An extragradient algorithm for solving general nonconvex variational inequalities

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1. Introduction

Variational inequalities theory, which was introduced by Stampacchia [1], provides us with a simple, general and unified framework in which to study a wide class of problems arising in pure and applied sciences. For the applications, physical formulation, numerical methods and other aspects of variational inequalities, see [2–17,1] and the references therein. Noor [12] has introduced and considered a new class of variational inequalities, called the general nonconvex variational inequalities on the uniformly prox-regular sets. It is well-known that the uniformly prox-regular sets are nonconvex and include the convex sets as a special case; see [3,16]. Using the projection operator, Noor [12] has established equivalence between the general nonconvex variational inequalities and the fixed point problem. The main aim of this work is to suggest and analyze an extragradient method (Algorithm 3.3) for solving the nonconvex variational inequalities. It is well-known that the convergence of the extragradient method requires that the operator is monotone and Lipschitz continuous. It is known that the evaluation of the Lipschitz continuity is itself a very difficult problem. To overcome this drawback, several modifications have been suggested; see, for example, [8,14] and the references therein. The main motivation of this work is to improve on these criteria. We note that the extragradient method is equivalent to an implicit iterative method. We use this equivalence between the extragradient method and the implicit method to show that the convergence of the extragradient method only requires pseudomonotonicity, which is a weaker condition than monotonicity. It is worth mentioning that we do not need the Lipschitz requiring continuity of the operator. In this sense, our result represents an improvement and refinement of the known results. Our method of proof is very simple as compared with other techniques.

2. Basic concepts

Let $H$ be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot , \cdot \rangle$ and $\| \cdot \|$ respectively. Let $K$ be a nonempty closed convex set in $H$. The basic concepts and definitions used in this work are exactly the same as those in Noor [11].
Poliquin et al. [16] and Clarke et al. [3] have introduced and studied a new class of nonconvex sets, which are called uniformly prox-regular sets.

**Definition 2.1.** The proximal normal cone of $K$ at $u \in H$ is given by

$$\mathcal{N}_K^p(u) := \{\xi \in H : \xi \in P_K[u + \alpha \xi]\},$$

where $\alpha > 0$ is a constant and

$$P_K[u] = \{u^* \in K : d_K(u) = \|u - u^*\|\}.$$  

Here $d_K(.)$ is the usual distance function for the subset $K$, that is

$$d_K(u) = \inf_{v \in K} \|v - u\|.$$  

The proximal normal cone $\mathcal{N}_K^p(u)$ has the following characterization.

**Lemma 2.1.** Let $K$ be a nonempty, closed and convex subset in $H$. Then $\zeta \in \mathcal{N}_K^p(u)$ if and only if there exists a constant $\alpha > 0$ such that

$$\langle \zeta, v - u \rangle \leq \alpha\|v - u\|^2, \quad \forall v \in K.$$ 

**Definition 2.2.** The Clarke normal cone, denoted by $\mathcal{N}_K^c(u)$, is defined as

$$\mathcal{N}_K^c(u) = \text{cl}[\mathcal{N}_K^p(u)],$$

where $\text{cl}$ means the closure of the convex hull. Clearly $\mathcal{N}_K^c(u) \subset \mathcal{N}_K^p(u)$, but the converse is not true. Note that $\mathcal{N}_K^c(u)$ is always closed and convex, whereas $\mathcal{N}_K^p(u)$ is convex, but may not be closed [16].

**Definition 2.3.** For a given $r \in (0, \infty]$, a subset $K_r$ is said to be a normalized uniformly $r$-prox-regular set if and only if every nonzero proximal cone normal to $K_r$ can be realized by an $r$-ball, that is, $\forall u \in K_r$ and $0 \neq \xi \in \mathcal{N}_K^p(u)$, one has

$$\langle (\xi)/\|\xi\|, v - u \rangle \leq (1/2r)\|v - u\|^2, \quad \forall v \in K_r.$$ 

It is clear that the class of normalized uniformly prox-regular sets is sufficiently large to include the class of convex sets, $p$-convex sets, $C^{1,1}$ submanifolds (possibly with boundary) of $H$, the images under a $C^{1,1}$ diffeomorphism of convex sets and many other nonconvex sets; see [3,16]. Obviously, for $r = \infty$, the uniform prox-regularity of $K_r$ is equivalent to the convexity of $K$. This class of uniformly prox-regular sets have played an important part in many nonconvex applications such as optimization, dynamic systems and differential inclusions. It is known that if $K_r$ is a uniformly prox-regular set, then the proximal normal cone $\mathcal{N}_K^p(u)$ is closed as a set-valued mapping.

We now recall a well-known proposition which summarizes some important properties of the uniformly prox-regular sets $K_r$.

**Lemma 2.2.** Suppose that $K$ is a nonempty closed subset of $H$, $r \in (0, \infty]$ and the set $K_r = \{u \in H : d_K(u) < r\}$. If $K_r$ is uniformly prox-regular, then

(i) $\forall u \in K_r$, $P_{K_r}(u) \neq \emptyset$.

(ii) $\forall r' \in (0, r)$, $P_{K_r} \text{ is Lipschitz continuous with constant } \frac{r}{r-r'} \text{ on } K_{r'}$.

For given nonlinear operators $T, g$, we consider the problem of finding $u \in H : g(u) \in K_r$ such that

$$\langle Tu, g(v) - g(u) \rangle \geq 0, \quad \forall v \in H : g(v) \in K_r,$$

which is called the general nonconvex variational inequality, introduced and studied by Noor [12]. See also [13,14].

We note that, if $K_r \equiv K$, the convex set in $H$, then problem (1) is equivalent to that of finding $u \in H : g(u) \in K$ such that

$$\langle Tu, g(v) - g(u) \rangle \geq 0, \quad \forall v \in H : g(v) \in K,$$

which is known as the general variational inequality, introduced and studied by Noor [7] in 1988. For the applications, numerical methods of solution, formulation and other aspects of the general variational inequalities (2), see [8,9,14] and the references therein.

If $g \equiv I$, the identity operator, then problem (1) is equivalent to that of finding $u \in K_r$ such that

$$\langle Tu, v - u \rangle \geq 0, \quad \forall v \in K_r,$$

which is called the nonconvex variational inequality. For the formulation and numerical methods of solution for the nonconvex variational inequalities, see Noor [12–15].
We note that, if \( K_r \equiv K \), the convex set in \( H \), and \( g \equiv I \), the identity operator, then problem (1) is equivalent to that of finding \( u \in K \) such that
\[
(Tu, v - u) \geq 0, \quad \forall v \in K.
\]
An inequality of type (4) is called the variational inequality, which was introduced and studied by Stampacchia [1] in 1964. It turned out that a number of unrelated obstacles and equilibrium problems arising in various branches of pure and applied sciences can be studied via variational inequalities; see [2–17,1] and the references therein.

If \( K_r \) is a nonconvex (uniformly prox-regular) set, then problem (1) is equivalent to that of finding \( u \in K_r \) such that
\[
0 \in \rho Tu + g(u) - g(u) + \rho N_{K_r}^p(g(u))
\]
where \( N_{K_r}^p(g(u)) \) denotes the normal cone of \( K_r \) at \( g(u) \) in the sense of nonconvex analysis. Problem (5) is called the general nonconvex variational inclusion problem associated with general nonconvex variational inequality (1). This equivalent formulation plays a crucial and basic role in this work. We would like to point out this equivalent formulation allows us to use the projection operator technique for solving the general nonconvex variational inequalities of the type (1).

**Definition 2.4.** An operator \( T : H \rightarrow H \) with respect to an arbitrary operator \( g \) is said to be:
(i) \( g \)-monotone iff
\[
\langle Tu - Tv, g(u) - g(v) \rangle \geq 0, \quad \forall u, v \in H.
\]
(ii) \( g \)-pseudomonotone if
\[
\langle Tu, g(v) - g(u) \rangle \geq 0, \quad \text{implies} \quad \langle Tv, g(v) - g(u) \rangle \geq 0, \quad \forall u, v \in H.
\]
It is well-known that \( g \)-monotonicity implies \( g \)-pseudomonotonicity, but the converse is not true.

3. Main results

It is known that the general nonconvex variational inequalities (1) are equivalent to the fixed point problem, which is given as follows.

**Lemma 3.1** ([12]). If \( u \in H : g(u) \in K_r \) is a solution of the general nonconvex variational inequality (1) if and only if \( u \in K_r \) satisfies the relation
\[
g(u) = P_{K_r}[g(u) - \rho Tu],
\]
where \( P_{K_r} \) is the projection of \( H \) onto the uniformly prox-regular set \( K_r \).

Lemma 3.1 implies that the general nonconvex variational inequality (1) is equivalent to the fixed point problem (6). This alternative equivalent formulation is very useful from the numerical and theoretical points of view. Using the fixed point formulation (6), we suggest and analyze the following iterative methods for solving the general nonconvex variational inequality (1).

**Algorithm 3.1.** For a given \( u_0 \in H \), find the approximate solution \( u_{n+1} \) by using the iterative scheme
\[
g(u_{n+1}) = P_{K_r}[g(u_n) - \rho Tu_n], \quad n = 0, 1, \ldots,
\]
which is called the explicit iterative method. For the convergence analysis of Algorithm 3.1, see Noor [12].

**Algorithm 3.2.** For a given \( u_0 \in H \), find the approximate solution \( u_{n+1} \) by using the iterative scheme
\[
g(u_{n+1}) = P_{K_r}[g(u_n) - \rho Tu_{n+1}], \quad n = 0, 1, \ldots
\]
Algorithm 3.2 is an implicit iterative method for solving the general nonconvex variational inequalities (1).

To implement Algorithm 3.2, we use the predictor–corrector technique. We use Algorithm 3.1 as predictor and Algorithm 3.2 as a corrector to obtain the following predictor–corrector method for solving the general nonconvex variational inequality (1).

**Algorithm 3.3.** For a given \( u_0 \in H \), find the approximate solution \( u_{n+1} \) by using the iterative scheme
\[
g(y_n) = P_{K_r}[g(u_n) - \rho Tu_n]
\]
\[
g(u_{n+1}) = P_{K_r}[g(u_n) - \rho Ty_n], \quad n = 0, 1, \ldots
\]
Algorithm 3.3 is known as the extragradient method. It is obvious that the implicit method (Algorithm 3.2) and the extragradient method (Algorithm 3.3) are equivalent. We use this equivalent formulation to study the convergence analysis of Algorithm 3.2 and this is the main motivation of this work.
If $K_r \equiv K$, then Algorithm 3.1 reduces to the well-known algorithm for solving the general variational inequality (2).

**Algorithm 3.4.** For a given $u_0 \in H$, find the approximate solution $u_{n+1}$ by using the iterative scheme

$$g(u_{n+1}) = P_K[g(u_n) - \rho Tu_{n+1}], \quad n = 0, 1, \ldots$$

We would like to mention that one can obtain several new and previously known iterative methods for solving the general nonconvex variational inequalities by making an appropriate choice of the operators and subspaces. For more details, see Noor [8, 14, 12].

We now consider the convergence analysis of Algorithm 3.2 and this is the main motivation for our next result.

**Theorem 3.1.** Let $u \in H : g(u) \in K_r$ be a solution of (1) and let $u_{n+1}$ be the approximate solution obtained from Algorithm 3.2. If the operator $T$ is $g$-pseudomonotone, then

$$\|g(u) - g(u_{n+1})\|^2 \leq \|g(u) - g(u_{n})\|^2 - \|g(u_{n+1}) - g(u_{n})\|^2. \quad (8)$$

**Proof.** Let $u \in H : g(u) \in K_r$ be solution of (1). Then, using the $g$-pseudomonotonicity of $T$, we have

$$\langle Tu, g(v) - g(u) \rangle \geq 0, \quad \forall v \in K_r. \quad (9)$$

Take $v = u_{n+1}$ in (9), we have

$$\langle Tu_{n+1}, g(u_{n+1}) - g(u) \rangle \geq 0. \quad (10)$$

Using Lemma 2.1, Eq. (7) can be written as

$$\langle \rho Tu_{n+1} + g(u_{n+1}) - g(u_{n}), g(v) - g(u_{n+1}) \rangle \geq 0, \quad \forall v \in H; g(v) \in K_r. \quad (11)$$

Taking $v = u$ in (11), we have

$$\langle \rho Tu_{n+1} + g(u_{n+1}) - g(u_{n}), g(u) - g(u_{n+1}) \rangle \geq 0. \quad (12)$$

From (10) and (12), we have

$$\langle g(u_{n+1}) - g(u_{n}), g(u) - g(u_{n+1}) \rangle \geq 0,$$

which implies that

$$\|g(u) - g(u_{n+1})\|^2 \leq \|g(u) - g(u_{n})\|^2 - \|g(u_{n+1}) - g(u_{n})\|^2,$$

the required result (8). \quad \square

**Theorem 3.2.** Let $u \in H : g(u) \in K_r$ be a solution of (1) and let $u_{n+1}$ be the approximate solution obtained from Algorithm 3.2. If $H$ is a finite dimensional space and $g^{-1}$ exists, then $\lim_{n \to \infty} u_n = u$.

**Proof.** Let $\tilde{u} \in H : g(\tilde{u}) \in K_r$ be a solution of (1). Then the sequence $\{\|g(u_n) - g(\tilde{u})\|\}$ is nonincreasing and bounded and

$$\sum_{n=0}^{\infty} \|g(u_{n+1}) - g(u_{n})\|^2 \leq \|g(u_0) - g(\tilde{u})\|^2,$$

which implies that

$$\lim_{n \to \infty} \|u_{n+1} - u_n\| = 0, \quad (13)$$

where we have used the fact that $g^{-1}$ exists.

Let $\hat{u}$ be a cluster point of $\{u_n\}$; there exists a subsequence $\{u_{n_i}\}$ such that $\{u_{n_i}\}$ converges to $\hat{u}$. Replacing $u_{n+1}$ by $u_{n_i}$ in (11) and taking the limits and using (13), we have

$$\langle T\hat{u}, g(v) - g(\hat{u}) \rangle \geq 0, \quad \forall v \in H; g(v) \in K_r.$$

This shows that $\hat{u} \in H : g(\hat{u}) \in K_r$ solves the general nonconvex variational inequality (1) and

$$\|g(u_{n+1}) - g(\hat{u})\|^2 \leq \|g(u_n) - g(\hat{u})\|^2,$$

which implies that the sequence $\{u_n\}$ has a unique cluster point and $\lim_{n \to \infty} u_n = \hat{u}$ is the solution of (1), the required result. \quad \square

**Remark 3.1.** We also remark that if the operator $g$ is a nonexpanding operator [9], namely,

$$\|g(u) - g(v)\| \geq \|u - v\|, \quad \forall u, v \in H,$$

then Theorem 3.2 continues to hold.
Acknowledgement

The author is grateful to the referee for his/her very constructive comments and suggestions. The author would like to express his sincere gratitude to Dr. M. Junaid Zaidi, Rector, CIIT, for providing excellent research facilities.

References