EMBEDDING OF REGULAR SEMIGROUPS IN WREATH PRODUCTS, III

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The purpose of this paper is to use parametrizations of the derived semigroup and
the Rhodes Expansion to embed several classes of regular semigroups in Wreath
products of more special semigroups. We continue the program of [30,31] here. If
S_1 and S_2 are semigroups, S_2 o S_1 denotes the wreath product of S_1 by S_2 (i.e.
p : S_2 o S_1 --> S_1 where p is the projection surmorphism) and S_1 <= S_2 means there
exists an isomorphism of S_1 into S_2. Let S and T be semigroups and let \Phi be a
homomorphism of S onto T. Tilson [15] constructed a semigroup D(\Phi) (the derived
semigroup of \Phi) such that S <= D(\Phi) o T. Rhodes [14] showed that a semigroup X
satisfies S <= X o T with p | S = \Phi if and only if X is a parametrization of D(\Phi) (i.e.
an image of the non-zero elements of D(\Phi) under a suitable partial homomor-
phism). The derived semigroup D(\Phi) of \Phi is (T' x S x T''(S\Phi))U\{0\} where T' is
'the monoid generated by T', 0 \in T' x S x T, and ((t_1, s_1, t_1(s_1 \Phi))(t_2, s_2, t_2(s_2 \Phi)) =
(t_1, s_1 s_2, t_2(s_2 \Phi)) or 0 according to whether t_1(s_1 \Phi) = t_2 or t_1(s_1 \Phi) \neq t_2 and
(t_1, s_1, t_1(s_1 \Phi)) = 0 = t_1(s_1 \Phi) \neq t_2 and
(t_1, s_1, t_1(s_1 \Phi)) = 0 - 0 = 0. R, L, H, J, and D will denote Green's
relations on an arbitrary semigroup S while E(S)(T(S)) will denote the set of idem-
potents (union of the maximal subgroups) of S. A semigroup S is termed combi-
natorial if each of its maximal subgroups is a singleton. In Section 1, we consider
the case S is a regular semigroup and T is an inverse semigroup. In Theorem 1.2,
we show D_X'(\Phi) = \{(t, s, t(s\Phi)) \in D(\Phi) : t \in X, a right ideal of T, and t \not\equiv t(s\Phi)
(in T)} \cup \{0\} is a regular subsemigroup of D(\Phi) which is a parametrization of D(\Phi).
Hence, S <= D_X'(\Phi) o T by Rhodes theorem. Let S be a natural regular semigroup
(i.e. T(S) is a subsemigroup, and, hence, T(S) is a semilattice A = T(S)/J of com-
pletely simple subsemigroups (T_\delta : \delta \in \Lambda). Let T = S/t where t is the smallest com-
binatorial inverse semigroup congruence on S (hence, E(T) = A by Theorem 1.6).
In Theorem 1.8, we show D_X'(t) is a natural regular semigroup and T(D_X'(t)) is a
direct sum of subsemigroups (A^0_u : u \in T \cap X) (A^0_0 is the semigroup A_0 with '0' ap-
pended) where A_0 = \bigcup (T_\delta : z \leq u^{-1} v). Let N (I) denote the set of natural numbers
(integers). Let N, (I_d) (d, a positive integer) denote N (I) (d = (0, 1, ..., d - 1))
under the reverse of the usual order. Next, further assume A = N_d x Y, where Y is
a semilattice with greatest element, under the lexicographic order and $e_{(n, \delta)} \not\preceq e_{(m, \lambda)}$ (in $S$) if and only if $\delta = \lambda$ ($S$ is termed a natural regular semigroup of type $\omega Y$). In Theorem 1.10, we show each element of $D^Y_X(t)$ is either a nilpotent element of degree two or an element of $T(D^Y_X(t))$, $T = N \times N \times Y$ (under a suitable multiplication), $A_{(p, q, i)} = \bigcup T_{(k, r)}, k < q, r \in Y \bigcup \bigcup T_{(q, a)} : \beta \leq \alpha$ and $S \leq D^Y_X(t) \circ Y \circ C$ where $Y$ is $Y$ with constant transformations appended and $C$ is the bicyclic semigroup. Next, let $S$ be a simple natural regular semigroup with $A = N_r$ (termed a simple natural regular semigroup of type $\omega$). We show (Lemma 1.11) that $S$ is a natural regular semigroup of type $\omega d_r$ where $d$ is a positive integer. Hence, in this case, $Y = d_r$ and $A_{(p, q, i)} = T_0 \cup T_1 \cup \cdots \cup T_{d_r}$. If $S$ is bisimple, $d = 1$ (Corollary 1.12). In Remark 1.13, we give an analogous structure theorem for $A = I_r$ (termed a simple natural regular semigroup of type $I$). Let $n$ be a positive integer and let $N_r^n$ denote the cartesian product of $N_r$ with itself $n$ times. Let $S$ be a bisimple natural regular semigroup with $A = N_r^n$ under the antilexicographic order (we term $S$ a bisimple natural regular semigroup of type $\omega^n$). In Theorem 1.14, we show $(x \in D^Y_X(t) : x^2 \neq \emptyset) \cup \{0\}$ is a direct sum of subsemigroups which are finite chains of $J$-classes of $T(S)$ and bisimple natural regular subsemigroups $V$ of type $\omega^j$ ($1 \leq i \leq n - 1$) of $S$ with each $J$-class of $T(V)$ a $J$-class of $T(S)$. We also show $S \leq D^Y_X(t) \circ C$ for $n = 1$ and $S \leq D^Y_X(t) \circ C^{n - 1} \circ C$ for $n \geq 2$ where $C^{n - 1} = C \circ \cdots \circ C$ ($n - 1$ times). In Remark 1.15, we give an analogous structure theorem for $A = N_r^n \times I_r$ under the antilexicographic order ($S$ is then termed a bisimple natural regular semigroup of type $\omega^n I$).

In Section 2, we use the Rhodes Expansion and a parametrization of the derived semigroup to clarify the structure of orthodox unions of groups. Let us briefly describe the Rhodes Expansion of an arbitrary semigroup $S$. If $a, b \in S$, $a \not\preceq b$ means $a \triangleleft b$ and $a \not\preceq b$ means $a \not\preceq b$ but $a \not\preceq b$. Let $\hat{S} = \{(s_n \subseteq \cdots \subseteq s_1) : 1, 2, \ldots, n \in N \setminus \emptyset$ and $s_i \in S$ for $i = 1, 2, \ldots, n\}$ under the product

$$(s_n \subseteq \cdots \subseteq s_1)(t_m \subseteq \cdots \subseteq t_1) = (\text{red}(s_n t_m \subseteq \cdots \subseteq s_1 t_m \subseteq t_m) \subseteq t_{m - 1} \subseteq \cdots \subseteq t_1)$$

where 'red' deletes the right most of $\subseteq$-equivalent elements. $\hat{S}$ is termed the Rhodes Expansion of $S$ after its inventor, John Rhodes. Let $\psi$ be a homomorphism of $\hat{S}$ onto a semigroup $T$. Then, $(s_n \subseteq \cdots \subseteq s_1)\hat{\psi} = \text{red}(s_n \psi \subseteq \cdots \subseteq s_1 \psi)$ defines a homomorphism of $\hat{S}$ onto $\hat{T}$ (Tilson [15,5]).

Let $S$ be an orthodox union of groups and let $\varrho$ denote the smallest inverse semigroup congruence on $S$. Then, $\hat{S} = (\hat{S}/\varrho)$ is a semilattice $Y = S/\varrho$ of orthodox completely simple semigroups (right groups) $(X_y : y \in Y)((U_y : y \in Y))$. We show $D_{\varrho}(\varrho) = \{(t, s, t(\varrho)) \in D(\varrho) : t \in U_y, y \leq y_0 \in Y, s \in X_z, y \leq z \} \cup \{0\}$ is a subsemigroup of $D(\varrho)$ which is a parametrization of $D(\varrho)$ (valid for arbitrary semilattices of semigroups with $X \subseteq U_y$ (Lemma 2.1)). Hence, $\hat{S} \leq D_{\varrho}(\varrho) \circ (\hat{S}/\varrho)$ by Rhodes Theorem and $D_{\varrho}(\varrho)$ is a subsemigroup of the direct sum of the Rees Matrix Semi-
groups $M^0(U_y, \bigcup (S_z : z \geq y), U_y, \text{id})(y \in Y, y \leq y_0)$. We show $D_{y_0}(\mathfrak{g})$ is a combinatorial semigroup consisting just of nilpotent elements of degree two and a subsemigroup of idempotent elements. Furthermore, $S \triangleright \mathcal{L} \triangleleft S$ where $(s_n, \ldots, s_1) \eta = s_n$ defines a homomorphism of $S$ onto $S$ such that $\eta^{-1} \subseteq E(\mathcal{S})$ for $e \in E(S)$ and, for $a, b \in S$, $a \mathcal{L} b \eta$ (in $S$) $(a \mathcal{R} b \eta$ (in $S$)) if and only if $a \mathcal{L} b$ (in $S$) $(a \mathcal{R} b$ (in $S$)). Finally,

$$(\mathcal{S}/\mathcal{Q}) \leq (E(\mathcal{S}/\mathcal{Q}))^{\delta}(E(\mathcal{S}/\mathcal{Q})))^{\delta}$$

where 'I' denotes an appended identity, the '□' denotes the reverse of $\circ$, and $(\mathcal{S}/\mathcal{Q})/\delta$ ($\delta$ is the smallest inverse semigroup congruence on $(\mathcal{S}/\mathcal{Q})$) is a semilattice $Y$ of maximal subgroups of $S$ (Theorem 2.6).

In Section 3, we give a much briefer proof of a fundamental result of [30] by using a parametrization of the associated derived semigroup and Rhodes Theorem (Theorem 3.1).

The class of natural regular semigroups has been extensively studied (see [28,29] and references therein) and a structure theorem for this class of semigroups was previously given in [28]. Previous structure theorems have been given for several classes of natural regular semigroups of type $\omega Y$: for example, by Reilly [13], Munn [10,11], Kočin [8], Clifford [3], and Warne [22,24,26,27,29]. In [30], we embedded $\omega Y$-inverse semigroups (each $T(n,\delta)$ is a group), $\omega Y$-$\mathcal{L}$-unipotent semigroups (each $T(n,\delta)$ is a left group), and generalized $\omega Y$-$\mathcal{L}$-unipotent bisimple semigroups ($\delta = 1$ and $E(S)$ is an $\mathcal{L}$-compatible band) in wreath products of more special classes of semigroups. We first gave structure theorems for simple (bisimple) (natural) inverse semigroups of type I (each $\mathcal{J}$-class of $T(S)$ is a group) in [20] ([18]) and embedded these classes of semigroups in wreath products in [30]. We first gave structure theorems for bisimple (natural) inverse semigroups of type $\omega^n$ ($\omega^n I$) (each $\mathcal{J}$-class of $T(S)$ is a group) in [17] ([19]) and embedded these classes of semigroups in wreath products in [30]. Structure theorems for orthodox unions of groups have been given by Fantham [6]. Preston (1961, unpublished; see [2]), Yamada [32], Warne [25], Petrich [12], and Clifford [4]. We used the Rhodes Expansion and the derived semigroup to clarify the structure of orthodox semigroups in [30]. The structure theorems for natural regular semigroups, simple natural regular semigroups of type $\omega$, and orthodox unions of groups given here are considerably simpler than those previously obtained for these classes of semigroups in [28], [29], and [25] respectively.

We next introduce and clarify the terminology and notation which will be used throughout this paper. See [15], [5, p. 356], and [14] for further details on the derived semigroup.

We adopt the following notation and definitions from [30, pp. 181–182]. Transformation semigroup (denoted by the pair $(Q, S)$ where $Q$ is a set and $S$ is a subsemigroup of $F(Q)$ the semigroup of all functions of $Q$ into $Q$ and $qs$ is the image of $q$ under $s$), dual $(S, Q)$ of $(Q, S)$ (i.e., if $s \in S$ and $q \in Q$, we write $sq$ as the image of $q$ under $s$), $S^1$ ($S$ with appended identity, 1) and $S^*$ where $S$ is a semigroup, monoid $S = (S^*, S)$ where $S$ is a semigroup and the image of $x \in S^*$ under $s \in S$ is just
(product in $S^\ast$), $(Q, S)$ where $(Q, S)$ is a transformation semigroup, wreath product of $(P, T)$ by $(Q, S)$ where $(P, T)$ and $(Q, S)$ are transformation semigroups (denoted by $(Q, S) \circ (P, T)$), $(T, P)$ of $(S, Q)$ where $(T, P)$ and $(S, Q)$ are the duals of $(P, T)$ and $(Q, S)$ respectively. $(Q, S)^n$ where $n$ is a positive integer, type $A$ semigroup congruence (for example inverse semigroup congruence), $aq$ ($a \in S$, a semigroup), ker $\varrho$ where $\varrho$ is a congruence relation on $S$ ($\varrho$ will also denote the natural homomorphism of $S$ onto $S/\varrho$), union of groups and Clifford’s theorem, $S \cong T$ and $S \cong T$ where $S$ and $T$ are semigroups and $\psi$ is a homomorphism of $S$ onto $T$, $\langle X \rangle$ where $X$ is a subset of a semigroup $S$, and lexicographic order.

We will use the following definitions and notation from Clifford and Preston [1] and/or Howie [7] and/or Krohn, Rhodes, and Tilson [9]. Right ideal, idempotent, zero, identity, Green’s relations ($\mathcal{R}$, $\mathcal{L}$, $\mathcal{H}$, $\mathcal{D}$, and $\mathcal{J}$), $\mathcal{J}$-class, semilattice, semilattice of semigroups, simple semigroup, completely simple semigroup, natural ordering of idempotents, regular element, inverse, regular semigroup, bisimple semigroup, maximal subgroup, inverse semigroup, right zero semigroup, right group, band, rectangular band, equivalent definitions of inverse semigroup, bicyclic semigroup, orthodox semigroup, equivalence between semilattice and commutative idempotent semigroup.

Let $S$ be a semigroup. If $S$ has a zero element, $0$, we write $S^0 = S$. If $S$ does not have a zero, let $S^0 = S \cup \{0\}$ under the multiplication $s_1 \cdot s_2 = s_1 s_2$ (product in $S$) if $s_1, s_2 \in S$, $s \cdot 0 = 0 \cdot s = 0$ for $s \in S$, and $0 \cdot 0 = 0$. $S^0$ is termed $S$ with zero appended. Let $(X^0_b : b \in B)$ be a collection of semigroups such that $\bigcap (X^0_b : b \in B) = \{0\}$. Let $X = \bigcup (X^0_b : b \in B)$ under the multiplication $x \cdot y = xy$ (product in $X^0_b$) if $x, y \in X^0_b$ and $x \cdot y = 0$ if $x \in X^0_b$ and $y \in X^0_c$ with $b \neq c$. $X$ is termed to be the direct sum of the semigroups $(X^0_b : b \in B)$ and we write $X = \sum (X^0_b : b \in B)$.

Let $\varphi$ be a homomorphism of a semigroup $S$ onto a semigroup $T$, and let $\theta$ be a mapping of $D(\varphi) - \{0\}$ into a semigroup $P$. Following Rhodes [14], we term $\theta : D(\varphi) - \{0\} \to P$ a parametrization of $D(\varphi)$ if (1) $\theta$ is a partial homomorphism of $D(\varphi) - \{0\}$ into $P$ (i.e. if $x, y \in D(\varphi) - \{0\}$ and $xy \neq 0$, then $(xy)\theta = x(y\theta)$ and (2) $\theta$ satisfies the embedding condition: $s_1 \varphi = s_2 \varphi$ and $(t, s_1, t(s_1 \varphi))\theta = (t, s_2, t(s_2 \varphi))\theta$ for all $t \in T^\ast$ imply $s_1 = s_2$. For brevity, we also term $P$ a parametrization of $D(\varphi)$.

Let $S$ be a semigroup and let $A$ be a non-empty set. We define $M^0(A, S, A, id)$ to be $\{(x, y) : x, y \in A, s \in S\}$ under the multiplication $(x, y)(u, v) = (xu, vy)$ if $y = u$; $(x, u, y)(u, t, v) = 0$ if $y \neq u$; $(x, r, y)0 = 0(x, r, y)$ if $a = b$; $(r, a)(s, b) = 0$ if $a \neq b$; $(r, a)0 = 0(r, a) = 0$. $M^0(A, S, A, id)$ is the Rees Matrix Semigroup over $S^0$ with identity sandwich matrix (see [1]). We define $S^0 \times A$ to be $(S \times A) \cup \{0\}$ under the multiplication $(r, a)(s, b) = (rs, a)$ if $a = b$; $(r, a)(s, b) = 0$ if $a \neq b$; $(r, a)0 = 0(r, a) = 0$.

Let $S$ be a semigroup with a zero, $0$, and $x \in S$. If $x \neq 0$ and $x^2 = 0$, $x$ is termed a nilpotent element of degree two. If $U$ is a semigroup and $u \in U$, let $U_u = \{x \in U : ux = u\}$. $U_u$ is a subsemigroup of $U$ and is called the stabilizer of $u$ in [5].

Let $\Phi$ be a homomorphism of a regular semigroup $S$ onto a semigroup $T$ and let $X$ be a right ideal of $T$. We let $(D_X(\Phi))_{\text{stab}} = \{(t, s, t) \in D_X(\Phi)\} \cup \{0\}$. Thus, $(D_X(\Phi))_{\text{stab}}$ is a subsemigroup of $D_X(\Phi)$. In fact, $D_X(\Phi)_{\text{stab}} = \sum (A^0_t : t \in T \cap X)$.
where \( A_T \equiv T_T \Phi^{-1} \) [5, p. 358] and \( D'_X(\Phi) - D'_X(\Phi)_{\text{stab}} \) is the collection of nilpotent elements of degree two of \( D'_X(\Phi) \).

A chain is a linearly ordered set. We term \( N_{[n]}(n, \text{a positive integer}) \) under the antilexicographic order on \( \omega^n \)-chain. If \( d \) is a positive integer, \( d_r \) is termed a \( d_r \)-chain. A semigroup \( S \) is termed \( \mathcal{R} \)-trivial if each \( \mathcal{R} \)-class of \( S \) is a singleton. If \( \mathcal{L} (\mathcal{D}) \) is a congruence on \( S \), \( S \) is termed an \( \mathcal{L} (\mathcal{D}) \)-compatible semigroup. The set of regular elements of \( S \) is denoted by \( S_{\text{reg}} \). If \( a \in S \), \( \mathcal{J}(a) \) will denote the collection of inverses of \( a \).

Let \( \theta \) be a homomorphism of a semigroup \( S \) onto a semigroup \( T \). We term \( \theta \) an \( E \)-epimorphism if \( e\theta^{-1} \subseteq E(S) \) for all \( e \in E(T) \). Following Rhodes [14], we term \( \theta \) an \( \mathcal{R} \)-map \( (\mathcal{J}-\text{map}) \) if \( a \mathcal{R} b (a \mathcal{J} b) \) (in \( S \)) if and only if \( a\theta \mathcal{R} b\theta (a\theta \mathcal{J} b\theta) \) (in \( T \)) for all \( a, b \in S \).

1. Solutions \( P \) of \( S \leq P \circ T \) where \( S(T) \) is a regular (inverse) semigroup

We first state Rhodes Theorem (Theorem 1.1). Let \( \Phi \) be a homomorphism of a semigroup \( S \) onto a semigroup \( T \). By Rhodes Theorem, for any parametrization \( P \) of \( D(\Phi) \), \( S \leq P \circ T \). In Theorem 1.2, for a regular semigroup \( S \), we exhibit a subsemigroup \( D'_X(\Phi) \) of \( D(\Phi) \) which is a parametrization of \( D(\Phi) \). If \( T \) is an inverse semigroup, we show \( D'_X(\Phi) \) is regular. In Corollary 1.4 and Remark 1.5, we clarify the structure of regular semigroups which are semilattices of semigroups as a consequence of Theorem 1.2. In Theorem 1.6, we state our previous characterization of the smallest combinatorial inverse semigroup congruence \( t \) on a natural regular semigroup. In Theorem 1.8, for \( S \) a natural regular semigroup, we show \( D'_X(t) \) is a natural regular semigroup, describe \( D'_X(t) \), and give a necessary and sufficient condition for \( (D'_X(t))_{\text{stab}} = T(D'_X(t)) \). In Theorem 1.10, we show \( T(D'_X(t)) = (D'_X(t))_{\text{stab}} \) for natural regular semigroups of type \( \omega Y \) and clarify their structure. In Corollary 1.12, we specialize Theorem 1.10 to simple (bisimple) natural regular semigroups of type \( \omega \), and in Remark 1.13, we give an analogous result for simple natural regular semigroups of type \( I \) (i.e. \( T(S)/\mathcal{J} = I_L \)). In Theorem 1.14, we give a structure theorem for bisimple natural regular semigroups of type \( \omega^n \). By Theorem 1.14, \( T(D'_X(t)) \subseteq (D'_X(t))_{\text{stab}} \) for \( n \geq 2 \) for this class of semigroups. Hence, their structure is more complicated than the structure of natural regular semigroups of type \( \omega Y \). In Remark 1.15, we give an analogous result for bisimple natural regular semigroups of type \( \omega^n I \).

If \( X \) and \( T \) are semigroups, let \( p : X \circ T \to T \) denote the projection surmorphism \( (f, t) \to t \). Theorem 1.1 will be used in the proof of Theorem 1.2 and Theorem 2.1.

**Theorem 1.1** (Rhodes [14]). Let \( \Phi \) be a homomorphism of a semigroup \( S \) onto a semigroup \( T \) and let \( U \) be a semigroup. Then, \( S \leq U \circ T \) and \( \Phi = p \mid S \) if and only if \( U \) is a parametrization of \( D(\Phi) \).
Theorem 1.2 will be used in the proofs of Corollary 1.4, Lemma 1.7, Theorem 1.8.

**Theorem 1.2.** Let \( \Phi \) be a homomorphism of a semigroup \( S \) onto a semigroup \( T \) and let \( X \) be a right ideal of \( T \). Then, \( D_X(\Phi) = \{(a, s, a(s\Phi)) \in D(\Phi) : a \in X \text{ and } a \not\sim a(s\Phi) \text{ (in } T)\} \cup \{0\} \) is a subsemigroup of \( D(\Phi) \). If, furthermore, \( S \) is a regular semigroup,

\[ (*) \quad S \leq D_X(\Phi) \circ T \]

and \( \Phi = p | S \).

If \( S \) is a regular semigroup and \( T \) is an inverse semigroup, then \( D_X(\Phi) \) is a regular subsemigroup of \( D(\Phi) \).

**Proof.** We first show \( D_X(\Phi) \) is a subsemigroup of \( D(\Phi) \). Let \( u = (a, s, a(s\Phi)) \) and \( v = (b, r, b(r\Phi)) \) be elements of \( D_X(\Phi) \). If \( a(s\Phi) \neq b \), \( uv = 0 \). If \( a(s\Phi) = b \), \( uv = (a, sr, a(sr)\Phi) \). Since \( a \in X \) and \( a \not\sim a(sr)\Phi \), \( uv \in D_X(\Phi) \). Let \( S \) be a regular semigroup. For \( (t, s, t(s\Phi)) \in D(\Phi) - \{0\} \), let \( (t, s, t(s\Phi)) \theta = (t, s, t(s\Phi)) \) if \( t \in X \) and \( t \not\sim t(s\Phi) \) and \( (t, s, t(s\Phi)) \theta = 0 \), otherwise. We will show \( \theta : D(\Phi) - \{0\} \to D_X(\Phi) \) is a parameterization of \( D(\Phi) \). Then, \( (*) \) will be a consequence of Theorem 1.1. We first show \( \theta \) is a partial homomorphism. Let \( x = (t_1, s_1, t_1(s_1\Phi)) \), \( y = (t_2, s_2, t_2(s_2\Phi)) \in D(\Phi) - \{0\} \) with \( t_1(s_1\Phi) = t_2 \). First assume \( t_1 \in X \) and \( t \not\sim t_1(s_1, s_2) \Phi \). Thus, there exists \( \bar{x} \in T' \) such that \( t_1 = t_1(s_1, s_2) \Phi \bar{x} = t_1(s_1) \Phi (s_2) \Phi \). Hence, \( t_1 \not\sim t_1(s_1) \Phi \). Furthermore, \( t_1(s_1) \Phi = t_1(s_1, s_2) \Phi(X(s_1)) \). Thus, \( t_1(s_1) \Phi \not\sim t_1(s_1, s_2) \Phi \). Hence, since \( t_1(s_1) \Phi \in X \), \( (xy) \theta = xy = x\theta y \theta \). If \( t_1 \not\in X \), \( (xy) \theta = 0 = x\theta y \theta \). If \( t_1 \not\sim t_1(s_1, s_2) \Phi \), then \( t \not\sim t_1(s_1) \Phi \) or \( t_1(s_1) \Phi \not\sim t_1(s_1, s_2) \Phi \). Thus, in this case, \( (xy) \theta = 0 = x\theta y \theta \). Hence, \( \theta \) is a partial homomorphism. We next establish the embedding condition. Let \( u \in X \), \( s_1 \Phi = s_2 \Phi \) and \( s_1^{-1} \in J(s_1) \). Then, \( (u(s_1^{-1}) \Phi, s_1, u(s_1 \Phi)) \theta = (u(s_1^{-1}) \Phi, s_1, u(s_1 \Phi)) \). Furthermore, \( (u(s_1^{-1}) \Phi, s_2, u(s_1^{-1}) \Phi) \theta = (u(s_1^{-1}) \Phi, s_2, u(s_1^{-1}) \Phi) \) since \( u(s_1^{-1}) \Phi = u(s_2) \Phi u(s_1^{-1}) \Phi \). Thus, \( (t_1, s_1, t_1(s_1) \Phi) \theta = (t_2, t_2(s_2) \Phi) \) for \( t \in T' \). If \( s \in J(s_1) \), \( s_1 \Phi \in J(s) \). Thus, \( a \not\sim a(s\Phi) \not\sim a(s\Phi) \Phi \) for \( a^{-1} \in J(a) \). Thus, \( a^{-1} = a^{-1} \Phi a^{-1} \Phi \). Hence, \( \Phi = a(s\Phi) \Phi \). Thus, \( (s \Phi, s', a) \in D_X(\Phi) \). By a straightforward calculation, \( (a(s\Phi), s', a) \in J(a, s, a(s\Phi)) \). Hence, \( D_X(\Phi) \) is a regular subsemigroup of \( D(\Phi) \).

**Remark 1.3.** Let \( \Phi \) be a homomorphism of a semigroup \( S \) onto a semigroup \( T \). In the notation of Theorem 1.2, let \( D'(\Phi) = D_T'(\Phi) \), i.e., \( D'(\Phi) = \{(t, s, t(s\Phi)) \in D(\Phi) : t \not\sim t(s\Phi)\} \). Let \( S \) be a regular semigroup and let \( T \) be an inverse semigroup. Then, using [30, Proposition 3.3(b)] and Theorem 1.2, \( D(\Phi)_{\text{reg}} = D'(\Phi) \).

**Corollary 1.4.** Let \( \Phi \) be a homomorphism of a regular semigroup \( S \) onto an \( \mathcal{R} \)-trivial semigroup \( T \). Then,
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\[ S \leq (\sum ((T_i \Phi^{-1})^0 : t \in T))^0 T. \]

**Proof.** Utilize Theorem 1.2 and the fact that for \( t \in T \), \( (t, s, t) \rightarrow s \) defines an isomorphism of \( \{(t, s, t) \in D_X(\Phi)\} \) onto \( T_i \Phi^{-1} \) by [5, p. 358].

**Remark 1.5.** Let \( S \) be a regular semigroup which is a semilattice \( Y \) of semigroups \( (S_y : y \in Y) \). For \( s \in S_y \), define \( s\Phi = y \). Then, \( \Phi \) defines a homomorphism of \( S \) onto the \( \mathcal{R} \)-trivial (inverse) semigroup \( Y \) and \( T_i \Phi^{-1} = \bigcup (S_y : y \geq t) \). For example, if \( S \) is an \( \omega \)-chain of the semigroups \( (S_n : n \in N) \), \( T_n \Phi^{-1} = S_0 \cup S_1 \cup \cdots \cup S_n \).

**Remark.** If \( S \) is a regular semigroup and \( T \) is an inverse semigroup, using Theorem 1.2 and [30, Proposition 3.3(b)], \( D_X(\Phi)_{\text{stab}} \) is a regular subsemigroup.

Theorem 1.6 will be used in the proofs of Theorem 1.8, Lemma 1.9, Theorem 1.10, Lemma 1.11, Remark 1.13, and Theorem 1.14.

**Theorem 1.6** [28, Theorem 1.17, Propositions 1.24, 1.19, and 1.21]. Let \( S \) be a natural regular semigroup and assume \( T(S) \) is a semilattice \( Y \) of completely simple semigroups \( (T_y : y \in Y) \). Then, \( t = \{(a, b) \in S \times S : aa', bb' \in T_y \text{ and } a' a, b' b \in T_z \text{ for some } y, z \in Y \text{ and } a' \in J(a) \text{ and } b' \in J(b)\} \) is the smallest combinatorial inverse semigroup congruence on \( S \). Furthermore, \( E(S/t) = Y \) and \( yt^{-1} = T_y \) for all \( y \in Y \).

Lemma 1.7 will describe Green's relation \( \mathcal{H} \) on \( D_X(\Phi) \) where \( \Phi \) is a combinatorial inverse semigroup congruence on a regular semigroup \( S \). Lemma 1.7 will be used in proof of Theorem 1.8.

**Lemma 1.7.** Let \( \Phi \) be a homomorphism of a regular semigroup \( S \) onto an inverse semigroup \( S\Phi \) and let \( X \) be a right ideal of \( S\Phi \). Let \( (t_1, s_1, t_1(s_1 \Phi)), (t_2, s_2, t_2(s_2 \Phi)) \in D_X(\Phi) \) and assume \( \Phi \) is a combinatorial semigroup. Then, \( (t_1, s_1, t_1(s_1 \Phi)) \not\sim (t_2, s_2, t_2(s_2 \Phi)) \) (in \( D_X(\Phi) \)) if and only if \( t_1 = t_2 \) and \( s_1 \not\sim s_2 \).

**Proof.** Let \( x = (t_1, s_1, t_1(s_1 \Phi)) \) and \( y = (t_2, s_2, t_2(s_2 \Phi)) \). First assume \( t_1 = t_2 \) and \( s_1 \not\sim s_2 \). We will show \( x \not\sim y \) (in \( D_X(\Phi) \)). Using Theorem 1.2 and [30, Proposition 3.3(b)], there exist \( s_1^{-1} \in J(s_1) \) and \( s_2^{-1} \in J(s_2) \) such that \( t_1(s_1 s_1^{-1})\Phi = t_1 \) and \( t_2(s_2 s_2^{-1})\Phi = t_2 \). Thus, \( (t_1(s_1 \Phi), s_1^{-1}s_2, t_1(s_2 \Phi)), (t_2(s_2 \Phi), s_2^{-1}s_1, t_2(s_1 \Phi)) \in D_X(\Phi) \). Furthermore, \( x(t_1(s_1 \Phi), s_1^{-1}s_2, t_1(s_2 \Phi)) = y \) and \( y(t_2(s_2 \Phi), s_2^{-1}s_1, t_2(s_1 \Phi)) = x \). Thus, \( x \not\sim y \) (in \( D_X(\Phi) \)). We next show \( x \not\sim y \) (in \( D_X(\Phi) \)). Using the fact \( S\Phi \) is combinatorial, \( t_1(s_1 \Phi) = t_2(s_2 \Phi) \). Since \( t_2(s_2 s_2^{-1})\Phi = t_1(s_1 s_1^{-1})\Phi = t_1 \) and \( t_1(s_1 s_2^{-1})\Phi = t_2(s_2 s_2^{-1})\Phi = t_2 \), \( t_1, t_2 \in D_X(\Phi) \). Furthermore, \( (t_1, s_1 s_2^{-1}, t_1)x = y \) and \( (t_1, s_1 s_2^{-1}, t_2)y = x \). Thus, \( x \not\sim y \) (in \( D_X(\Phi) \)), and, hence, \( x \not\sim y \) (in \( D_X(\Phi) \)). A routine calculation establishes the converse.

Theorem 1.8 will be used in the proofs of Theorem 1.10, Remark 1.13, and Theorem 1.14.
Theorem 1.8. Let $S$ be a natural regular semigroup and let $T(S)$ be a semilattice $W$ of completely simple semigroups ($T_w: w \in W$). Then,

$$S \leq D'_X(t) \circ S/t$$

(1)

where $t$ is the smallest combinatorial inverse semigroup congruence on $S$ and $D'_X(t)$ (notation of Theorem 1.2) is a natural regular semigroup with

$$T(D'_X(t)) = \sum (A_v: v \in S/t \cap X)$$

(2)

where $A_v = \bigcup (T_z: z \geq v^{-1}v)$. Furthermore,

$$(D'_X(t))_{st} = T(D'_X(t))$$

(3)

if and only if $(S/t)z \subseteq E(S/t)$ for all $z \in S/t \cap X$.

**Proof.** Using Theorems 1.2 and 1.6, (1) is valid with $D'_X(t)$ a regular semigroup. We next establish (2). As a consequence of (2), $D'_X(t)$ is a natural regular semigroup. For $v \in S/t \cap X$, let $A_v = \{(v, s, v) \in D'_X(t): s \in T_z$ for some $z \geq v^{-1}v\}$. We first show that $A_v \neq \emptyset$. Let $s \in (v^{-1}v)t^{-1}$. Thus, $s \in T_0v^{-1}$ by Theorem 1.6, and furthermore, $v = v(v^{-1}v) = v(st)$. Hence, $(v, s, v) \in A_v$. Let $U = \bigcup (T_z: z \geq v^{-1}v)$. Using Theorem 1.6, it is easily checked that $(v, s, v) \rightarrow s$ defines an isomorphism of $A_v$ onto $U$. We next show that $T(D'_X(t)) = \sum (A_v: v \in S/t \cap X)$. Let $(v, s, v) \in T(D'_X(t))$. Thus $(v, s, v) \not\sim (u, e, u)$ (in $D'_X(t)$) for some $e \in E(S)$. Hence, using Theorem 1.6 and Lemma 1.7 or directly, $s \not\sim e$ (in $S$). Thus, $s \in T_z$ for some $z \in W$. Using [30, Proposition 3.3(b)], $v^{-1}v \leq (ss^{-1})t$ for some $s^{-1} \in S$. Let $s^*$ denote the group inverse of $s$ in $H_e$. Hence, using Theorem 1.6, $(ss^{-1})t = (ss^*)t = z$, and, thus $(v, s, v) \in A_v$. Conversely, let $(v, s, v) \in A_v$. Thus, $s \not\sim e$ for some $e \in E(T_z)$. Hence, using Theorem 1.6, $v(et) = v(st) = v$. Thus, $(v, e, v) \in E(D'_X(t))$. Hence, using Theorem 1.6 and Lemma 1.7, $(v, s, v) \not\sim (v, e, v)$ (in $D'_X(t)$). Thus, $(v, s, v) \in T(D'_X(t))$. Finally, we establish (3). First, we assume $(S/t)z \subseteq E(S/t)$ for all $z \in S/t \cap X$. Let $(z, s, z) \in (D'_X(t))_{st}$. Thus, $z(st) = z$ implies $st = w$ for some $w \in E(S/t)$. Hence, using Theorem 1.6, $s \in T_w$. Thus, $s \not\sim f$ for some $f \in E(S)$. Hence, using Theorem 1.6 and Lemma 1.7, $(z, s, z) \not\sim (z, f, z)$ (in $D'_X(t)$). Thus, $(z, s, z) \not\in T(D'_X(t))$. Hence, $T(D'_X(t)) = (D'_X(t))_{st}$. Conversely, assume $T(D'_X(t)) = (D'_X(t))_{st}$. Let $xt \in (S/t)z$ where $z \in S/t \cap X$. Then, $z(xt) = z$ and, thus, $(z, x, z) \not\in (D'_X(t))_{st}$. Hence, $(z, x, z) \not\sim (r, e, r)$ (in $D'_X(t)$) for some $e \in E(S)$. Thus, $x \not\sim e$ (in $S$). Hence, $xt = et \in E(S/t)$. Thus, $(S/t)_z \subseteq E(S/t)$. □

In Lemma 1.9, we show that for any natural semigroup $S$, the $\mathcal{D}$-classes of $S$ and the $\mathcal{D}$-classes of $S/t$ are in a one-to-one correspondence. Lemma 1.9 will be used in the proofs of Theorem 1.10 and Lemma 1.11.

Lemma 1.9. Let $S$ be a natural regular semigroup. Let $at = a^*$. Then, $(a, b) \in \mathcal{D}$ (in $S$) if and only if $(a^*, b^*) \in \mathcal{D}$ (in $S/t$).
Proof. It is immediate that \((a, b) \in D\) (in \(S\)) implies \((a^*, b^*) \in D\) (in \(S/t\)). Let \((a^*, b^*) \in D\) (in \(S/t\)). Hence, there exists \(c^* \in S/t\) such that \(a^* \not\in c^* \not\in b^*.\) Thus, \((aa^*)^* \not\in (cc^*)^*\) and \((c'c)^* \not\in (b' b)^*\) for any \(a' \in J(a)\), \(b' \in J(b)\), and \(c' \in J(c)\). Thus, using Theorem 1.6, \((aa^*, cc^*) \in t\) and \((c'c, b'b) \in t\). Hence, \(a \not\in aa' \not\in cc' \not\in c' \not\in b'b \not\in b \) (in \(S\)). \(\blacksquare\)

Theorem 1.10 will be used in the proof of Corollary 1.12.

**Theorem 1.10.** Let \(S\) be a natural regular semigroup of type \(\omega Y\). Then,

\[
S \cong D'_{X}(t) \circ \tilde{Y} \circ C
\]

where \(C\) is the bicyclic semigroup and \(D'_{X}(t)\) (notation of Theorems 1.2 and 1.8) is a natural regular semigroup such that each element of \(D'_{X}(t)\) is either a nilpotent element of degree two or an element of \(T(D'_{X}(t))\). Furthermore,

\[
T(D'_{X}(t)) = \sum (A^{0}_{(p, q, \beta)} : (p, q, \beta) \in (N \times N \times Y) \cap X)
\]

where

\[
A^{0}_{(p, q, \beta)} = \bigcup (T(k, y) : k < q, y \in Y) \cup (\bigcup (T(q, a) : \beta \leq \alpha)).
\]

**Proof.** Let \(S\) be a natural regular semigroup of type \(\omega Y\). Using Theorem 1.6 and Lemma 1.9, \(S/t\) is a combinatorial \(\omega Y\)-inverse semigroup (i.e. \(E(S/t)\) is an \(\omega Y\)-semilattice \((e_{n, \delta}) : (n, \delta) \in N \times Y\) with \(e_{n, \delta} \not\in e_{m, \lambda}\) (in \(S/t\)) if and only if \(\delta = \lambda\)). Thus, using [24, Theorem 2.3], \(S/t = ((n, k, y) : n, k \in N, y \in Y)\) under the multiplication

\[
(n, k, y)(r, s, z) = (n + r - \min(k, r), k + s - \min(k, r), f(k, r))
\]

where \(f(k, r) = y, y \wedge z,\) or \(z\) according to whether \(k > r, k = r,\) or \(r > k\) (i.e. \(S/t = Y \circ C\) (notation of [30])).

Hence, using Theorem 1.8 and [30, Lemma 2.7, Lemma 2.5, Definition 2.2, Lemma 2.1 and Lemma 1.6], \(S \cong D'_{X}(t) \circ \tilde{Y} \circ C.\) Using Theorem 1.8, \(D'_{X}(t)\) is a natural regular semigroup. We next apply Theorem 1.8 to show that \(T(D'_{X}(t)) = (D'_{X}(t))_{\text{stab}},\) i.e. that each element of \(D'_{X}(t)\) is either a nilpotent element of degree 2 or an element of \(T(D'_{X}(t))\). Let \((n, m, y), (p, q, z) \in S/t\) and assume \((n, m, y)(p, q, z) = (n, m, y).\) Thus, by a straightforward calculation, \(p = \min(m, p) = q.\) Hence, \((p, q, z) \in E(S/t).\) Thus, \(T(D'_{X}(t)) = (D'_{X}(t))_{\text{stab}}\) by Theorem 1.8. Let \(v = (p, q, \beta).\) Then, \(v^{-1} = (q, p, \beta)\) by [24, Corollary 2.4] or directly. Hence, \(v^{-1}v = (q, q, \beta).\) Thus, using Theorem 1.8 and its notation, \(A_{(p, q, \beta)} = \bigcup (T(k, v) : k < q, v \in Y) \cup (\bigcup (T(q, a) : \alpha \geq \beta))\) and \(T(D'_{X}(t)) = \sum (A^{0}_{(p, q, \beta)} : (p, q, \beta) \in (N \times N \times Y) \cap X).\) \(\blacksquare\)

In Lemma 1.11, we show that \(S\) is a simple natural regular semigroup of type \(\omega\) if and only if \(S\) is a natural regular semigroup of type \(\omega d\), for some positive integer \(d\). Thus, we may obtain a structure theorem for simple natural regular semigroups (Corollary 1.12) by specializing Theorem 1.10. Lemma 1.11 is used in the proof of Corollary 1.12.
Lemma 1.11. S is a simple natural regular semigroup of type $\omega$ if and only if $S$ is a natural regular semigroup of type $\omega d_r$, where $d$ is a positive integer. $S$ is bisimple if and only if $d=1$.

Proof. Let $S$ be a natural simple semigroup of type $\omega$. Thus, using Theorem 1.6, $S/t$ is a combinatorial simple inverse semigroup of type $\omega$ (i.e. $E(S/t)$ is an $\omega$-chain). Hence, using [24, Lemma 7.5], $S/t$ is a combinatorial $\omega d_r$-inverse semigroup for some positive integer $d$. Thus, using Theorem 1.6 and Lemma 1.9, $S$ is natural regular semigroup of type $\omega d_r$. Conversely, assume $S$ is a natural regular semigroup of type $\omega d_r$. Hence, $T(S)$ is an $\omega$-chain of completely simple semigroups $(T_n : n \in N_r)$. Using Theorem 1.6 and Lemma 1.9, $S/t$ is an $\omega d_r$-inverse semigroup. Thus, using [24, Lemma 7.3], $S/t$ is a simple semigroup.

Let $a, b \in S$. Hence, there exist $x, y \in S$ such that $xtaty = bt$. Thus, using Theorem 1.6 (definition of $t$), there exists $(xay)' \in A(xay)$ and $b' \in A(b)$ such that $(xay)(xay)', bb' \in T_n$ for some $n \in N_r$. Hence, there exist $u, v \in T_n$ such that $b = u(xay)(xay)v$ and $b' = u$. Thus, $S$ is a simple semigroup. The last line of the lemma is a consequence of [24, Corollary 2.4(e)] and Lemma 1.9. □

Corollary 1.12. Let $S$ be a simple natural regular semigroup of type $\omega$. Then,

$$S \leq D'_X(t) \circ \delta_c \circ C$$

where $D'_X(t)$ and $C$ are as in Theorem 1.10. Furthermore,

$$T(D'_X(t)) = \sum (A_{(p,q,i)}^0 : (p,q,i) \in N \times N \times d)$$

where

$$A_{(p,q,i)} = T_0 \cup T_1 \cup \cdots \cup T_{i+qd}.$$ 

If $S$ is bisimple, $d=1$.

Proof. Apply Lemma 1.11 and Theorem 1.10. □

Remark 1.13. $I_r$ is termed an $I_r$-chain. A regular semigroup $S$ is termed a natural regular semigroup of type $I$ if $T(S)$ is an $I_r$-chain of completely simple semigroups $(T_k : k \in I_r)$. Let $C^*$ denote $I \times I$ under the multiplication

$$(m,n)(p,q) = (m+p-\min(n,p), n+q-\min(n,p)).$$

If we replace $\omega$ by $I$, $C$ by $C^*$, $N$ by $I$ and let $A_{(p,q,i)} = \bigcup (T_k : k \leq i+qd)$ in Corollary 1.12, we obtain a structure theorem for simple natural regular semigroups of type $I$. To prove this theorem, we use Theorem 1.6, [20, Theorem 1.1], and [21, Remark 1.2] to show $S/t = C^* \times d_r$ under the multiplication $((m,n),i)((p,q),j) = ((m,n)(p,q), f(n,p))$ where $f(n,p) = i, \max(i,j)$, or $j$ according to whether $n>p$, $n=p$, or $p>n$. Using this multiplication and Theorem 1.8, we show $T(D'_X(t)) = (D'_X(t))_{stab} = \sum (A_{(p,q,i)}^0 : (p,q,i) \in I \times I \times d)$ and $A_{(p,q,i)} = \bigcup (T_k : k \leq i+qd)$.  

Using Theorem 1.8, [30, Lemma 1.6] and the proof of [30, Theorem 2.14], \( S \leq D'_X(t) \circ \sigma_C \). The last line of the theorem is a consequence of [21, Remark 1.2] and Lemma 1.9.

**Theorem 1.14.** Let \( S \) be a bisimple natural regular semigroup of type \( \omega^n \). Then, \( D'_X(t) \) (notation of Theorem 1.2 and 1.8) is a natural regular semigroup such that

\[
D'_X(t)_{\text{stab}} = \sum (A^0_v : v \in (N^{[2]})^n \times X)
\]

where \( A_v \) is a \( \bar{D} \)-compatible natural regular subsemigroup of \( S \).

If \( v = ((a_0, b_0), \ldots, (a_{n-1}, b_{n-1})) \), \( A_v / \bar{D} \) is a \( (b_0 + 1)_r + (b_1)_r + \cdots + (b_{n-1})_r \)-chain such that for \( x \in (b_0 + 1)_r \), \( x \bar{D}^{-1} \) is a \( f \)-class of \( T(S) \), and for \( x \in (b_i)_r \), where \( 1 \leq i \leq n-1 \), \( x \bar{D}^{-1} \) is a bisimple natural regular subsemigroup of type \( \omega^i \) of \( S \) and each \( f \)-class of \( T(x \bar{D}^{-1}) \) is a \( f \)-class of \( T(S) \). Furthermore,

\[
S \leq D'_X(t) \circ C \quad \text{for} \quad n = 1,
\]

\[
(*) \quad S \leq D'_X(t) \circ \bar{C}^{n-1} \circ C \quad \text{for} \quad n \geq 2
\]

where \( C \) is the bicyclic semigroup.

**Proof.** Let \( S \) be a bisimple natural regular semigroup of type \( \omega^n \). Using Theorem 1.6 and [16, Corollary 2.2], we will proceed to describe the multiplication on \( S/t \).

Let \( X \) be an arbitrary semigroup and define \( X \circ C \) to be \( X \times C \) under the multiplication \((a, (m, n))(b, (p, q)) = (f(n, p), (m, n)(p, q)) \) where \( f(n, p) = a, b, \) or \( ab \) according to whether \( n > p, p > n, \) or \( p = n \). Define \( C_1 = C, C_2 = C \circ C, \ldots, C_n = C_{n-1} \circ C \). Then, \( S/t = C_n \). For \( v \in C_n \), let \( A_v = \{(u, s, v) : u, s \in D'_X(t)\} \). Hence, \( D'_X(t)_{\text{stab}} = \sum (A^0_v : v \in C_n \cap X) \) where \( A_v \equiv (C_v)t^{-1} \) under the mapping \((v, s, v) \rightarrow s \). If \( v = ((a_0, b_0), \ldots, (a_{n-1}, b_{n-1})) \), we use the multiplication on \( C_n \) and mathematical induction on \( n \) to show that

\[
(C_v)_0 = \{(x, x), (b_1, b_1), \ldots, (b_{n-1}, b_{n-1}) : x \leq b_0\}
\]

\[
\cup \{z, (d_1, d_1), (b_2, b_2), \ldots, (b_{n-1}, b_{n-1}) : z \in C, d_1 < b_1\}
\]

\[
\cup \cdots \cup \{z, (d_i, d_i), (b_{i+1}, b_{i+1}), \ldots, (b_{n-1}, b_{n-1}) : z \in C, d_i < b_i\}
\]

\[
\cup \cdots \cup \{z, (d_{n-1}, d_{n-1}) : z \in C_{n-1}, d_{n-1} < b_{n-1}\}.
\]

For \( v \in C_n \), let \( t_v = vt^{-1} \). Hence, \( A_v \equiv \bigcup \{t_x : x \in (C_v)_0\} \). We assume \( T(S) \) is the \( \omega^n \)-chain of completely simple semigroups \( (T_v : v \in \omega^n) \). We will clarify the structure of \( A_{((a_0b_0), \ldots, (a_{n-1}, b_{n-1}))} \). Let

\[
A^{\omega^0} = \bigcup \{t_{(x, x), (b_1, b_1), \ldots, (b_{n-1}, b_{n-1})} : x \in b_0 + 1\}.
\]

Thus, using Theorem 1.6, \( A^{\omega^0} \) is the \( (b_0 + 1)_r \)-chain of completely simple semigroups \( (T_{(x, b_1, \ldots, b_{n-1})} : x \in b_0 + 1) \). For \( 1 \leq i \leq n-1 \) and \( j \in b_i \), let

\[
A^{\omega^j}_f = \bigcup \{t_{(z, z, (b_{i+1}, b_{i+1}), \ldots, (b_{n-1}, b_{n-1}))} : z \in C_i\}.
\]
We will show that $A_{\omega^j}$ is a bisimple natural regular semigroup of type $\omega^j$. Using Theorem 1.6, $T(A_{\omega^j})$ is the $\omega^j$-chain of completely simple semigroups

$$(T(d_0, d_1, \ldots, d_{i-1}, b_i, b_{i+1}, \ldots, b_{n-1}) : d_0, d_1, \ldots, d_{i-1} \in \mathbb{N}).$$

We next show that $A_{\omega^j}$ is a regular bisimple semigroup. It is easily seen that $A_{\omega^j}$ is a semigroup. Let $u, v \in A_{\omega^j}$. Since $S$ is bisimple, there exists $c \in S$ such that $u R c L v$. Let

$$ut = ((x_1, y_1), \ldots, (x_{i-1}, y_{i-1}), (j, j), (b_{i+1}, b_{i+1}), \ldots, (b_{n-1}, b_{n-1}))$$

and

$$vt = ((r_1, s_1), \ldots, (r_{i-1}, s_{i-1}), (j, j), (b_{i+1}, b_{i+1}), \ldots, (b_{n-1}, b_{n-1})).$$

Thus, since $ut \mathcal{R} ct \mathcal{L} vt$,

$$ct = ((x_1, s_1), \ldots, (x_{i-1}, s_{i-1}), (j, j), (b_{i+1}, b_{i+1}), \ldots, (b_{n-1}, b_{n-1})).$$

Hence, $c \in A_{\omega^j}$. Let $u' \in I(u)$. Thus,

$$u't = ((y_1, x_1), \ldots, (y_{i-1}, x_{i-1}), (j, j), (b_{i+1}, b_{i+1}), \ldots, (b_{n-1}, b_{n-1})).$$

Hence, $u' \in A_{\omega^j}$. Thus, $A_{\omega^j}$ is a regular semigroup. Similarly, $c' \in A_{\omega^j}$. Thus, since $u = c(c'u)$ and $c = u(u'c)$, $u \mathcal{R} c$ (in $A_{\omega^j}$). Similarly, $c \mathcal{L} v$ (in $A_{\omega^j}$). Hence, $A_{\omega^j}$ is bisimple semigroup. Let $A_{\omega^j} = \bigcup_{i \neq j} A_{\omega^j}$ with $A_{\omega^j} \cap A_{\omega^j} = \emptyset$ for $i \neq j$. Thus, $A_{\omega^j}$ is a $(b_0 + 1)_r + (b_1)_r + \cdots + (b_i)_r + \cdots + (b_{n-1})_r$-chain of the semigroups

$$\{T(j, b_i, \ldots, b_{n-1}) : j \in b_0 + 1\}, \{A_{\omega^j} : j \in b_1\}, \ldots, \{A_{\omega^j} : j \in b_i\}, \ldots, \{A_{\omega^{n-1}} : j \in b_{n-1}\}.$$

Since each semigroup in the above chain is bisimple, $\mathcal{D}$ is a congruence on $A_{\omega^j}$ and $A_{\omega^j}/\mathcal{D}$ is the above chain. Using Theorem 1.8 and [30, Lemma 1.6, Lemma 2.8, Lemma 2.5, Lemma 2.1 and Definition 2.2] we obtain ($*$). \hfill $\square$

**Remark 1.15.** In the statement of Theorem 1.14, replace "n" by "n + 1", "\omega^{n+1}" by "\omega^n I", "(\mathbb{N}^{[2]} \times \mathbb{N}^{[2]}) + (\mathbb{N}^{[2]} \times \mathbb{N}^{[2]}) + (\mathbb{N}^{[2]} \times \mathbb{N}^{[2]})" by "(\mathbb{N}^{[2]} \times \mathbb{N}^{[2]}) + (\mathbb{N}^{[2]} \times \mathbb{N}^{[2]}) + (\mathbb{N}^{[2]} \times \mathbb{N}^{[2]})" by "(\mathbb{N}^{[2]} \times \mathbb{N}^{[2]})"; under the reverse of usual order", and "(\ast)" by "$S \subseteq D_X(t) \circ C^n \circ C^\ast". We then obtain a structure theorem for bisimple natural regular semigroups of type $\omega^n I$. We outline a proof of this theorem. Define $C_{n+1}^\ast = C_n \circ C^\ast$ to be $C_n \times C^\ast$ under the multiplication $(x, (m, n))(y, (p, q)) = (f(n, p), (m, n)(p, q))$ where $f(n, p) = x$, $xy$, or $y$ according to whether $n > p$, $n = p$, or $p > n$. Using [17, Theorem 4.6], $S/t = C_{n+1}^\ast$. Proceed as in the proof of Theorem 1.14. Using Theorem 1.8 and the proof of [30, Theorem 2.13], $S \subseteq D_X(t) \circ C^n \circ C^\ast$.

2. Orthodox unions of groups

In this section, we will use the Rhodes Expansion and a parametrization of the
derived semigroup to clarify the structure of orthodox unions of groups (Theorem 2.6).

We first describe the Rhodes Expansion $\hat{S}$ of an arbitrary semigroup $S$. Let $S_+=\{(s_n,\ldots,s_1):s_i\in S \text{ for } 1\leq i\leq n \text{ and } s_n\cdots s_1\}$. If $x=(s_n,\ldots,s_1), y=(t_m,\ldots,t_l)$, define $xy=(s_nt_m,\ldots,s_1t_l,t_m,\ldots,t_l)$. Then, $S_+$ is a semigroup under this multiplication. If $a=(s_n,\ldots,s_1)\in S_+$ and $s_{k+1}\not\in s_k$ for some $1\leq k\leq n-1$, delete $s_k$ to obtain $a_1\in S_+$ and denote the deletion by $a\rightarrow a_1$. Perform $a\rightarrow a_1\rightarrow\cdots\rightarrow a_k$ where $a_k=(s_n,s_{n-1},\ldots,s_1)$ (such an $a_k$ is termed an irreducible element of $S_+$). Write $a_k=\text{red}a$ and $a\sim b$ if $\text{red}a=\text{red}b$ for $a,b\in S_+$. The equivalence relation $\sim$ is a congruence relation on $S_+$. Let $\hat{S}=S_+/_\sim$. $\hat{S}$ may be treated as the set of irreducible elements of $S_+$ under the multiplication $ab=\text{red}(ab)$. If $(s_n,\ldots,s_1)\in \hat{S}, (s_n,\ldots,s_1)\eta=s_n$ defines a homomorphism of $\hat{S}$ onto $S$ [30, Theorem 3.1(h)]. Let $\psi$ be a homomorphism of a semigroup $S$ onto a semigroup $T$. Then, $(s_n,\ldots,s_1)\psi=\text{red}(s_n\psi,\ldots,s_1\psi)$ defines a homomorphism of $\hat{S}$ onto $\hat{T}$ [30, Theorem 3.1(b)]. For further details on the Rhodes Expansion see [15], [5, pp. 367-369] and [14].

We will make considerable use of [30, Theorem 3.1]. This result is a collection of results of Rhodes and Tilson with appropriate references given in [30]. Theorem 2.6 will be a consequence of Lemmas 2.1–2.4 and Remark 2.5.

Let $S(T)$ be a semilattice $Y$ of semigroups $(S_y: y \in Y)((T_y: y \in Y))$ and let $\Phi$ be a homomorphism of $S$ onto $T$ such that $S_y \Phi \subseteq T_y$ for all $y \in Y$ and let $X$ be a right ideal of $T$. In Lemma 2.1, we show

$$\hat{D}_X(\Phi) = \{(a, s, a(s\Phi)) \in D(\Phi): a \in T_y \cap X, s \in S_z, y \leq z\} \cup \{0\}$$

is a subsemigroup of $D(\Phi)$ which is a parametrization of $D(\Phi)$ (hence, $S \leq \hat{D}_X(\Phi) \circ T$ by Rhodes Theorem (Theorem 1.1)); $\hat{D}_X(\Phi)$ is a direct sum of subsemigroups $A_y^0 (y : T_y \cap X \neq \emptyset \leq M^0(T_y \cap X, \bigcup (S_z : z \geq y), T_y \cap X, \text{id})$; and, under suitable conditions, $E(A_y^0)$ is a band with $E(A_y^0)/\mathfrak{f} \leq \{z \in Y | y \leq z\}^0 \times (T_y \cap X)$. Let $S$ be a union of groups, i.e., $S$ is a semilattice $Y$ of completely simple semigroups $(S_y: y \in Y)$. In Lemma 2.2, we show $\hat{S}$ is a semilattice $Y$ of completely simple semigroups $(X_y: y \in Y)$ where $X_y = S_y\eta^{-1}$. Let $S$ be an orthodox union of groups and let $\rho, (\delta)$ denote the smallest inverse semigroup congruence on $S ((S/\rho))$. In Lemma 2.3, we show $S/\rho$ and $(\hat{S}/\mathfrak{g})/\delta$ are both semilattices $S/\mathfrak{f} = Y$ of the maximal subgroups $(G_y: y \in Y)$ of $S$ and that $(\hat{S}/\mathfrak{g})$ is a semilattice $Y$ of right groups $(U_y: y \in Y)$ where $U_y = G_y\eta^{-1}(\eta : (\hat{S}/\mathfrak{g}) \rightarrow S/\rho)$. Let $S$ be a union of groups admitting a congruence $\rho$ such that $ep^{-1} \subseteq E(S)$ for all $e \in E(S/\rho)$ (for example, let $S$ be an orthodox union of groups (see Remark 2.5)). In Lemma 2.4, we show $D(\rho)$ consists just of idempotents and nilpotents of degree two.

**Lemma 2.1.** Let $S$ be a semilattice $Y$ of semigroups $(S_y: y \in Y)$, let $T$ be a semilattice $Y$ of semigroups $(T_y: y \in Y)$, and let $X$ be a right ideal of $T$. Let $\Phi$ be a homomorphism of $S$ onto $T$ such that $S_y \Phi \subseteq T_y$ for all $y \in Y$. Then,

$$\hat{D}_X(\Phi) = \{(a, s, a(s\Phi)) \in D(\Phi): a \in X, a \in T_y, s \in S_z \text{ with } y, z \in Y \text{ and } y \leq z\} \cup \{0\}$$
is a subsemigroup of $D(\Phi)$ and

$$S \leq D_X(\Phi) \circ T.$$  \hfill (1)

Let $Y' = \{ y \in Y \mid T_y \cap X \neq \emptyset \}$, and, for $y \in Y'$, let $A^0_y = \{(a, s, a(s\Phi)) \in D_X(\Phi) : a \in T_y \cap X \} \cup \{0\}$. Then, $D_X(\Phi)$ is a direct sum of the subsemigroups $(A^0_y : y \in Y')$ and $A^0_y \leq M^0(T_y \cap X, \bigcup (S_z : z \geq y), T_y \cap X, \text{id})$. \hfill (2)

If $E(S)$ is a band, $E(A^0_y)$ is a band. If, furthermore, $e, f \in E(S_y) (z \in Y)$ implies $e \not\sim f$ (in $E(S)$), then

$$E(A^0_y) / J \leq Y^0_x \times (T_y \cap X) \quad \text{where} \quad Y_y = \{ z \in Y, y \leq z \}.$$  

**Proof.** First, we show $D_X(\Phi)$ is a subsemigroup of $D(\Phi)$. Let $u = (a, s, a(s\Phi))$ and $v = (b, q, b(q\Phi))$ be elements of $D_X(\Phi)$. If $a(s\Phi) \neq b$, $uv = 0$. If $a(s\Phi) = b$, $uv = (a, sq, a(sq\Phi))$. Suppose $a \in T_y$, $s \in S_z$ with $y \leq z$, $b \in T_r$ and $q \in S_w$ with $w \leq r$. Since $a(s\Phi) = b$, $y = yz = w$. Thus, $sq \in S_w$, with $y \leq z$, and $w \leq r$. Hence, since $a \in X$, $uv \in D_X(\Phi)$. We will utilize Theorem 1.1 to show $S \leq D_X(\Phi) \circ T$. For $(t, s, t(s\Phi)) \in D(\Phi) - \{0\}$, define $(t, s, t(s\Phi))\theta = (t, s, t(s\Phi))$ if $t \in T_y \cap X$, $s \in S_z$, and $y \leq z$, and $(t, s, t(s\Phi))\theta = 0$, otherwise. We will show that $D(\Phi) - \{0\} \rightarrow D_X(\Phi)$ is a parametrization of $D(\Phi)$. Clearly, $\theta$ defines a mapping of $D(\Phi) - \{0\}$ into $D_X(\Phi)$. We first show that $\theta$ defines a partial homomorphism. Let $a = (t_1, s_1, t_1(s_1\Phi))$, $b = (t_2, s_2, t_2(s_2\Phi))$, $t_1 \in T_y$, $s_1 \in S_u$, $t_2 \in T_z$, $s_2 \in S_v$, and $t_1(s_1\Phi) = t_2$. Then, $ab = (t_1, s_1 s_2, t_1(s_1 s_2\Phi))$. First assume that $t_1 \in T_y \cap X$ and $y \leq uv$. Thus, $y \leq uv \leq u$, and, hence, $ab = a$. Using the fact that $X$ is a right ideal, $t_2 = t_1(s_1\Phi) \in T_y \cap X = T_y \cap X$. Furthermore, $y \leq uv \leq v$. Hence, $b\theta = b$. Thus, $(ab)\theta = ab\theta$. If $t_1 \notin X$, then $ab\theta = 0 \cdot b\theta = 0 = (ab)\theta$. Finally, we assume $y \leq uv$. First, suppose $y \leq u$. Then, $y \leq v$ (if $y \leq v$, $y \leq uv$). Since $t_2 = t_1(s_1\Phi)$, $z = yu = y \leq v$. Hence, $ab\theta = a\theta = (ab)\theta$. If $y \leq u$, $ab\theta = 0 \cdot b\theta = (ab)\theta$. Next, we establish the embedding condition. Suppose $s_1\Phi = s_2\Phi$ and $(t, s_1, t(s_1\Phi))\theta = (t, s_2, t(s_2\Phi))\theta$ for all $t \in T^*$. Let $u \in X$. Then, $u \in T_y$ for some $y \in Y$. Suppose $s_1, s_2 \in S_z$, say. Thus, $u(s_1\Phi) = u(s_2\Phi) \in T_y$. Hence, $(u(s_1\Phi), s_1, u(s_1\Phi))\theta = (u(s_1\Phi), s_2, u(s_2\Phi))\theta$ implies $s_1 = s_2$. Thus, $D_X(\Phi)$ is a parametrization of $D(\Phi)$, and, hence, using Theorem 1.1, $S \leq D_X(\Phi) \circ T$. By easy calculations $A^0_y (y \in Y')$ is a subsemigroup of both $D_X(\Phi)$ and $M^0(T_y \cap X, \bigcup (S_z : z \leq y), T_y \cap X, \text{id})$. Since $T_y \cap T_z = \emptyset$ for distinct $y, z \in Y'$, $D_X(\Phi)$ is a direct sum of the $(A^0_y : y \in Y')$. It is easily checked that $E(A^0_y) = \{(a, s, a) \in A^0_y : s \in E(S)\} \cup \{0\}$. Hence, if $E(S)$ is a band, $E(A^0_y)$ is a band. We will show that if $E(A^0_y)$ is a band and if $e, f \in E(S_y) (z \in Y)$ implies $e \not\sim f$ (in $E(S)$), then $E(A^0_y) / J \leq Y^0_x \times T_y \cap X$. We first just assume $E(A^0_y)$ is band. We then show that $E(A^0_y) / J = (a, e, a) \not\sim (b, f, b) (E(A^0_y))$ if and only if $a = b$ and $e \not\sim f$ (in $E(S)$). Since $\not\sim$ (in $E(S)$), there exists $g \in E(S)$ such that $e \not\sim g \not\sim f$ (in $E(S)$). Thus, $a(\Phi) = (a(f\Phi))(g\Phi) = a(f\Phi) = a$. Furthermore, since $f \in S_z$, say, with $y \leq z$, $g \in S_z$. Hence, $(a, g, a) \in E(A^0_y)$. It is easily checked that $(a, e, a) \not\sim (a, g, a) \not\sim (a, f, a)$ (in
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Thus, \( (a,e,a) \not\sim (b,f,b) \) (in \( E(A_y) \)). By a straightforward calculation, \( (a,e,a) \not\sim (b,f,b) \) (in \( E(A_y) \)) implies \( a=b \) and \( e \not\sim f \) (in \( E(S) \)). Let \( x \sim y \) denote the natural homomorphism of \( E(A_y) \) onto \( E(A_y)/\mathcal{J} \). For \( (a,e,a) \in E(A_y) \), define \( (a,e,a)y = (z,a) \) if \( e \in E(S_z) \) and define \( 0_y = 0 \). We will show \( y \) defines a homomorphism of \( E(A_y)/\mathcal{J} \) into \( Y_y \times (\tau_y \cap X) \). By a straightforward calculation, \( y \) is a homomorphism. To show \( y \) is one-to-one, we further assume that \( (a,e,a) \in E(S_y) \) implies \( a \not\sim b \) (in \( E(S) \)). Suppose \( (a,e,a)y = (b,f,b)y \). Thus, \( a=b \) and \( e \not\sim f \) (in \( E(S) \)). Since \( e \not\sim f \) (in \( E(S) \)), \( z=w \). Hence, \( (a,e,a) = (b,f,b) \). \( \square \)

Lemma 2.2. Let \( S \) be a semilattice \( Y \) of completely simple semigroups \( (S_y : y \in Y) \). Then, \( \hat{S} \) is a semilattice \( Y \) of completely simple semigroups \( (X_y : y \in Y) \) where \( X_y = \{(s_n,s_{n-1},...,s_1)n \in \mathbb{N}\setminus\{0\}, s_n \in S_y \) and \( s_j \in S \) for \( 1 \leq j \leq n-1 \). If \( S_y \) is a group, \( X_y \) is a right group.

Proof. By a straightforward calculation, \( X_y \) is a semigroup for \( y \in Y \). Let \( a = (s_n,s_{n-1},...,s_1) \in X_y \) and \( b = (t_k,t_{k-1},...,t_1) \in X_y \). Let \( c = (w_n,s_{n-1},...,s_1) \in X_y \) where \( s_n \not\sim w_n \not\sim t_k \) (in \( S_y \)). Hence, using \([30, \text{Theorem 3.1(g)} \) and \( 3.1(i)]\), \( a \not\sim c \not\sim b \) (in \( X_y \)) and \( X_y \) is a bisimple (hence, simple) semigroup. Let \( e, f \in E(X_y) \) and suppose \( ef = fe = e \). Using \([5, \text{Ch. 12, Proposition 12.1}] \) and \([30, \text{Theorem 3.1(f)} \) (both due to Tilson \([15] \)), \( e = (s_n,...,s_m,s_{m-1},...,s_1) \) and \( f = (t_m,t_{m-1},...,s_1) \) where \( m \leq n \), \( s_m \not\sim t_m \) (in \( S \)) and \( s_n,t_m \in E(S) \). Since \( S_y \) is completely simple, \( s_n = t_m \) by an easy calculation. Thus, \( e = f \) and hence, \( X_y \) is a completely simple subsemigroup of \( \hat{S} \). By an easy calculation, \( \hat{S} \) is the semilattice \( Y \) of the \( X_y \). Finally, suppose \( S_y \) is a right group. Using \([30, \text{Theorem 3.1(f)} \) and an easy calculation, \( E(X_y) \) is a right zero semigroup. Hence, \( X_y \) is a right group. \( \square \)

Lemma 2.3. Let \( S \) be an orthodox union of groups (i.e. \( S \) is a semilattice \( Y \) of orthodox completely simple semigroups \( (S_y : y \in Y) \)). Let \( \mathcal{Q} \) be the smallest inverse semigroup congruence on \( S \). Then, \( S/\mathcal{Q} \) is a semilattice \( Y \) of groups \( (G_y : y \in Y) \) where \( G_y \) is a maximal subgroup of \( S_y \) and \( S_y \cap \mathcal{Q} \subset G_y \) for all \( y \in Y \). (\( S/\mathcal{Q} \)) is a semilattice \( Y \) of right groups \( (U_y : y \in Y) \) where \( U_y = \{(g_n,...,g_1) : g_n \in G_y \) and \( g_j \in S_y \) for \( 1 \leq j \leq n-1 \). Let \( \delta \) denote the smallest inverse semigroup congruence on \( (S/\mathcal{Q}) \). Then, \( (S/\mathcal{Q})/\delta \) is a semilattice \( Y \) of groups \( (G_y : y \in Y) \) and ker \( \delta = E((S/\mathcal{Q})) \).

Proof. Let \( S \) be an orthodox union of groups. Thus, \( S \) is a semilattice \( Y \) of orthodox completely simple semigroups \( (S_y : y \in Y) \). Hence, \( S_y = G_y \times B_y \) (algebraic direct product) where \( G_y \) is a maximal subgroup of \( S_y \) and \( B_y \) is a rectangular band. Let \( \mathcal{Q} \) be the smallest inverse semigroup congruence on \( S \). Hence, \( \mathcal{Q} = \{(x,y) \in S^2 : \mathcal{A}(x) = \mathcal{A}(y) \} \) ([7, Theorem 1.12, p. 190] due to T.E. Hall). Thus, a multiplication \( \cdot \) may be defined on \( G = \bigcup (G_y : y \in Y) \) such that \( a \cdot b = ab \) for
where \((g, b)\delta = g\) for \(g \in G_y\) and \(b \in B_y\), \(S_y \subseteq G_y\) for all \(y \in Y\). Using Lemma 2.2, \((\widetilde{S}/\delta)\) is a semilattice \(Y\) of right groups \((U_y: y \in Y)\) where \(U_y = \{(g_n, g_{n-1}, \ldots, g_1): g_n \in G_y\) and \(g_j \in G\) for \(1 \leq j \leq n - 1\}\). Let \(G'_y = \{(g): g \in G_y\}\). Thus, \(G'_y \equiv G_y\).

Since \((g, b)\delta = g\) for \(g \in G_y\) and \(b \in E(U_y)\), \(E(S/\delta) = E((S/Q))\). []

Lemma 2.4. Let \(S\) be a regular semigroup admitting a congruence \(\varrho\) such that \(e\varrho^{-1} \subseteq E(S)\) for all \(e \in E(S/\varrho)\). Then,

(a) \(D(\varrho)_{\text{reg}}\) consists just of idempotents and nilpotents of degree two.

(b) If furthermore, \(S\) is a union of groups, \(D(\varrho)\) consists just of idempotents and nilpotents of degree two.

Proof. (a) Let \((a, s, a(s\varrho)) \in D(\varrho)_{\text{reg}}\) with \(a(s\varrho) = a\). Using [30, Proposition 3.3(b)], there exists \(s^{-1} \in \mathcal{H}(s)\) such that \(a(ss^{-1}\varrho) = a\). Hence, \(a(s^{-1}\varrho) = a(ss^{-1})\varrho = a\). Thus, \(s\varrho, s^{-1}\varrho, (ss^{-1})\varrho \in (\widetilde{S}/\varrho)_a\) and \(s\varrho \equiv (ss^{-1})\varrho\) (in \((\widetilde{S}/\varrho)_a\)). Hence, using [30, Theorem 3.1(e)], \(s\varrho = (ss^{-1})\varrho\). Let \(s = (a_n, \ldots, a_1) \in \mathcal{S}\). Thus, \(s\varrho = \text{red}(a_n\varrho, \ldots, a_1\varrho) \in E(S/\varrho)\). Hence, using [30, Theorem 3.1(f)], \(a_n\varrho \in E(S)\). Thus, \(a_n \in E(S)\) and hence \(s \in E(\mathcal{S})\) by [30, Theorem 3.1(f)]. Thus, \((a, s, a(s\varrho)) \in E(D(\varrho))\). If \(a(s\varrho) \neq a\), \((a, s, a(s\varrho))^2 = 0\).

(b) Next, assume \(S\) is a union of groups. Let \((a, s, a(s\varrho)) \in D(\varrho)\) with \(a(s\varrho) = a\). Using Lemma 2.2, \(\mathcal{S}\) is a union of groups. Hence, it is easily checked that \(s \neq e\) for some \(e \in E(\mathcal{S})\). Let \(s^{-1}\) denote the group inverse of \(s\) in the group \(H_e\). Thus, \(a(ss^{-1}\varrho) = a(e\varrho) = (a(s\varrho))(e\varrho) = a(s\varrho) = a\). Hence, using [30, Proposition 3.3(b)], \((a, s, a(s\varrho)) \in D(\varrho)_{\text{reg}}\). Thus, using Lemma 2.4(a), \((a, s, a(s\varrho)) \in E(D(\varrho))\). If \(a(s\varrho) \neq a\), \((a, sa(s\varrho))^2 = 0\). []

Remark 2.5. Let \(\varrho\) denote the smallest inverse semigroup congruence on an orthodox semigroup \(S\). Using, for example, [23, proof of theorem] or [30, proof of Theorem 3.4], \(e\varrho^{-1} \subseteq E(S)\) for all \(e \in E(S/\varrho)\). Hence, Lemma 2.4(a) may be applied to refine [30, Theorem 3.4].

Theorem 2.6. Let \(S\) be an orthodox union of groups (i.e. \(S\) is a semilattice \(Y\) of orthodox completely simple semigroups \((S_y: y \in Y)\)). Then,

\[
S \xrightarrow{\eta} \mathcal{S} \leq D_{\gamma_0}(\mathcal{Q}) \circ (\widetilde{S}/\varrho)
\]

where \((s_n, \ldots, s_1)\eta = s_n\) defines an \(E\)-epimorphism of \(\mathcal{S}\) onto \(S\) which is both a \(\mathcal{F}\)-map and an \(\mathcal{R}\)-map. \(\mathcal{S}\) is an orthodox union of groups which is a semilattice \(Y\) of orthodox completely simple semigroups \((X_y: y \in Y)\), \(\varrho\) is the smallest inverse semigroup congruence on \(S\), \((S/\varrho)\) is a semilattice \(Y\) of right groups \((U_y: y \in Y)\), and \(y_0\)
is a fixed element of \( Y \). \( D_{y_0}(\hat{\sigma}) \) is a combinatorial semigroup consisting just of idempotents and nilpotents of degree two and \( D_{y_0}(\hat{\sigma})_{\text{reg}} \) is an orthodox subsemigroup of \( D_{y_0}(\hat{\sigma}) \). Furthermore, \( D_{y_0}(\hat{\sigma}) \) is a direct sum of subsemigroups \( (A^0_y : y \in Y, y \leq y_0) \) such that

\[
A^0_y \leq M^0(U_y, \bigcup(X_z : y \leq z), U_y, \text{id}). \tag{2}
\]

\((A^0_y)_{\text{reg}} \) is an orthodox subsemigroup of \( A^0_y \) such that

\[
E(A^0_y) \leq Y \times U_y \quad \text{where} \quad Y_y = \{z \in Y \mid y \leq z\}. \tag{3}
\]

If \((a, e, a) \sim (b, f, b) \) in \( E(A^0_y) \), then \((e, f) \in \hat{\sigma}^{-1} \circ \hat{\sigma} \). Furthermore,

\[
(S/\hat{\sigma}) \leq ((S/\hat{\sigma})/\hat{\delta}) 1 \delta (E((S/\hat{\sigma}))) \tag{4}
\]

where \( \delta \) is the smallest inverse semigroup congruence on \( (S/\hat{\sigma}) \). \( (S/\hat{\sigma})/\hat{\delta} \) is a semilattice \( Y \) of groups \( (G_y : y \in Y) \) where \( G_y \) is a maximal subgroup of \( S_y \). \( E((S/\hat{\sigma})) \) is a semilattice \( Y \) of right zero semigroups.

**Proof.** Using [30, Proposition 3.3(a)] and Lemma 2.2, \( S \) is an orthodox union of groups which is a semilattice \( Y \) of orthodox completely simple semigroups \((X_y : y \in Y)\) where \( X_y = S_y \eta^{-1} \). Using [30, Theorem 3.4], \( \eta \) defines an \( E \)-epimorphism and an \( R \)-map of \( S \) onto \( S \). Using Lemma 2.2, \( \eta \) defines a \( \hat{\sigma} \)-map. Using Lemma 2.3, \( S/\hat{\sigma} \) is a semilattice \( Y \) of groups \((G_y : y \in Y)\) where \( G_y \) is a maximal subgroup of \( S_y \), and \((S/\hat{\sigma}) \) is a semilattice \( Y \) of right groups \((U_y : y \in Y)\) where \( U_y = \{(g_1, \ldots, g_1) : g_n \in G_y \text{ and } g_j \in S/\hat{\sigma} \text{ for } 1 \leq j \leq n - 1\} \). Using [30, Theorem 3.1(b)] and Lemma 2.3, \((x_1, \ldots, x_1) \hat{\sigma} = \text{red}(x_0 \hat{\sigma}, \ldots, x_0 \hat{\sigma}) \) defines a homomorphism of \( S \) onto \((S/\hat{\sigma}) \) such that \( X_y \hat{\sigma} \subseteq U_y \) for all \( y \in Y \). Let

\[
D_{y_0}(\hat{\sigma}) = \{(t, s, t(s\hat{\sigma})) \in D(\hat{\sigma}) : t \in U_y, y \leq y_0, s \in X_z, y \leq z \} \cup \{0\}
\]

(note, \( \bigcup(U_y : y \in Y, y \leq y_0) \) is an ideal of \( (S/\hat{\sigma}) \)). Hence, using Lemma 2.1, \( D_{y_0}(\hat{\sigma}) \) is a subsemigroup of \( D(\hat{\sigma}) \) and \( S \leq D_{y_0}(\hat{\sigma}) \circ (S/\hat{\sigma}) \). Thus, (1) is established. Using [30, Theorem 3.3(a)], \( D(\hat{\sigma}) \) is a combinatorial semigroup. Hence, \( D_{y_0}(\hat{\sigma}) \) is a combinatorial semigroup. Using Remark 2.5 and Lemma 2.4(b), \( D(\hat{\sigma}) \) and, hence, \( D_{y_0}(\hat{\sigma}) \) consists just of idempotents and nilpotents of degree two.

For \( y \in Y \) with \( y \leq y_0 \), let \( A^0_y \) be the direct sum of the subsemigroups \((A^0_y : y \in Y, y \leq y_0)\) and (2) and (3) are valid. We next show \( D_{y_0}(\hat{\sigma}) \) \( D_{y_0}(\hat{\sigma})_{\text{reg}} \) is a regular subsemigroup of \( D_{y_0}(\hat{\sigma}) \). Let

\[
(a, s, a(s\hat{\sigma})) \in D_{y_0}(\hat{\sigma}) \quad \text{and suppose} \quad a \in U_y, y \leq y_0 \text{ and } s \in X_z \text{ with } y \leq z \text{. Thus, by [30, Proposition 3.3(b)], there exists } s^{-1} \in \mathcal{J}(a) \text{ such that } a(s(s^{-1})\hat{\sigma}) = a \text{. Since } s^{-1} \in X_z \text{ with } y \leq z \text{ and } a(s\hat{\sigma}) \in U_y, (a(s\hat{\sigma}), s^{-1}, a) \in D_{y_0}(\hat{\sigma}) \text{. By a routine calculation, } (a(s\hat{\sigma}), s^{-1}, a) \in \mathcal{J}(a, s, a(s\hat{\sigma})) \text{. Hence, } (a(s\hat{\sigma}), s^{-1}, a) \in D_{y_0}(\hat{\sigma})_{\text{reg}} \text{. Let } u, v \in D_{y_0}(\hat{\sigma})_{\text{reg}} \text{. Since } D(\hat{\sigma})_{\text{reg}} \text{ is an (orthodox) subsemigroup of } D(\hat{\sigma}) \text{ by [30, Theorem 3.4], it follows that } uv \in D_{y_0}(\hat{\sigma}) \cap D(\hat{\sigma})_{\text{reg}} \text{. Thus, as above, there exists } (uv)^{-1} \in \mathcal{J}(uv) \cap D_{y_0}(\hat{\sigma})_{\text{reg}} \text{. Hence, } D_{y_0}(\hat{\sigma})_{\text{reg}} \text{ is a regular subsemigroup of } D_{y_0}(\hat{\sigma}) \text{. Since } D(\hat{\sigma})_{\text{reg}} \text{ is orthodox, } D_{y_0}(\hat{\sigma})_{\text{reg}} \text{ is orthodox. Similarly for } y \in Y \text{ and } y \leq y_0, (A^0_y)_{\text{reg}} \text{ is an orthodox subsemigroup of }
A°

We next show \((a,e,a)\not\approx (b,f,b)\) (in \(E(A°)\)) implies \((e,f)\in \delta^{-1} \circ \delta\). Suppose \((a,e,a)\not\approx (b,f,b)\) (in \(E(A°)\)). Thus, \(a=b\) and \(e,f\in E(X_z)\) for some \(z\in Y\). Thus, \(e \not\approx g \not\approx f\) for some \(g \in E(X_z)\). Since \(g \not\approx f\), \(g\delta = f\delta\). Hence, \(a(g\delta) = a(f\delta) = a\). Thus, \(g\delta, e\delta \in (S/\delta)_a\) and \(g\delta \not\approx e\delta\) (in \((S/\delta)_a\)). Hence, using [30, Theorem 3.1(e)], \(g\delta = e\delta\). Thus, \(e\delta = f\delta\) and, hence, \((e,f) \in \delta^{-1} \circ \delta\). Finally, let \(\delta\) denote the smallest inverse semigroup congruence on \((S/\delta)\). Hence, using Lemma 2.3, \((S/\delta)/\delta\) is a semilattice \(Y\) of the groups \((G_y : y \in Y)\) and \(\ker \delta = E((S/\delta))\). To obtain (4), apply the left-right dual of [30, Theorem 1.8].

3. Left regular pairs

Let \(S\) be a regular semigroup admitting a congruence \(\varrho\) such that \(e\varrho\) is a left group for all \(e\in E(S)\). We term \((S, \varrho)\) a left regular pair. We showed in [30, Theorem 1.7] that, if \((S, \varrho)\) is a left regular pair, then \(S \leq \langle \ker \varrho \rangle \circ S/\varrho\) (if \(S\) has an identity). We will give a much briefer proof of this result here by showing \(\langle \ker \varrho \rangle\) is a parametrization of \(D(\varrho)\) and then applying Rhodes Theorem (Theorem 1.1). It is easy to dispense with the requirement that \(S\) have an identity.

**Theorem 3.1** [30, Theorem 1.7]. Let \((S, \varrho)\) be a left regular pair. Then,

\[
(*) \quad S \leq \langle \ker \varrho \rangle^1 \circ (S/\varrho)^1
\]

If \(S\) has an identity, the superscript ‘1’ in \((*)\) may be deleted.

**Proof.** Let \((S, \varrho)\) be a left regular pair. We first assume that \(S\) has an identity. For each \(s\in S/\varrho\), select \(s' \in f(s)\) and select a representative element \(u_s \in s\varrho^{-1}\). Then, using [30, Lemma 1.1], every element of \(S\) may be uniquely expressed in the form \(g_{ss'}u_s\) where \((ss')\varrho \in \varrho^{-1}\). For \(t \in S/\varrho\) and \(s \in z\varrho^{-1}\), we may write \(u_{ts} = f(t, s)u_s\) where \(f(t, s) \in ((ts)(ts)')\varrho^{-1}\). For \((t, s, t(s\varrho)) \in D(\varrho) - \{0\}\), define \((t, s, t(s\varrho))\theta = f(t, s).\)

We will show \(\theta : D(\varrho) - \{0\} \to \langle \ker \varrho \rangle\) is a parametrization of \(D(\varrho)\). It is easily checked that \(\theta\) defines a mapping of \(D(\varrho) - \{0\}\) into \(\langle \ker \varrho \rangle\). Next, we show \(\theta\) defines a partial homomorphism. Let \((t_1, s_1, t_1(s_1\varrho)), (t_2, s_2, t_2(s_2\varrho)) \in D(\varrho) - \{0\}\) with \(t_1(s_1\varrho) = t_2\). We must show \(f(t_1, s_1)f(t_2, s_2) = f(t_1, s_1s_2)\). Suppose \(s_1 \in z_1\varrho^{-1}\) and \(s_2 \in z_2\varrho^{-1}\). Then, \(u_{t_1}(s_1s_2) = f(t_1, s_1s_2)u_{t_1s_1z_1z_2}\) where \(f(t_1, s_1s_2) \in ((t_1z_1z_2)(t_1z_1z_2))\varrho^{-1}\). However, since \(t_1z_1 = t_2, (u_{t_1}(s_1s_2) = f(t_1, s_1s_2)u_{t_1s_1z_1z_2} = f(t_1, s_1)f(t_2, s_2)u_{t_1s_1z_1z_2}.\) Since \(f(t_1, s_1)f(t_2, s_2) \in ((t_1z_1z_2)(t_1z_1z_2))\varrho^{-1}, f(t_1, s_1)f(t_2, s_2) = f(t_1, s_1s_2).\) We next show the embedding condition is satisfied. Let \(e\) denote the identity of \(S/\varrho\) and let \(u_e = 1\), the identity of \(S\). Thus, \(s_1\varrho = s_2\varrho = z\) and \(f(e, s_1) = f(e, s_2)\) imply \(s_1 = u_s s_1 = f(e, s_1)u_s = f(e, s_2)u_s = u_s s_2 = s_2.\) Hence, using Theorem 1.1, \(S \leq \langle \ker \varrho \rangle \circ S/\varrho\). If \(S\) has no identity, consider \(S^1.\) Define \(\varrho^* = \varrho \cup (1, 1)\). Then, \((S^1, \varrho^*)\) is a left regular pair. Hence, \(S^1 \leq \langle \ker \varrho^* \rangle \circ S^1/\varrho^*\). Since \(\langle \ker \varrho^* \rangle \cong \langle \ker \varrho \rangle^1\) and \(S^1/\varrho^* \cong (S/\varrho)^1\), using [30, Lemma 1.6], \(S \leq S^1 \leq \langle \ker \varrho \rangle^1 \circ (S/\varrho)^1.\) \(\square\)
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Remark 3.2. Many of the results of [30] were consequences of Theorem 3.1 (i.e. of [30, Theorem 1.7]). Thus, the present method of proof further illustrates the importance of the derived semigroup and its parametrizations in the structure theory.

References
