Solving a non-linear stochastic pseudo-differential equation of Burgers type

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Abstract

In this paper, we study the initial value problem for a class of non-linear stochastic equations of Burgers type of the following form

$$\partial_t u + q(x, D)u + \partial_x f(t, x, u) = h_1(t, x, u) + h_2(t, x, u) F_{t, x}$$

for $u : (t, x) \in (0, \infty) \times \mathbb{R} \mapsto u(t, x) \in \mathbb{R}$, where $q(x, D)$ is a pseudo-differential operator with negative definite symbol of variable order which generates a stable-like process with transition density, $f, h_1, h_2 : [0, \infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are measurable functions, and $F_{t, x}$ stands for a Lévy space-time white noise. We investigate the stochastic equation on the whole space $\mathbb{R}$ in the mild formulation and show the existence of a unique local mild solution to the initial value problem by utilising a fixed point argument.

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1. Introduction

Starting with a given complete probability space $(\Omega, \mathcal{F}, P; \{\mathcal{F}_t\}_{t \geq 0})$, we are concerned with the initial value problem for the following non-linear stochastic equation in one space dimension...

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\begin{equation}
\begin{aligned}
\partial_t u(t, x; \omega) + q(x, D_x)u(t, x; \omega) + \partial_x f(t, x, u(t, x; \omega)) &= h_1(t, x, u(t, x, \omega)) \\
+ h_2(t, x, u(t, x, \omega))F_{t,x}(\omega), & \quad (t, x, \omega) \in (0, \infty) \times \mathbb{R} \times \Omega \\
u(0, x, \omega) &= u_0(x, \omega), & \quad (x, \omega) \in \mathbb{R} \times \Omega
\end{aligned}
\end{equation}

on the domain \([0, \infty) \times \mathbb{R}\), where \(q(x, D_x)\) is a pseudo-differential operator with negative definite symbol of variable order of the form \(q(x, k) := -|k|^\alpha(x)\) for \(\alpha : \mathbb{R} \to (0, 2)\) being measurable, which generates a stable-like process with transition density \(\{G(s, x; t, y)\}_{0 < x < t < \mathbb{R}, x, y \in \mathbb{R}}\) (see Section 2.1), \(f : \mathbb{R} \to \mathbb{R}\) is at least a measurable function, \(h_1, h_2 : [0, \infty) \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}\) are measurable functions, the initial data \(u_0 : \mathbb{R} \times \Omega \rightarrow \mathbb{R}\) is \(F_0\)-measurable, and \(F_{t,x}\) is the Lévy space-time white noise (see [40] for a detailed account of \(F_{t,x}\)). These kinds of equations arise from modeling anomalous diffusion in continuum mechanics and are generalisations of the classical Burgers equation. We also refer to related problems using fractional derivatives and their calculus, see e.g. [17, 32, 43, 35]. Recently there has been an increasing interest in studying the Eq. (1.1) both from the stochastic partial differential equations (SPDEs) aspect and from the non-linear partial differential equations aspect, see, e.g. [2, 31, 6–12, 14, 16] and references therein. In [40] (as well as in [38, 39] for some special cases), the initial value problem for Eq. (1.1) where \(q(x, D_x)\) is replaced by certain integro-differential operators governed by constant stable index \(\alpha \in (0, 2)\), was investigated and existence and uniqueness results of local mild solutions were obtained. In [24], we consider the deterministic equation (with \(h_1 \equiv h_2 \equiv 0\)) in any space dimension \(n \in \mathbb{N}\), and show under certain conditions the existence of a unique (global) mild solution on \(\mathbb{R}^n\). Our object of interest in the present article is a stochastic extension of non-linear equation of Burgers type involving a suitable pseudo-differential operator with variable order index \(\alpha : \mathbb{R} \to (0, 2)\) in \(\mathbb{R}^n\). The existence and uniqueness of (local) mild solutions for the initial value problem (1.1). Our Eq. (1.1) features more on the anomalous effects as part of the mathematical modeling on continuum mechanics and it is natural to study this kind of nonlinear SPDEs mathematically. On the other hand, as we encounter a stable-like Markov generator with variable order \(\alpha(x)\), we have to carry out new transition density estimates (with different techniques compared with those used in [40]) for such a generator and we have to treat some extra terms (as the estimates of the fundamental solution are very different from the case when \(\alpha(x)\) is a constant) for proving the existence and uniqueness result towards (1.1).

Due to the appearance of the Lévy space-time white noise which consists of a Gaussian space-time white noise and certain Poisson space-time white noise as an independent sum, the whole consideration in this paper is limited to one space dimension. The reason for this is, as pointed out initially in [41] (cf. pp. 328–329), the relevant stochastic integral with respect to the Gaussian space-time white noise diverges when the space dimension is greater than or equal to two, and one only has Schwartz distributional solutions. As our Eq. (1.1) involves a Burgers type non-linear term, it does not make sense to seek distributional solutions for (1.1). Thus, we restrict ourselves in this paper to one space dimension.

However, we would like to point out that in order to study Eq. (1.1) in higher spatial dimensions, one has to modify the (general) Lévy space-time white noise into a suitable special form. Mueller proves in [33] the existence and uniqueness of a short time solution for the initial value problem for stochastic (fractional) heat equation driven by a (special) pure Poisson space-time white noise. In a recent paper [42], Wu and Xie establish the existence of a unique \(L^p\) (for \(p \in (1, 2]\)) local solution to the initial value problem (1.1) in \(\mathbb{R}^n\) for any \(n \in \mathbb{N}\) driven by a pure (general) Poisson compensated martingale measure (namely, a pure jump Lévy space-time white noise).
noise without the Gaussian part). In fact, it is the absence of Gaussian noise that allows one to consider the problem (1.1) in more than one space dimension. It was showed in [42] that, under the following condition
\[ \alpha_L := \inf_{x \in \mathbb{R}^n} \alpha(x) \in \left[ \frac{n(p-1)}{p} + 1, 2 \right] \]
for certain exponent index \( p \in (1, 2] \) characterizing the intensity measure of the Poisson noise, there exists a unique local \( L^p \) solution to (1.1). The main steps of the proof in [42] go along the same line as in the current paper, with the difference of replacing \( L^2 \) estimates in the present paper by \( L^p \) bounds in the whole derivation there. Indeed, the above condition gives the full range of parameters in the absence of Gaussian space-time white noise for which the existence and uniqueness hold for the initial value problem (1.1) in \( \mathbb{R}^n \).

Let us finish this section by giving a more general remark regarding mathematical modeling here. Modeling physical phenomena using jump-type processes is relatively new compared to approaches using diffusions. In parallel, although much progress has been made towards the analysis and stochastic analysis of jump-type processes in the last two or three decades, they are still not as satisfactory and comfortable to work with compared to their counterparts for diffusions. This naturally leads to restrictions when modeling with jump-type processes. Therefore, simplified models handled with new or modified techniques may contribute to enlarging the toolbox available for further investigations. In this sense, the present note combined with its “deterministic” part [24] and considerations in [27], has as an additional minor aim to contribute to the toolbox when modeling with jump-type processes: we emphasise certain techniques that have their origin in the theory of integro-differential and pseudo-differential operators, and consider them in the context of stochastic calculus for jump-type processes.

The paper is organised as follows. In Section 2, we present basic results related to the pseudo-differential operator \( q(x, D_x) \) and certain estimates for its fundamental solutions. We also give a brief account of Lévy space-time white noise. We end the section with the main result of the paper. Section 3 is then devoted to presenting all relevant proofs.

2. Preliminaries and notations

2.1. The fundamental solution and some estimates

We first elucidate our operator \( q(x, D_x) \) in more detail by following [3,34,23,26], while we refer [36,37,20–22,18] for general and systematical account of pseudo-differential operators and the associated Markov processes. As the Markov generator of a stable-like process, the operator \( q(x, D_x) \) has the following representation
\[ - (q(x, D_x)\varphi)(x) = \int_{\mathbb{R}\setminus\{0\}} \left[ \varphi(x + z) - \varphi(x) - \frac{z\varphi'(x)1_{|z|<1}(z)}{1 + |z|^2} \right] \frac{dz}{|z|^{1+\alpha(x)}} \]  
for certain suitable functions \( \varphi : \mathbb{R} \to \mathbb{R} \) (for instance, \( \varphi \) could be a Schwartz test function on \( \mathbb{R} \)), where \( \alpha : \mathbb{R} \to (0, 2) \) is a measurable function.

The symbol of \( q(x, D_x) \), usually denoted by the same notation \( q \) as a complex-valued bivariate function \( q : \mathbb{R} \times \mathbb{R} \to \mathbb{C} \), can be calculated as follows
\[ q(x, k) := e^{ik} (q(x, D_x)e^{-ik})(x) \]
which implies that the symbol for our $q(x, D_x)$ is real-valued and

$$q(x, k) = q(x, -k), \quad (x, k) \in \mathbb{R} \times \mathbb{R}.$$  

Thus, for $k \geq 0$, we have

$$q(x, k) = \rho_{\alpha(x)} k^{\alpha(x)}$$

where $\rho_{\alpha(x)}$ is a negative valued function of $x$ determined by (cf. e.g. \cite{[1]})

$$\rho_{\alpha(x)} := \int_{\mathbb{R}\setminus\{0\}} (\cos z - 1)\frac{dz}{|z|^{1+\alpha(x)}} < 0.$$  

Hence, we get

$$q(x, k) = \rho_{\alpha(x)} |k|^\alpha(x), \quad (x, k) \in \mathbb{R} \times \mathbb{R}.$$  

Let us point out that $\rho_{\alpha(x)}$ is nothing but $\frac{1}{w_{\alpha(x)}}$ in \cite{[34]}. With such a calculation, our operator $q(x, D_x)$ becomes

$$-q(x, D_x) = \rho_{\alpha(x)}(-\Delta)^{\frac{\alpha(x)}{2}} = \rho_{\alpha(x)} \left(-\frac{d^2}{dx^2}\right)^{\frac{\alpha(x)}{2}},$$

where $(-\Delta)^{\frac{\alpha(x)}{2}}$ is defined as

$$(-\Delta)^{\frac{\alpha(x)}{2}} \varphi(x) := -(2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{i\alpha x} |k|^\alpha(x) \hat{\varphi}(k) dk$$

with $\hat{\varphi}$ being the Fourier transform of the function $\varphi$. Moreover, the associated semigroup is a Feller semigroup having a transition density (cf. e.g. \cite{[26,34,23]}).

On the other hand, let us represent our operator $q(x, D_x)$ in polar coordinates as in the framework of \cite{[27]}. Following the notations of \cite{[27]}, let $S^0 := \{-1, 1\}$ and $\mathcal{B} := \mathcal{P}(S^0) = \{\phi, \{-1\}, \{1\}, S^0\}$. Given any finite, centrally symmetric measure $m$ on $(S^0, \mathcal{B})$, we define $s := \frac{z}{|z|}$ for $z \in \mathbb{R}\setminus\{0\}$. Then, (2.1) can be rewritten as

$$-(q(x, D_x)\varphi)(x) = \int_0^\infty \int_{S^0} \left[\varphi(x + z) - \varphi(x) - \frac{\varphi'(x)1_{z<1}(z)}{1 + z^2}\right] \frac{dz}{z^{1+\alpha(x)}} m(ds)$$

which is in the form of the generators for stable-like processes considered in \cite{[27]} (cf. Section 5 there). Thus, by Theorem 5.1 of \cite{[27]} (see page 759 there), $-q(x, D_x)$ generates a stable-like process with probability transition density

$$\{G(s, x_0; t, x)\}_{0 < s < t < \infty, x_0, x \in \mathbb{R}}$$

which is nothing but the fundamental solution of the following parabolic equation

$$\partial_t u(t, x) = -q(x, D_x)u(t, x), \quad (t, x) \in (s, \infty) \times \mathbb{R}$$

namely,
\[
\begin{align*}
\partial_t G(s, x_0; t, x) &= -q(x, D_x)G(s, x_0; t, x), \quad (t, x) \in (s, \infty) \times \mathbb{R} \\
\lim_{t \downarrow s} G(s, x_0; t, x) &= \delta_{x_0}(x), \quad x \in \mathbb{R} 
\end{align*}
\] (2.3)

for \((s, x_0) \in [0, \infty) \times \mathbb{R}\), such that

\[
u(t, x_0) = \int_\mathbb{R} G(s, x_0; t, x)\varphi(x)dx
\] (2.4)

is a weak solution to the Cauchy problem for the parabolic equation (2.2) with initial data \(u(s, x_0) = \varphi(x_0)\). Furthermore, for \(t > 0\),

\[
(T_t \varphi)(x) = \int_\mathbb{R} G(0, x; t, y)\varphi(y)dy, \quad x \in \mathbb{R}.
\] (2.5)

The study of stable-like Markov generators with variable order can be traced back to the seminal paper [3] where Bass studied pure jump Markov processes associated with such Lévy type Markov generators. Further works on sample path behaviors as well as transition densities related to stable-like processes can be found in [34,26,27,13,5,4].

The topic of estimating the transition densities (or equivalently, the fundamental solutions) \(G\) associated with Lévy stable type Markov generators \(q(x, D_x)\) (with fixed \(\alpha \in (0, 2)\)) was started in [28,29]. There are further investigations for \(q(x, D_x)\) with variable order in [34,26,27,5,13,4], where under certain further assumptions on \(\alpha(x)\), interesting estimates for transition densities of the stable-like processes have been achieved.

As explicated in Section 9 of [27] (cf. page 766), the estimates for stable-like transition densities obtained by Kolokoltsov carry over to our one-dimensional stable-like transition density \(G(s, z; t, x)\) (see Proposition 3.1, Theorem 5.1 and its corollary in [27]) which we summarise as follows: Let

\[
\alpha_L := \inf_{x \in \mathbb{R}} \alpha(x) \quad \text{and} \quad \alpha^U := \sup_{x \in \mathbb{R}} \alpha(x).
\]

**Proposition 2.1.** If \(0 < \alpha_L \leq \alpha^U < 2\) and the derivative of \(\alpha : \mathbb{R} \to [\alpha_L, \alpha^U]\) is uniformly continuous and bounded, then there exists a constant \(C\) only depending on \(\alpha_L\) and \(\alpha^U\) such that for any \(0 \leq s < t\) and \(x, z \in \mathbb{R}\), the following estimates hold

\[
G(s, z; t, x) \leq C \left( (t-s)^{-\frac{\beta}{2}} \left| \frac{|x-z|}{|x-s|} \right|^{\theta} \right) \left[ (t-s)^{\frac{\lambda}{2}} \left| \frac{|x-z|}{|x-s|} \right|^{\theta} \right]
\]

\[
\times \left[ (1 + (t-s)^\theta) + \frac{C(t-s)^\lambda}{1 + |x-z|^{1+\theta}} \right]
\] (2.6)

and

\[
|\partial_z G(s, z; t, x)| \leq C \left( (t-s)^{-\frac{\beta}{2}} \wedge |x-z|^{-1} \right) \left[ (t-s)^{\frac{\lambda}{2}} \left| \frac{|x-z|}{|x-s|} \right|^{\theta} \right]
\]

\[
\times \left[ (1 + (t-s)^\theta) + \frac{C(t-s)^\lambda|x-z|^\theta}{(1 + |x-z|^{1+\theta})^2} \right]
\] (2.7)
for any \( \theta \in [\alpha_L, \alpha^U] \), \( \beta \in \left(0, \frac{1}{1+\theta}\right) \), and \( \lambda \in (0, 1 - \beta(1 + \theta)) \). Here the constant \( C \) is independent of \( \theta, \beta, \) and \( \lambda \).

Based on the above estimates (2.6) and (2.7), we show the following integral inequalities for convolutions involving the fundamental solution \( G \) and its partial derivative \( \partial_z G \) which will be used to prove our main result of the paper.

**Lemma 2.2.** Let \( T > 0 \) be arbitrary but fixed. For any \( \theta \in [\alpha_L, \alpha^U] \subset (0, 2) \) and for every \( u : [0, T] \times \mathbb{R} \to \mathbb{R} \), the following estimates hold\(^1\)

\[
\int_{\mathbb{R}} \left( \int_0^t \int_{\mathbb{R}} [\partial_z G(s, z; t, x)]u(s, z)dzds \right)^2 dx \leq C \left( \int_0^t (t-s)^{-\frac{3}{2\theta}} \int_{\mathbb{R}} |u(s, z)|dzds \right)^2 
\]  
(2.8)

and

\[
\int_{\mathbb{R}} \left( \int_0^t \int_{\mathbb{R}} G(s, z; t, x)u(s, z)dzds \right)^2 dx \leq C \left[ \int_0^t \left( \int_{\mathbb{R}} |u(s, z)|^2dz \right)^{\frac{1}{2}} ds \right]^2 
\]
(2.9)

in particular,

\[
\left| \int_0^t \int_{\mathbb{R}} G(s, z; t, x)u(s, z)dzds \right| \leq C \int_0^t (t-s)^{-\frac{1}{2\theta}} \left( \int_{\mathbb{R}} |u(s, z)|^2dz \right)^{\frac{1}{2}} ds.
\]
(2.10)

The proofs of Proposition 2.1 and Lemma 2.2 are deferred to Section 3.

2.2. Lévy space-time white noise

Following [40], let us briefly recollect basic facts about Lévy space-time white noise on \([0, \infty) \times \mathbb{R}\). For an arbitrary \( \sigma \)-finite measure space \((U, \mathcal{B}(U), \mu)\), by [19], Theorem I.8.1, there always exists a canonical Poisson random measure on the product measure space

\[
\left( [0, \infty) \times \mathbb{R}^n \times U, \mathcal{B}([0, \infty) \times \mathbb{R}^n) \times \mathcal{B}(U), dt \, dx \otimes \mu \right),
\]
i.e. a measure

\[
N : \mathcal{B}([0, \infty) \times \mathbb{R}^n) \times \mathcal{B}(U) \times \Omega \to \mathbb{N} \cup \{0\} \cup \{\infty\}.
\]
(2.11)

We will from now on denote by \( m(A) \) the Lebesgue measure of any Borel set \( A \) in any dimension and by \( \mathcal{P}(A) \) the power set of \( A \).

Let \((\mathcal{F}_t)_{t \geq 0}\) be a right-continuous increasing family of sub \( \sigma \)-algebras of \( \mathcal{F} \), each containing all \( P \)-null sets of \( \mathcal{F} \), such that the canonical Poisson random measure \( N \) has the properties

(i) \( N([0, t] \times A, B, \cdot) : \Omega \to \mathbb{N} \cup \{0\} \cup \{\infty\} \) is \( \mathcal{F}_t \setminus \mathcal{P}(\mathbb{N} \cup \{0\} \cup \{\infty\}) \)-measurable for all \( (t, A, B) \in [0, \infty) \times \mathcal{B}([0, \infty) \times \mathbb{R}^n) \times \mathcal{B}(U) \),

(ii) \( \{N([0, t+s] \times A, B, \cdot) - N([0, t] \times A, B, \cdot)\}_{s \geq 0, (A,B) \in \mathcal{B}(\mathbb{R}^n) \times \mathcal{B}(U)} \) is independent of \( \mathcal{F}_t \) for any \( t \geq 0 \).

---

\(^1\) For simplicity, here and in the sequel in this paper the constant \( C \) is a generic positive constant whose value may vary from line to line.
For instance we may directly take

\[ \mathcal{F}_t := \sigma \left( \{\mathcal{N}([0, t] \times A, B, \cdot) : (A, B) \in \mathcal{B}(\mathbb{R}^n) \times \mathcal{B}(U)\} \right) \vee \mathcal{N}, \quad t \in [0, \infty) \quad (2.12) \]

where \( \mathcal{N} \) denotes all \( P \)-null sets of \( \mathcal{F} \). Next we introduce the compensating \( \mathcal{F}_t \)-martingale measure

\[ M(t, A, B, \omega) := N([0, t], A, B, \omega) - tm(A)\mu(B) \quad (2.13) \]

for any \( (t, A, B) \in [0, \infty) \times \mathcal{B}(\mathbb{R}^n) \times \mathcal{B}(U) \) with \( m(A)\mu(B) < \infty \).

For the Poisson random measure \( N \) and its compensating martingale measure \( M \), we may now set heuristically

\[ N_{t,x}(B, \omega) := \frac{N(dr \, dx, B, \omega)}{dr \, dx}(t, x) \quad (2.14) \]

and

\[ M_{t,x}(B, \omega) := \frac{M(dr \, dx, B, \omega)}{dr \, dx}(t, x) = N_{t,x}(B, \omega) - \mu(B), \quad (2.15) \]

for \( (t, x) \in [0, \infty) \times \mathbb{R}^n \) and \( (B, \omega) \in \mathcal{B}(U) \times \Omega \). Of course, we shall understand \( N_{t,x} \) and \( M_{t,x} \) rigorously in the sense of distributions. We call \( M_{t,x} \) Poisson space-time white noise. A Lévy space-time white noise has the following structure:

\[ F_{t,x}(\omega) = W_{t,x}(\omega) + \int_{U_0} c_1(t, x; y)M_{t,x}(dy, \omega) + \int_{U \setminus U_0} c_2(t, x; y)N_{t,x}(dy, \omega), \quad (2.16) \]

for \( \omega \in \Omega \) where \( c_1, c_2 : [0, \infty) \times \mathbb{R}^n \times U \to \mathbb{R} \) are measurable, \( W_{t,x} \) is a Gaussian space-time white noise on \([0, \infty) \times \mathbb{R}^n\) (cf. e.g. [41]), and \( U_0 \in \mathcal{B}(U) \) with \( \mu(U \setminus U_0) < \infty \).

2.3. Formulation of the solution and the main result

Now we return to Eq. (1.1). A solution to Eq. (1.1) at \( (t, x) \) will be a random variable \( u(t, x, \omega) \). Hence, as usual, we will suppress \( \omega \) and write instead \( u(t, x) \). Let us present a mild formulation of Eq. (1.1) in terms of the fundamental solution \( G(s, z; t, x) \) for \( q(x, D_x) \). Using (2.3) and (2.16), we have

\[
\begin{align*}
    u(t, x) = & \int_{\mathbb{R}} G(0, z; t, x)u_0(z)dz + \int_0^t \int_{\mathbb{R}} [\partial_z G(s, z; t, x)]f(s, z, u(s, z))dzds \\
    & + \int_0^t \int_{\mathbb{R}} G(s, z; t, x)h_1(s, z, u(s, z))dzds \\
    & + \int_0^t \int_{\mathbb{R}} G(s, z; t, x)h_2(s, z, u(s, z))W(ds, dz) \\
    & + \int_0^{t+} \int_{\mathbb{R}} \int_{U_0} G(s, z; t, x)h_2(s, z, u(s-, z); y)c_1(s, z; y)M(ds, dz, dy) \\
    & + \int_0^{t+} \int_{\mathbb{R}} \int_{U \setminus U_0} G(s, z; t, x)h_2(s, z, u(s-, z); y)c_2(s, z; y)N(ds, dz, dy). \quad (2.17)
\end{align*}
\]
Recall that $G(s, z; t, x)$ is the fundamental solution starting from $(s, z) \in [0, \infty) \times \mathbb{R}$ and it satisfies the following
\begin{align*}
\frac{\partial t}{u}(t, x) &= (-q(x, D_x)v)(t, x), \quad (t, x) \in (s, \infty) \times \mathbb{R} \\
v(s, x) &= \delta_0(x), \quad x \in \mathbb{R}.
\end{align*}
(2.18)

Here and in the sequel, in order that the stochastic integrals against $M$ and $N$ are well defined, an $\{\mathcal{F}_t\}$-predictable version of $u$ is taken as $u$ is càdlàg in $t \in [0, \infty)$ (cf. e.g. [19,25]).

Let us reformulate Eq. (2.17) by the following consideration. Observing that $v(U \setminus U_0) < \infty$, we have by Itô isometry
\[
\int_0^{t+} \int_{U \setminus U_0} G(s, z; t, x) h_2(s, z, u(s-, z)) c_2(s, z; y) N(ds, dz, dy)
= \int_0^{t+} \int_{U \setminus U_0} G(s, z; t, x) h_2(s, z, u(s-, z)) c_2(s, z; y) M(ds, dz, dy)
- \int_0^{t+} \int_{U \setminus U_0} [G(s, z; t, x) h_2(s, z, u(s, z)) c_2(s, z; y) v(dy)] dz ds.
\]

Thus, without loss of generality, we shall consider the equation in the following form
\[
u(t, x) = \int_{\mathbb{R}} G(0, z; t, x) u_0(z) dz + \int_0^t \int_{\mathbb{R}} [\frac{\partial t}{u} G(s, z; t, x)] f(s, z, u(s, z, \omega)) dz ds
+ \int_0^t \int_{\mathbb{R}} G(s, z; t, x) h_1(s, z, u(s, z)) dz ds
+ \int_0^t \int_{\mathbb{R}} G(s, z; t, x) h_2(s, z, u(s, z)) W(ds, dz)
+ \int_0^{t+} \int_{U} G(s, z; t, x) h(s, z, u(s-, z); y) M(ds, dz, dy)
\]
(2.19)
where $h_1, h_2 : [0, \infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $h : [0, \infty) \times \mathbb{R} \times \mathbb{R} \times U \to \mathbb{R}$ are measurable, and $f : [0, \infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a measurable function satisfying the following growth condition
\[
|f(s, z, u)| \leq K(z) + C|u|^2
\]
(2.20)
for all $(t, z, u) \in [0, \infty) \times \mathbb{R} \times \mathbb{R}$, for some nonnegative functions $K \in L^1(\mathbb{R})$. The case of the classical Burgers nonlinearity where $f(s, z, u) = u^2$ clearly satisfies the above growth condition.

Obviously, Eq. (2.19) is a mild formulation of the following (formal) equation
\[
(\frac{\partial t}{u} + q(x, D_x))u(t, x) + \frac{\partial x}{u}(f(t, x, u(t, x))) = h_1(t, x, u(t, x)) + h_2(t, x, u(t, x)) W_{t,x}
+ \int_{U} h(t, x, u(t, x); y) M_{t,x}(dy).
\]

Let us now give a precise definition of what we understand as a solution of Eq. (2.19). By a (global) solution of (2.19) in the setting $(\Omega, \mathcal{F}, P; \{\mathcal{F}_t\}_{t \in [0, \infty)})$, we mean an $\{\mathcal{F}_t\}$-adapted function $u : [0, \infty) \times \mathbb{R} \times \Omega \to \mathbb{R}$ which is càdlàg in the variable $t \in [0, \infty)$ for all $x \in \mathbb{R}$ and for almost all $\omega \in \Omega$ such that (2.19) holds. Furthermore, we say that the solution is (pathwise) unique, if $u^{(1)}$ and $u^{(2)}$ are any two solutions of (2.19) then for all $(t, x) \in [0, \infty) \times \mathbb{R}$, $u^{(1)}(t, x, \cdot) = u^{(2)}(t, x, \cdot) P$-a.s. Moreover, one can formulate a (global) solution over a finite time interval $[0, T]$ for any $0 < T < \infty$ in the same pattern. In addition, we say Eq. (2.19)
has a unique solution if there exists an \( \{\mathcal{F}_t\} \)-stopping time \( \tau : \Omega \to (0, T) \) and an \( \{\mathcal{F}_t\} \)-adapted function \( u : [0, T] \times \mathbb{R} \times \Omega \to \mathbb{R} \) which is càdlàg in \( t \in [0, T] \) such that Eq. (2.19) has a unique solution on the (random) interval \([0, T]\) such that Eq. (2.19) has a (pathwise) unique local solution \( \tilde{u} : [0, T] \times \mathbb{R} \times \Omega \to \mathbb{R} \) for all \((t, x, \omega)\) in \([0, \tau \wedge \tilde{\tau}] \times \mathbb{R} \times \Omega : = \{(t, x, \omega) : 0 \leq t < \tau(\omega) \wedge \tilde{\tau}(\omega)\}

Our main result reads as follows:

**Theorem 2.3.** Let \( \frac{3}{2} < \alpha_L := \inf_{x \in \mathbb{R}} \alpha(x) \leq \alpha_U := \sup_{x \in \mathbb{R}} \alpha(x) < 2 \) and let \( T > 0 \) be arbitrary but fixed. Assume that there exist (positive) functions \( K, L_1, L_2, L_3 \in L^1(\mathbb{R}) \) such that the growth condition (2.20) for \( f \) and the following growth conditions

\[
|h_1(t, x, z)|^2 \leq L_1(x) + C|z|^2, \tag{2.21}
\]

\[
|h_2(t, x, z)|^2 + \int_U |h(t, x, z; y)|^2 v(dy) \leq L_2(x) + C|z|^2 \tag{2.22}
\]

as well as the following Lipschitz conditions

\[
|f(t, x, z_1) - f(t, x, z_2)|^2 + |h_1(t, x, z_1) - h_1(t, x, z_2)|^2 \leq [L_3(x) + C(|z_1|^2 + |z_2|^2)]|z_1 - z_2|^2 \tag{2.23}
\]

and

\[
|h_2(t, x, z_1) - h_2(t, x, z_2)|^2 + \int_U |h(t, x, z_1; y) - h(t, x, z_2; y)|^2 v(dy) \leq C|z_1 - z_2|^2 \tag{2.24}
\]

hold for all \((t, x) \in [0, T] \times \mathbb{R} \) and \( z, z_1, z_2 \in \mathbb{R} \). Then for every \( \mathcal{F}_0 \)-measurable \( u_0 : \mathbb{R} \times \Omega \to \mathbb{R} \) with \( \mathbb{E}\left( \int_\mathbb{R} |u_0(x, \cdot)|^2 dx \right) < \infty \), there exists a unique local solution \( u \) to Eq. (2.19) with the property

\[
\mathbb{E}\left( \int_\mathbb{R} |u(t \wedge \tau(\cdot), x, \cdot)|^2 dx \right) < \infty
\]

for any \( t \in [0, T] \).

3. Proofs

This section is devoted to the proof of our main result. We start by giving proofs for Proposition 2.1 and Lemma 2.2.

3.1. Proof of Proposition 2.1

**Proof.** Let \( G_\theta(s, x; t, y) \) be the fundamental solution of (2.3) with \( \alpha(x) \equiv \theta \in \{\alpha_L, \alpha_U\} \subset (0, 2) \), then (i) and (ii) of Proposition 3.1 in [27] (see (3.5) and (3.6) on page 739 with \( K = 1 \) and \( A = 0 \) there) yield that there exists a constant \( C \) depending only on \( \alpha_L \) and \( \alpha_U \) such that

\[
G_\theta(s, x; t, y) \leq C \left[ (t - s)^{-\frac{3}{2}} \mathbb{1}_{\left\{ |x - y| \leq \frac{1}{2} \right\}} + \frac{t-s}{|x-y|^{1+\theta}} \mathbb{1}_{\left\{ \frac{1}{2} < |x-y| < \frac{1}{2} + \frac{1}{2} \right\}} \right] \tag{3.1}
\]

and (iii) of Proposition 3.1 in [27] (see (3.8) and (3.9) on page 739 with \( K = 1 \) and \( A = 0 \) there) yields
we get the estimate we and and (3.1) we can reformulate inequalities and Proof.

3.2. Proof of obtain the estimate Now combining in equality 

\[
\frac{\partial}{\partial G} G(x, t, y) = [\partial G(s, x; t, y)](1 + O((t - s)^{\beta})) + \frac{O((t - s)^{\lambda})}{1 + |x - y|^{1+\theta}}.
\]

(3.4) for any \(0 < \beta < \frac{1}{1+\theta}\) and \(0 < \lambda < 1 - \beta(1+\theta)\), where the standard notation \(O(f)\), for any positive function \(f\), stands for a function that does not exceed in magnitude the function \(C \cdot f\) for some constant \(C > 0\). Furthermore, we would like to clarify that the factors \(O((t - s)^{\beta}), O((t - s)^{\lambda})\), etc. appeared above and in the sequel are indeed functions that do not depend on the space variables \(x, y, \text{etc.}\).

By taking partial derivatives with respect to \(y\) (as well as to \(x\), as \(G(s, x; t, y) = G(s, y; t, x)\)) in equality (3.3), we derive

\[
\frac{\partial}{\partial y} G(s, x; t, y) = [\partial G(s, x; t, y)](1 + O((t - s)^{\beta})) + \frac{O((t - s)^{\lambda}|x - y|^\theta}{1 + |x - y|^{1+\theta}^2}.
\]

(3.5) Now combining (3.1) and (3.3) we get the estimate (2.6), and combining (3.2) and (3.4) we obtain the estimate (2.7). \(\square\)

3.2. Proof of Lemma 2.2

Proof. Since

\[
|x - z| \leq (t - s)^{\frac{1}{\theta}} \equiv |x - z|^{-1} \leq (t - s)^{-\frac{1}{\theta}}
\]

and

\[
|x - z| > (t - s)^{\frac{1}{\theta}} \equiv |x - z|^{-1} < (t - s)^{-\frac{1}{\theta}},
\]

we can reformulate inequalities (2.6) and (2.7), respectively, as follows

\[
G(s, z; t, x) \leq C(t - s)^{-\frac{1}{\theta}}(1 + (t - s)^{\beta})1_{\{|x - z| \leq (t - s)^{\frac{1}{\theta}}\}} + \frac{C(t - s)(1 + (t - s)^{\beta})}{|x - z|^{1+\theta}}1_{\{|x - z| > (t - s)^{\frac{1}{\theta}}\}} + \frac{C(t - s)^{\lambda}}{1 + |x - y|^{1+\theta}}
\]

(3.5)

and

\[
|\partial y G(s, z; t, x)| = |\partial_x G(s, z; t, x)|
\]

\[
\leq C[1 + (t - s)^{\beta}] \left[ (t - s)^{-\frac{2}{\theta}}1_{\{|x - z| \leq (t - s)^{\frac{1}{\theta}}\}} + \frac{t - s}{|x - z|^{2+\theta}}1_{\{|x - z| > (t - s)^{\frac{1}{\theta}}\}} \right] + \frac{C(t - s)^{\lambda}|x - z|^\theta}{(1 + |x - z|^{1+\theta})^2}.
\]

(3.6)
Now by using (in turn) the translation \( x - z \to x' \), the facts that \( \{ |x| \leq (t-s)^{\frac{1}{\theta}} \} \cup \{ |x| > (t-s)^{\frac{1}{\theta}} \} = \mathbb{R} \) and \( \{ |x| \leq (t-s)^{\frac{1}{\theta}} \} \cap \{ |x| > (t-s)^{\frac{1}{\theta}} \} = \phi \), the scaling \( x = (t-s)^{\frac{1}{\theta}} x' \), and the inequality \((a+b)^2 \leq 2a^2 + 2b^2\), we have

\[
\int_{\mathbb{R}} \left( (t-s)^{-\frac{2}{\theta}} + (t-s)^{\beta-\frac{2}{\theta}} \right) 1_{\{ |x-z| \leq (t-s)^{\frac{1}{\theta}} \}} + \frac{(t-s)(1+(t-s)^\beta)}{|x-z|^{2+\theta}} 1_{\{ |x-z| > (t-s)^{\frac{1}{\theta}} \}} \frac{|x-z|^\theta}{(1+|x-z|^{1+\theta})^2} \ dx
\]

\[
= \int_{|x| \leq (t-s)^{\frac{1}{\theta}}} \left( (t-s)^{-\frac{2}{\theta}} + (t-s)^{\beta-\frac{2}{\theta}} + \frac{(t-s)^\lambda |x|^\theta}{(1+|x|^{1+\theta})^2} \right) \ dx
\]

\[
+ \int_{|x| > (t-s)^{\frac{1}{\theta}}} \left( \frac{(t-s)(1+(t-s)^\beta)}{|x|^{2+\theta}} + \frac{(t-s)^\lambda |x|^\theta}{(1+|x|^{1+\theta})^2} \right) \ dx
\]

\[
\leq 2 \left[ (t-s)^{-\frac{2}{\theta}} + (t-s)^{2\beta-\frac{2}{\theta}} \right] \int_{|x| \leq 1} \ dx
\]

\[
+ 2(t-s)^{-\frac{2}{\theta}} (1+(t-s)^\beta)^2 \int_{|x| > 1} \frac{dx}{|x|^{4+2\theta}} + 2(t-s)^{2\lambda} \int_{\mathbb{R}} \frac{|x|^{2\theta} dx}{(1+|x|^{1+\theta})^4}
\]

\[
\leq C (t-s)^{-\frac{3}{\theta}} \left[ 1 + (t-s)^{2\beta} + (1+(t-s)^\beta)^2 + (t-s)^{2\lambda} \right],
\]

where in the above derivation we have used the facts that

\[
\int_{|x| \leq 1} \ dx < \infty, \quad \int_{|x| > 1} \frac{1}{|x|^{4+2\theta}} dx < \infty,
\]

and

\[
\int_{\mathbb{R}} \frac{|x|^{2\theta} dx}{(1+|x|^{1+\theta})^4} \leq 2 \int_0^1 x^{2\alpha} \ dx + 2 \int_1^{\infty} x^{-4-2\alpha} \ dx < \infty.
\]

The above calculation indicates that these integrals with respect to \( x \) are bounded by constants depending on \( \alpha_L \), i.e., the constants are independent of \( \theta \).

Now in order to prove inequality (2.8), by Minkowski’s inequality (cf. e.g. p. 47 of [30]) and the inequality (3.6) in turn, we have

\[
\int_{\mathbb{R}} \left( \int_{\mathbb{R}} [\partial_z G(s, z; t, x)]^2 u(s, z) dz \right) dx
\]

\[
\leq \left[ \int_{\mathbb{R}} \left( \int_{\mathbb{R}} [\partial_z G(s, z; t, x)]^2 dx \right)^{\frac{1}{2}} dz \right]^2
\]

\[
\leq C \left[ \int_0^t \int_{\mathbb{R}} |u(s, z)| \left( \int_{\mathbb{R}} (t-s)^{-\frac{2}{\theta}} + (t-s)^{\beta-\frac{2}{\theta}} \right) 1_{\{ |x-z| \leq (t-s)^{\frac{1}{\theta}} \}} \right]
\]

\[
+ \frac{(t-s)(1+(t-s)^\beta)}{|x-z|^{2+\theta}} 1_{\{ |x-z| > (t-s)^{\frac{1}{\theta}} \}} + \frac{(t-s)^\lambda |x-z|^\theta}{(1+|x-z|^{1+\theta})^2} \ dx \right)^{\frac{1}{2}} dz \ dx \right]^2
\]
can be verified by utilizing Fubini’s theorem, Minkowski’s inequality, in-
by shifting and scaling
\[ \int_0^t \int_{\mathbb{R}} |u(s, z)|(t-s)^{-\frac{3}{p}} \left[ 1 + (t-s)^{2\beta} + (1 + (t-s)^{\beta})^2 \right] \] 
\[ \leq C (1 + T^{2\beta} + (1 + T^{\beta})^2 + T^{2\lambda+\frac{1}{p}})^{\frac{1}{2}} \left[ \int_0^t \int_{\mathbb{R}} |u(s, z)|(t-s)^{-\frac{3}{p}} dz ds \right]^2 \] 
\[ \leq C \left[ \int_0^t (t-s)^{-\frac{3}{p}} \int_{\mathbb{R}} |u(s, z)| dz ds \right]^2 \]
which roves inequality (2.8).

Inequality (2.9) can be verified by utilizing Fubini’s theorem, Minkowski’s inequality, inequality (2.6), and Young’s inequality (cf. e.g. inequality (4) on p. 99 of [30] with \( p = r = 2 \) and \( q = 1 \) there). In fact, we have

\[
\int_{\mathbb{R}} \left( \int_0^t \int_{\mathbb{R}} G(s, z; t, x)u(s, z)dz ds \right)^2 dx 
\leq C \left\{ \int_0^t \left[ \int_{\mathbb{R}} |u(s, z)| \left( (t-s)^{-\frac{1}{p}} (1 + (t-s)^{\beta}) \right) \frac{1}{|x-z|^{1+\theta}} \right] ds \right\}^2 
\leq C \left\{ \int_0^t \left[ \int_{\mathbb{R}} |u(s, z)|^2 ds \right]^{\frac{1}{2}} dx \right\}^2 
\leq C \left\{ \int_0^t [1 + (t-s)^{\beta} + (t-s)^{\lambda}] \left[ \int_{\mathbb{R}} |u(s, z)|^2 dz \right]^{\frac{1}{2}} ds \right\}^2 
\leq C \left\{ \int_0^t \left[ \int_{\mathbb{R}} |u(s, z)|^2 dz \right]^{\frac{1}{2}} ds \right\}^2 
\]
as \( 0 \leq t-s \leq T \), where we have used the following calculation in deriving the third inequality:

by shifting and scaling \( x-z = (t-s)^{\frac{1}{\beta}} x' \) we obtain

\[
\int_{\mathbb{R}} \left( (t-s)^{-\frac{1}{p}} (1 + (t-s)^{\beta}) \right) \frac{1}{|x-z|^{1+\theta}} \left( (t-s)^{\lambda} \right) dx \leq C (1 + (t-s)^{\beta} + (t-s)^{\lambda}).
\]
Finally, inequality (2.10) can be derived as follows. By inequality (2.6), the Fubini theorem, the Schwarz inequality, and then again by shifting and scaling, we have

\[
\left| \int_0^t \int \mathcal{G}(s, z; t, x) u(s, z) dz dx \right|
\leq C \int_0^t \int \left| u(s, z) \right| (t - s)^{-\frac{1}{\beta}} \left( 1 + (t - s)^{\beta} \right) 1_{\{|x-z| \leq (t-s)^{\frac{1}{\beta}}\}} dz dx
\]

\[
+ \frac{(t - s)(1 + (t - s)^{\beta})}{|x-z|^{1+\theta}} 1_{\{|x-z| > (t-s)^{\frac{1}{\beta}}\}} dz dx
\]

\[
\leq C \int_0^t \left[ (t - s)^{-\frac{1}{\beta}} (1 + (t - s)^{\beta}) \left( \int \left| u(s, z) \right|^2 dz \right)^{\frac{1}{2}} \left( \int 1_{\{|x-z| \leq (t-s)^{\frac{1}{\beta}}\}} dz \right)^{\frac{1}{2}} \right] dz dx
\]

\[
+ (t - s)(1 + (t - s)^{\beta}) \left( \int \left| u(s, z) \right|^2 dz \right)^{\frac{1}{2}} \left( \int \frac{dz}{(1 + |x-z|^{1+\theta})^{2}} \right)^{\frac{1}{2}} dz dx
\]

\[
+ (t - s)^{\lambda} \left( \int \left| u(s, z) \right|^2 dz \right)^{\frac{1}{2}} \left( \int \frac{dz}{(1 + |x-z|^{1+\theta})^{2}} \right)^{\frac{1}{2}} dz dx
\]

\[
= C \int_0^t \left[ \left( \int \left| u(s, z) \right|^2 dz \right)^{\frac{1}{2}} \right] \left[ (t - s)^{-\frac{1}{\beta}} (1 + (t - s)^{\beta}) \left( \int 1_{\{|x| \leq 1\}} dz \right)^{\frac{1}{2}} \right]
\]

\[
+ (t - s)^{1+\theta} (1 + (t - s)^{\beta}) \left( \int 1_{\{|z| > 1\}} dz \right)^{\frac{1}{2}} + (t - s)^{\lambda} \left( \int \frac{dz}{(1 + |x|^{1+\theta})^{2}} \right)^{\frac{1}{2}} dz dx
\]

\[
\leq C \int_0^t (t - s)^{-\frac{1}{\beta}} \left[ \int \left| u(s, z) \right|^2 dz \right]^{\frac{1}{2}} dz dx. \quad \Box
\]

4. Proof of Theorem 2.3

Our proof to Theorem 2.3 follows the argumentation in [40], but here we have to treat some extra terms as the estimates of the fundamental solution in Proposition 2.1 are very different from the case when \( \alpha(x) \) is a constant which was handled in [40]. To this end, we need some preparation. For any fixed \( n \in \mathbb{N} \), let the mapping

\[
\pi_n : L^2(\mathbb{R}) \to B_n := \left\{ u \in L^2(\mathbb{R}) : \| u \|_{L^2} := \left( \int \left| u^2(x) \right| dx \right)^{\frac{1}{2}} \leq n \right\}
\]

be defined by

\[
\pi_n(u) = \begin{cases} 
 u, & \text{if } \| u \|_{L^2} \leq n \\
 \frac{nu}{\| u \|_{L^2}}, & \text{if } \| u \|_{L^2} > n.
\end{cases}
\]
Clearly, for any \( n \in \mathbb{N} \), we have
\[
\| \pi_n(u) \|_{L^2} \leq n.
\]
Moreover, it is clear that the norm
\[
\| \pi_n \|_{\text{Lip}(L^2)} := \sup_{u_1, u_2 \in L^2, u_1 \neq u_2} \frac{\| \pi_n(u_2) - \pi_n(u_1) \|_{L^2}}{\| u_2 - u_1 \|_{L^2}} \leq 1,
\]
i.e., \( \pi_n : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}) \) is a contraction since
\[
\| \pi_n(u) \|_{L^2} = \| \pi_n(u) - \pi_n(0) \|_{L^2} \leq \| \pi_n \|_{\text{Lip}(L^2)} \| u - 0 \|_{L^2} \leq \| u \|_{L^2}.
\]
Notice that if \( u \) is a solution to Eq. (2.19), then \( u \) is an \( L^2(\mathbb{R}) \)-valued, \( \{ F_t \} \)-progressive process. Thus, by Theorem 2.1.6 in [15] (page 55), for any \( n \in \mathbb{N} \),
\[
\tau_n(\omega) := \inf \left\{ t \in [0, T] : \int_0^t u^2(t, x, \omega)dx \geq n^2 \right\}, \quad \omega \in \Omega
\]
defines a stopping time. It is clear that \( \{ \tau_n \}_{n \in \mathbb{N}} \) is an increasing sequence of stopping times determined by \( u \). Moreover, for any fixed \( n \in \mathbb{N} \), the stopped process \( u(t \wedge \tau_n) \) satisfies the following equation
\[
u(t, x, \omega) = \int_{\mathbb{R}} G(0, z; t, x)u_0(z, \omega)dz
\]
\[
+ \int_0^t \int_{\mathbb{R}} [\partial_z G(s, z; t, x)] f(s, z, (\pi_n u)(s, z, \omega))dzds
\]
\[
+ \int_0^t \int_{\mathbb{R}} G(s, z; t, x)h_1(s, z, (\pi_n u)(s, z, \omega))dzds
\]
\[
+ \int_0^t \int_{\mathbb{R}} G(s, z; t, x)h_2(s, z, (\pi_n u)(s, z, \omega))W(ds, dz)
\]
\[
+ \int_0^{t+} \int_U G(s, z; t, x)h(s, z, (\pi_n u)(s, z, \omega); y)M(ds, dz, dy, \omega). (4.1)
\]
On the other hand, any solution of Eq. (4.1) is a local solution of Eq. (2.19). Therefore, the existence of a unique local solution of Eq. (2.19) is equivalent to the existence of a unique solution of Eq. (4.1). Hence, we will focus our attention to showing the existence of a unique solution of Eq. (4.1).

**Proof of Theorem 2.3.** We will carry out the proof by the following three steps.

**Step 1.** Suppose that \( u : [0, T] \times \mathbb{R} \times \Omega \rightarrow \mathbb{R} \) is an \( L^2(\mathbb{R}) \)-valued, \( \{ F_t \} \)-adapted, càdlàg process. For any fixed \( n \in \mathbb{N} \), set
\[
(Ju)(t, x, \omega) := \int_{\mathbb{R}} G(s, z; t, x)u_0(z, \omega)dz + \sum_{k=1}^4 (J_ku)(t, x, \omega)
\]
with
\[
(J_1u)(t, x, \omega) := \int_0^t \int_{\mathbb{R}} [\partial_z G(s, z; t, x)] f(s, z, (\pi_n u)(s, z, \omega))dzds
\]
\[
(J_2u)(t, x, \omega) := \int_0^t \int_{\mathbb{R}} G(s, z; t, x)h_1(s, z, (\pi_n u)(s, z, \omega))dzds
\]

\[
(J_3u)(t, x, \omega) := \int_0^t \int_{\mathbb{R}} G(s, z; t, x)h_2(s, z, (\pi_n u)(s, z, \omega))W(ds, dz, \omega)
\]

and

\[
(J_4u)(t, x, \omega) := \int_0^{t+} \int_{\mathbb{R}} \int_{\mathbb{R}} G(s, z; t, x)h(s, z, (\pi_n u)(s, z, \omega); y)M(ds, dz, dy, \omega).
\]

By (2.8) in Lemma 2.2, the condition (2.20), and the Schwarz inequality, we have

\[
\int_{\mathbb{R}} [(J_1u)(t, x, \omega)]^2 dx \leq C \left( \int_0^t (t-s)^{-\frac{3}{2n}} \int_{\mathbb{R}} f(s, z, (\pi_n u)(s, z, \omega))dzds \right)^2
\]

\[
\leq C \left[ \int_0^t (t-s)^{-\frac{3}{2n}} \left( \int_{\mathbb{R}} K(z)dz + C \int_{\mathbb{R}} |(\pi_n u)(s, z, \omega)|^2 dzdz \right) ds \right]^2
\]

\[
\leq C \int_0^t (t-s)^{-\frac{3}{2n}} ds \int_0^t (t-s)^{-\frac{3}{2n}} \times \left[ \left( \int_{\mathbb{R}} K(z)dz \right)^2 + C \int_{\mathbb{R}} |(\pi_n u)(s, z, \omega)|^2 dzdz \right] ds
\]

\[
\leq C t^{1-\frac{3}{2n}} \left[ \int_0^t (t-s)^{-\frac{3}{2n}} \left[ \left( \int_{\mathbb{R}} K(z)dz \right)^2 + C n^4 \right] ds \right]
\]

\[
\leq C t^{2-\frac{3}{2n}} \leq C T^{2-\frac{3}{2n}} < \infty.
\]

Notice that here and in what follows the constant C also depends on \( n \) (of course C depends on \( T \) as well). By inequality (2.9) in Lemma 2.2, we get

\[
\int_{\mathbb{R}} [(J_2u)(t, x, \omega)]^2 dx \leq C \left[ \int_0^t \left( \int_{\mathbb{R}} \left| h_1(s, z, (\pi_n u)(s, z, \omega)) \right|^2 dz \right) ds \right]^2
\]

\[
\leq C \left\{ \int_0^t \left[ \int_{\mathbb{R}} \left( L_1(z) + C |(\pi_n u)(s, z, \omega)|^2 \right) dz \right] \frac{1}{2} ds \right\}^2
\]

\[
\leq C \left[ \int_0^t \left( \int_{\mathbb{R}} L_1(z)dz + C n^2 \right) \frac{1}{2} ds \right]^2 \leq C t^2 \leq C T^2 < \infty.
\]

On the other hand, by Itô’s isometry property for stochastic integrals with respect to (both continuous and càdlàg) martingales (cf. e.g. [19]), we have

\[
E \left[ \int_{\mathbb{R}} |(J_3u)(t, x, \cdot)|^2 dx \right]
\]

\[
= E \left[ \int_{\mathbb{R}} \left( \int_0^t \int_{\mathbb{R}} G^2(s, z; t, x)h_2^2(s, z, (\pi_n u)(s, z, \cdot))dzds \right) dx \right]
\]

and

\[
E \left[ \int_{\mathbb{R}} |(J_4u)(t, x, \cdot)|^2 dx \right] = E \left\{ \int_{\mathbb{R}} \int_{\mathbb{R}} G^2(s, z; t, x) \right\} dx
\]
Thus, by inequality (2.7) and the condition (2.22), we get
\[
\mathbb{E} \left[ \int_\mathbb{R} |(\mathcal{J}u)(t, x, \cdot)|^2 \, dx + \int_\mathbb{R} |(\mathcal{J}u)(t, x, \cdot)|^2 \, dx \right] \\
\leq C \mathbb{E} \left[ \int_0^t (t - s)^{-\frac{1}{\beta}} \left( \int_\mathbb{R} h_2^2(s, z, (\pi_n u)(s, z, \cdot); y) \, d\nu(dy) \right) \, dz \, ds \right] \\
\leq C \int_0^t (t - s)^{-\frac{1}{\beta}} \left( \int_\mathbb{R} L_2(z) \, dz + C \int_\mathbb{R} \mathbb{E}[(\pi_n u)(s, z, \cdot)]^2 \, dz \right) \, ds \\
\leq C t^{1-\frac{1}{\beta}} \left( \int_\mathbb{R} L_2(z) \, dz + C n^2 \right) \leq C T^{1-\frac{1}{\beta}} < \infty.
\]
Therefore, we obtain that
\[
\mathbb{E} \left( \int_\mathbb{R} |(\mathcal{J}u)(t, x, \cdot)|^2 \, dx \right) \leq C \left( T^{2-\frac{3}{\beta}} + T^2 + T^{1-\frac{1}{\beta}} \right) < \infty
\]
for any fixed \( t \in [0, T] \).

Step 2. Now let \( \rho > 0 \) be arbitrary but fixed. For any \( L^2(\mathbb{R}) \)-valued, \( \{\mathcal{F}_t\} \)-predictable, and square integrable process \( u : [0, T] \times \mathbb{R} \times \Omega \rightarrow \mathbb{R} \) (i.e., \( \mathbb{E} \left( \int_\mathbb{R} u^2(t, x, \cdot) \, dx \right) < \infty \) for all \( t \in [0, T] \), with initial condition \( u(0, x, \omega) = u_0(x, \omega) \)), we define
\[
\|u\|_\rho^2 := \int_0^T e^{-\rho t} \mathbb{E} \left( \int_\mathbb{R} u^2(t, x, \cdot) \, dx \right) \, dt.
\]
Clearly, \( \| \cdot \|_\rho \) is a norm.\(^2\) Let \( B \) denote the completion of the collection of all \( L^2(\mathbb{R}) \)-valued, \( \{\mathcal{F}_t\} \)-predictable processes \( u : [0, T] \times \mathbb{R} \times \Omega \rightarrow \mathbb{R} \) with the initial condition \( u(0, x, \omega) = u_0(x, \omega) \), such that
\[
\|u\|_\rho^2 = \int_0^T e^{-\rho t} \mathbb{E} \left( \int_\mathbb{R} u^2(t, x, \cdot) \, dx \right) \, dt < \infty
\]
with respect to the norm \( \| \cdot \|_\rho \), namely, \( (B, \| \cdot \|_\rho) \) is a Banach space. Obviously, we have that our initial data \( u_0 \in B \). Now for all \( u \in B \), \( \mathcal{J}u \) is well defined and for any fixed \( t \in [0, T] \)
\[
\mathbb{E} \left( \int_\mathbb{R} |(\mathcal{J}u)(t, x, \cdot)|^2 \, dx \right) \leq C \left( T^{2-\frac{3}{\beta}} + T^2 + T^{1-\frac{1}{\beta}} \right) < \infty.
\]
Thus, by the formula
\[
\int_0^\infty t^{\lambda-1} e^{-st} \, dt = \Gamma(\lambda)s^{-\lambda}
\]

\(^2\) cf. e.g. page 794 of [16], or one can verify the definition of a norm directly from elementary properties of the Laplace transform, see e.g. page 145 of [14].
we have
\[ \|\mathcal{J}u\|_\rho^2 \leq C \int_0^\infty \left( t^{\frac{3}{\rho}} + t^2 + t^{1-\frac{1}{\rho}} \right) e^{-\rho t} \, dt \leq C \left[ \rho^{-\left(\frac{3}{\rho} - 1\right)} + \rho^{-3} + \rho^{-\left(2 - \frac{1}{\rho}\right)} \right] < \infty, \]

i.e., \( \mathcal{J}u \in B \), which implies that \( \mathcal{J}(B) \subset B \).

Step 3. Now for any \( u, v \in B \), by (2.8)–(2.10) in Lemma 2.2 and the Lipschitz condition (2.23), together with Fubini’s theorem, and the Schwarz inequality, we get for any \( t \in [0, T] \)
\[
E\left( \int_\mathbb{R} |(\mathcal{J}_1 u)(t, x, \cdot) - (\mathcal{J}_1 v)(t, x, \cdot)|^2 \, dx \right) 
\leq CE\left( \int_0^t (t-s)^{-\frac{3}{\rho}} \int_\mathbb{R} |f(s,z,(\pi_n u)(s,z,\cdot)) - f(s,z,(\pi_n v)(s,z,\cdot))| \, dz \, ds \right)^2 
\leq CE\left[ \int_0^t (t-s)^{-\frac{3}{\rho}} \int_\mathbb{R} (L_3(z) + C[(\pi_n u)^2](s,z,\cdot) + (\pi_n v)^2(s,z,\cdot)) |(\pi_n u)(s,z,\cdot) - (\pi_n v)(s,z,\cdot)| \, dz \, ds \right]^2 
\leq C \int_0^t (t-s)^{-\frac{3}{\rho}} \, ds \, E\left[ \int_0^t (t-s)^{-\frac{3}{\rho}} \int_\mathbb{R} (L_3(z) + C[(\pi_n u)^2](s,z,\cdot) + (\pi_n v)^2(s,z,\cdot)) |(\pi_n u)(s,z,\cdot) - (\pi_n v)(s,z,\cdot)| \, dz \, ds \right]^2 
\leq C \int_0^t (t-s)^{-\frac{3}{\rho}} \, ds \, E\left( \int_\mathbb{R} |u(s,z,\cdot) - v(s,z,\cdot)|^2 \, dz \right) \, ds,
\]
where in the derivation of the fourth inequality above, we have used the fact that for \( \theta > \frac{3}{2} \)
\[ \int_0^t (t-s)^{-\frac{3}{\rho}} \, ds = \left( 1 - \frac{3}{2\theta} \right) T^{1-\frac{3}{\rho}} \leq \left( 1 - \frac{3}{2\theta} \right) T^{1-\frac{3}{\rho}} < \infty. \]

Moreover,
\[
E\left( \int_\mathbb{R} |(\mathcal{J}_2 u)(t, x, \cdot) - (\mathcal{J}_2 v)(t, x, \cdot)|^2 \, dx \right) 
\leq E \left\{ \int_\mathbb{R} \int_0^t \int_\mathbb{R} G(s, z; t, x) [L_3(z) + C[(\pi_n u)^2](s,z,\cdot) + (\pi_n v)^2(s,z,\cdot)] |(\pi_n u)(s,z,\cdot) - (\pi_n v)(s,z,\cdot)| \, dz \, ds \right\}^2 \, dx 
\leq CE \int_\mathbb{R} \int_0^t \left\{ \int_\mathbb{R} G(s, z; t, x) [L_3(z) + C[(\pi_n u)^2](s,z,\cdot) + (\pi_n v)^2(s,z,\cdot)] \right\} \, dx \, ds.
\]
Combining all the above together, we obtain

\[ + (\pi_n v^2(s, z, \cdot)) \right)^{1/2} |(\pi_n u)(s, z, \cdot) - (\pi_n v)(s, z, \cdot)| dz \right) dx ds \]

\[ = CE \int_0^t \int_\mathbb{R} \left\{ \int_\mathbb{R} G(s, z; t, x) \right. \left[ L_3(z) + C((\pi_n u)^2(s, z, \cdot) 
+ (\pi_n v)^2(s, z, \cdot)) \right]^{1/2} |(\pi_n u)(s, z, \cdot) - (\pi_n v)(s, z, \cdot)| dz \right) dx ds \]

\[ \leq CE \int_0^t \int_\mathbb{R} \left\{ \int_\mathbb{R} G(s, z; t, x) dx \right. \left[ L_3(z) + C((\pi_n u)^2(s, z, \cdot) 
+ (\pi_n v)^2(s, z, \cdot)) \right]^{1/2} |(\pi_n u)(s, z, \cdot) - (\pi_n v)(s, z, \cdot)|^2 dz dx \]

\[ \leq CE \int_0^t \int_\mathbb{R} \left[ L_3(z) + C((\pi_n u)^2(s, z, \cdot) 
+ (\pi_n v)^2(s, z, \cdot)) \right] ds \int_\mathbb{R} |(\pi_n u)(s, z, \cdot) - (\pi_n v)(s, z, \cdot)|^2 dz ds \]

\[ \leq C \int_0^t \mathbb{E} \int_\mathbb{R} |u(s, z, \cdot) - v(s, z, \cdot)|^2 dz ds \]

where we have used Fubini’s theorem and the property that

\[ \int_\mathbb{R} G(s, z; t, x) dx = \int_\mathbb{R} G(s, z, t) dx = 1, \quad 0 \leq s < t < \infty \]

in the above derivation. Furthermore, by Itô’s isometry for both stochastic integrals with respect to \(W(ds, dz)\) and \(M(ds, dz, dy, \omega)\), we have

\[ \mathbb{E} \int_\mathbb{R} |(J_3 u)(t, x, \cdot) + (J_4 u)(t, x, \cdot) - (J_3 v)(t, x, \cdot) - (J_4 v)(t, x, \cdot)|^2 dx \]

\[ \leq CE \int_\mathbb{R} \left( \int_0^t \int_\mathbb{R} G^2(s, z; t, x)|(\pi_n u)(s, z, \cdot) - (\pi_n v)(s, z, \cdot)|^2 dz dx \right) \]

\[ \leq CE \int_\mathbb{R} \left( \int_0^t \int_\mathbb{R} G^2(s, z; t, x)|u(s, z, \cdot) - v(s, z, \cdot)|^2 dz ds \right) \]

\[ = CE \int_0^t \int_\mathbb{R} \left( \int_\mathbb{R} G^2(s, z; t, x) dx \right) |u(s, z, \cdot) - v(s, z, \cdot)|^2 dz ds \]

\[ \leq C \int_0^t (t - s)^{-\frac{1}{2}} \mathbb{E} \int_\mathbb{R} |u(s, z, \cdot) - v(s, z, \cdot)|^2 dz ds. \]

Combining all the above together, we obtain
\[
\|J(u - v)\|_\rho^2 = \int_0^T e^{-\rho t} E \left( \int_{\mathbb{R}} |(J u)(t, x, \cdot) - (J v)(t, x, \cdot)|^2 dx \right) dt \\
\leq C \int_0^T e^{-\rho t} \int_0^t \left( (t - s)^{-\frac{3}{2\theta}} + 1 + (t - s)^{-\frac{1}{\theta}} \right) \\
\times E \left( \int_{\mathbb{R}} |u(s, z, \cdot) - v(s, z, \cdot)|^2 dz \right) ds dt \\
= C \int_0^T \left[ \int_s^T e^{-\rho t} \left( (t - s)^{-\frac{3}{2\theta}} + 1 + (t - s)^{-\frac{1}{\theta}} \right) dt \right] \\
\times E \left( \int_{\mathbb{R}} |u(s, z, \cdot) - v(s, z, \cdot)|^2 dz \right) ds \\
\leq C \int_0^T \left[ \int_s^\infty e^{-\rho t} \left( (t - s)^{-\frac{3}{2\theta}} + 1 + (t - s)^{-\frac{1}{\theta}} \right) dt \right] \\
\times E \left( \int_{\mathbb{R}} |u(s, z, \cdot) - v(s, z, \cdot)|^2 dz \right) ds \\
= C \left( \int_0^\infty e^{-\rho t} t^{-\frac{3}{2\theta}} dt + \int_0^\infty e^{-\rho t} t dt + \int_0^\infty e^{-\rho t} t^{-\frac{1}{\theta}} dt \right) \\
\times E \left( \int_{\mathbb{R}} |u(s, z, \cdot) - v(s, z, \cdot)|^2 dz \right) ds \\
\leq C \left( \frac{\Gamma \left( 1 - \frac{3}{2\theta} \right)}{\rho^{1 - \frac{3}{2\theta}}} + \frac{1}{\rho} + \frac{\Gamma \left( 1 - \frac{1}{\theta} \right)}{\rho^{1 - \frac{1}{\theta}}} \right) \|u - v\|_\rho^2 \\
\leq C \left( \frac{1}{\rho^{1 - \frac{3}{2\theta}}} + \frac{1}{\rho} + \frac{1}{\rho^{1 - \frac{1}{\theta}}} \right) \|u - v\|_\rho^2.
\]

Here the constant \( C \) also depends on both \( u \) and \( v \). Notice that \( \theta \in [\alpha_L, \alpha_U] \subset \left( \frac{3}{2}, 2 \right) \), we have \( \frac{3}{2} < \theta < 2 \) and hence \( 1 - \frac{3}{2\theta} > 0 \) and \( 1 - \frac{1}{\theta} > 0 \). Thus, let us take \( \rho \) large enough so that

\[
C \left( \frac{1}{\rho^{1 - \frac{3}{2\theta}}} + \frac{1}{\rho} + \frac{1}{\rho^{1 - \frac{1}{\theta}}} \right) < 1
\]

which implies that \( J : B \to B \) is a contraction. Therefore there must be a unique fixed point in \( B \) for \( J \) and this fixed point is the unique solution for Eq. (4.1). In order to see that this gives us a local solution to Eq. (2.19), let us denote by \( u_n \) the unique solution of the Eq. (4.1) for each \( n \in \mathbb{N} \). For this \( u_n \), let us set the stopping time

\[
\tau_n(\omega) := \inf \left\{ t \in [0, T] : \int_{\mathbb{R}} u_n^2(t, x, \omega) dx \geq n^2 \right\}, \quad \omega \in \Omega.
\]

Clearly by the contraction property of \( J \), we have for all \( j \geq n \) and for almost all \( \omega \in \Omega \)

\[
u_j(t, \cdot, \omega) = u_n(t, \cdot, \omega), \quad \text{for all } (t, \omega) \in [0, \tau_n) \times \Omega.
\]

Therefore we define

\[
u(t, x, \omega) := u_n(t, x, \omega)
\]
for any \((t, x, \omega) \in [0, \tau_n) \times \mathbb{R} \times \Omega\) and
\[
\tau_\infty(\omega) := \sup_{n \in \mathbb{N}} \tau_n(\omega).
\]
Then
\[
\{u(t, x, \omega) : (t, x, \omega) \in [0, \tau_\infty) \times \mathbb{R} \times \Omega\}
\]
is a local solution of Eq. (2.19).

Finally, for the uniqueness of the local solution to Eq. (2.19), suppose that there are two local solutions \(u\) and \(v\) to Eq. (2.19). Then \(u\) and \(v\) must satisfy Eq. (4.1) for any fixed \(n \in \mathbb{N}\). On the other hand, by the uniqueness of the solution to Eq. (4.1), we get
\[
u(t, x, \omega) = v(t, x, \omega), \quad \text{for all } (t, x, \omega) \in [0, \tau_n) \times \mathbb{R} \times \Omega.
\]
Now let \(n \to \infty\), we deduce
\[
u(t, x, \omega) = v(t, x, \omega), \quad \text{for all } (t, x, \omega) \in [0, \tau_\infty) \times \mathbb{R} \times \Omega.
\]
Hence we obtain the uniqueness. \(\square\)

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