{j+1}}$ are nice. Let σ be a P_{x_j}-nice permutation and σ' be a $P_{x_{j+1}}$-nice permutation. So $\text{dist}_{P_j}(x_1, \sigma(x_1)) = 1$, $\text{dist}_{P_j}(x_j, \sigma(x_j)) \in \{0, 1\}$, and $\text{dist}_{P_{j+1}}(x_{j+1}, \sigma(x_{j+1})) = 1$, $\text{dist}_{P_{j+1}}(x_n, \sigma(x_n)) \in \{0, 1\}$. Then the composition of these two permutations (not commutative) gives a P-nice permutation $\sigma'' = \sigma \circ \sigma'$ such that $\text{dist}_P(x_1, \sigma''(x_1)) = 1$ and $\text{dist}_P(x_n, \sigma''(x_n)) \in \{0, 1\}$. ∎

3.2. Proof of Proposition 8

Let $S_1 = K_{1,q}$ be of center x_1 and end-vertices $z_1, z_2, \ldots, z_q$, with $q \geq 1$ and $S_2 = K_{1,q'}$ of center x_p and of end-vertices $y_1, y_2, \ldots y_{q'}$, with $q' \geq 1$. Let $P = (x_1, x_2, \ldots, x_p)$, with $p \geq 2$.

According to Proposition 7, we may assume that S_1 has at least two end-vertices.

CASE 1. $q' = 1$.

In case $2 \leq p \leq 4$, by Lemma 14, T has a two-placement into T^3. If $p = 5$, by Lemma 13, $P \cdot S$ is nice and x_1 is not a fixed vertex. To obtain a two-placement σ of T into T^3 in this case, we put $\sigma(z_i) = z_i$, for $1 \leq i \leq q$.

For $p \geq 6$, from $T - x_3 x_4$, the neighbor trees T_{x_3} and T_{x_4} are, respectively, a star-path of diameter three and a path of length at least three. By Lemma 14, there exists a permutation such that $\text{dist}(x_3, \sigma_{T_{x_3}}(x_3)) = 1$, and by Proposition 7, T_{x_4} is nice. So we obtain a two-placement of T in T^3.

CASE 2. $q' \geq 2$.

If $p = 2$, T is a double-star and a two-placement is done by Lemma 15. In case $p = 3$, let $P' = (z_1, x_1, x_2, x_3)$. By Lemma 13, P' is nice, $\text{dist}(x_1, \sigma_{P'}(x_1)) = 2$ and $\text{dist}(x_3, \sigma_{P'}(x_3)) = 1$. So by putting $\sigma(z_i) = z_i$, for $2 \leq i \leq q$ and $\sigma(y_i) = y_i$, for $1 \leq i \leq q'$, we obtain a two-placement of T in T^3.

In case $p = 4$, by Lemma 13, P is nice and $\text{dist}_P(x_1, \sigma_P(x_1)) = \text{dist}_P(x_4, \sigma_P(x_4)) = 1$. So put $\sigma(z_i) = z_i$, for $1 \leq i \leq q$ and $\sigma(y_i) = y_i$, for $1 \leq i \leq q'$. Thus, $S_1 \cdot P \cdot S_2$ admits a two-placement σ in its third power. If $p = 5$, by Lemma 13, the path $P' = (x_1, \ldots, x_5, y_1)$ is nice such that $\text{dist}(x_1, \sigma_{P'}(x_1)) = 1$, $\text{dist}(y_1, \sigma_{P'}(y_1)) = 0$, and $\text{dist}(x_5, \sigma_{P'}(x_5)) \neq 0$. Then we put $\sigma(z_i) = z_i$, for $1 \leq i \leq q$ and $\sigma(y_i) = y_i$, for $2 \leq i \leq q'$.

If $p = 6, 7, 8, 9$, we consider T_j and T_{j+1} the neighbor trees obtained by deletion of the edge $x_j x_{j+1}$, with $j = p/2$ if p is even and $j = (p+1)/2$ otherwise. By Lemma 14, T_j and T_{j+1} admit two-placements into their third power such that $\text{dist}(\sigma_{T_j}(x_j), \sigma_{T_{j+1}}(x_{j+1})) \leq 3$.

For $p \geq 10$, let $U_1 = S_1 \cdot (x_1, x_2, x_3)$ and $U_2 = (x_{p-2}, x_{p-1}, x_p) \cdot S_2$. Define σ on $(x_4, x_5, \ldots, x_{p-3})$ according to Proposition 7 and on U_1 and U_2 according to Lemma 14. So $\text{dist}(\sigma(x_2), \sigma(x_3)) \leq 3$ and $\text{dist}(\sigma(x_{p-1}), \sigma(x_{p-2})) \leq 3$. ∎

3.3. Proof of Proposition 9

We prove for any internal vertex $x \in V(T)$ that T is x-good. Let x be any internal vertex of $V(T)$. Let $i(T)$ be the number of internal vertices in T. The proof is by induction on $i(T)$. Since T is not a star, then $i(T) > 1$. If $i(T) = 2$, then T is a double-star. Therefore, by Lemma 15, T is good.

Assume for any nonstar $T' \subseteq T$, with x as an internal vertex in T' and $i(T') < i(T)$, that T' is x-good.

Let $v \in V(T)$ ($\neq x$) be an internal vertex whose neighbors are all end-vertices except one. Call $v_1, v_2, \dots, v_k$ the end-vertices adjacent to e, with $k \geq 2$. Let $T' = T - \{v_1, v_2, \dots, v_k\}$. Observe that T' is not a star. Moreover, e becomes an end-vertex in T'. So $i(T') = i(T) - 1$ and by hypothesis T' is x-good. Let σ' be a (T', x)-good permutation. We construct a permutation σ on T by combining σ' and the cycle $(v_1, v_2, \dots, v_k)$. Observe that $\mathrm{dist}_T(\sigma(v_i), \sigma(v_j)) = 2$, for each $1 \leq i \neq j \leq k$. $\mathrm{dist}_T(\sigma(v), \sigma(v_i)) = \mathrm{dist}_T(\sigma'(v), \sigma(v_i)) = 3$. Moreover, for each vertex y of T', $\sigma(y) = \sigma'(y)$. Therefore, T is x-good. Finally, since it is true for each internal vertex, then T is good. ∎

3.4. Proof of Proposition 10

The proofs of T is r-good (and T is r'-good) are similar to that given in the proof of Proposition 9. ∎

3.5. Proof of Proposition 12

Let $P = (x_1, x_2, \dots, x_p)$, with $p \geq 2$. If both T_1 and T_2 are stars, Proposition 8 is applied. Assume that T_2 is not a star. It is easy to see that if T_i has, at least, three edges, it is not a star and has no vertex of degree 2 then T_i is good, for $i = 1, 2$. Let $T_1' = T_1 \cdot (x_1, x_2)$ and $T_2' = (x_{p-1}, x_p) \cdot T_2$. Note that T_2' is x_p-good. We have four cases.

CASE 1. $p = 2$.

The tree $T = T_1 \cdot (x_1, x_2) \cdot T_2$ has no vertex of degree two. So by Proposition 9, T is good.

CASE 2. $p = 3$.

So x_2 is the unique vertex of degree 2 in T. By Proposition 10, T is x_2-good.

CASE 3. $p = 4$.

If T_1 is a star, then by Lemma 14, $T_1 \cdot (x_1, x_2, x_3)$ admits a permutation σ' such that $\sigma'(T_1 \cdot (x_1, x_2, x_3)) \subset (T_1 \cdot (x_1, x_2, x_3))^3$ and $\mathrm{dist}(x_3, \sigma'(x_3)) = 1$. Since T_2 is not a star, then it is x_4-good and any x_4-good permutation of T_2 provides an extension of σ' to T.

If T_1 is not a star, then T_1' satisfies Proposition 9. Let σ be an x_1-good permutation of this tree. We may extend it to T by letting $\sigma(x_3) = x_3$ and taking for σ_{T_2} any x_4-good permutation.

CASE 4. $p = 5$.

If T_1 is a star, a permutation is obtained by considering $T_1 \cdot (x_1, x_2, x_3, x_4)$ as in Case 3 and T_2. In case T_1 is not a star, use an x_1-good permutation for T_1', and x_5-good permutation for T_2' and put $\sigma(x_3) = x_3$.

CASE 5. $p \geq 6$.

In case T_1 is a star, let $T_{x_{p-1}}$ and $T_{x_p} = T_2$ be the two components of $T - x_{p-1}x_p$. As for Case 1 of the proof of Proposition 8, $T_{x_{p-1}}$ has a packing σ into its third power such that $\mathrm{dist}(x_{p-1}, \sigma_{T_{x_{p-1}}}(x_{p-1})) = 0$ or 1. So any x_p-good permutation of T_2 gives an admissible extension of σ to T.

If T_1 is not a star, let $P' = P \setminus \{x_1, x_p\}$. Proposition 7 provides a permutation of P' such that $\mathrm{dist}(x_2, \sigma_{P'}(x_2)) \in \{0, 1\}$ and $\mathrm{dist}(x_{p-1}, \sigma_{P'}(x_{p-1})) = 1$. Extend it to T using any x_1-good permutation of T_1, and any x_p-good permutation of T_2. ∎

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