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Leading logarithms for the nucleon mass

Johan Bijnens *, Alexey A. Vladimirov

Department of Astronomy and Theoretical Physics, Lund University, Sölvegatan 14A, SE 223 62 Lund, Sweden

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Abstract

Within the heavy baryon chiral perturbation theory approach, we have studied the leading logarithm behavior of the nucleon mass up to four-loop order exactly and we present some results up to six-loop order as well as an all-order conjecture. The same methods allow to calculate the main logarithm multiplying the terms with fractional powers of the quark mass. We calculate thus the coefficients of $m^{2n+1}\log^{(n-1)}(\mu^2/m^2)$ and $m^{2n+2}\log^{n}(\mu^2/m^2)$, with m the lowest-order pion mass. A side result is the leading divergence for a general heavy baryon loop integral.

1. Introduction

The calculation of high order terms in low-energy effective field theories (EFTs) is a difficult task. Nowadays, most interesting observables have been calculated at the second order of the expansion, and the difficulty of these calculations shows little hope for any further expansion. The main problem which restricts the potential of EFTs is their non-renormalizability. The non-renormalizability does not bring any problem, in principle, for the calculation by means of counting schemes for EFTs, first introduced in [1]. However, the rapidly increasing number of low-energy coupling constants (LECs), makes very high order applications practically of little use in general.

* Corresponding author.
E-mail addresses: bijnens@thep.lu.se (J. Bijnens), vladimirov.aleksey@gmail.com (A.A. Vladimirov).

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Nevertheless, there are contributions of the higher order terms which are free from higher order LECs. In particular this is true for the leading logarithmical (LLog) contributions. LLogs are not in general dominant for a generic observable. However, for some observables the LLog contribution is dominant. Examples of such observables are the generalized parton distributions at small-x [2,3] and certain $\pi\pi$ scattering lengths [4]. In addition, the LLog terms are of great theoretical interest because they allow us to judge the behavior of a whole series of corrections in EFTs. We therefore consider the calculation of LLog terms in EFTs as an interesting and useful task.

In renormalizable field theories the LLog terms can be calculated to all orders using the renormalization group (RG) and (simple) one-loop calculations of the beta functions. In EFTs, as e.g. chiral perturbation theory (ChPT), they can also be calculated using one-loop calculations as was suggested already in [1], and proven in [5]. In contrast to renormalizable theories, in EFTs the LLog terms cannot be obtained simultaneously for all orders, and every order of the perturbative expansion requires an additional calculation. However, the evaluation of LLog terms is considerably simpler than a full calculation. As an example, the full two-loop leading logarithms in bosonic ChPT were known long before the full results [6].

Within bosonic EFTs the LLogs have been studied extensively. It has been shown that for EFTs with massless particles the LLog behavior is described by a closed set of equations with known kernels, which were elaborated in [7–10]. Although, the analytical solution of these equations is not known, one can generate numerically the first few hundreds of coefficients rather fast, and use the approximate numerical solution in applications. An example is the exploration of the "chiral inflation" of the pion radius within ChPT [11]. Taking into account the mass of the fields allows for non-zero tadpole diagrams, which leads to a rapidly increasing number of equations with the chiral order since one has to consider one-loop diagrams with an ever increasing number of external legs. Therefore, one needs to incorporate new processes at every new order. As a result, the difficulty of the calculation grows extremely fast with the chiral order. By automatizing the procedure for a large number of processes the LLogs are known up to seven loops for some quantities [12–15].

The main goal of this paper is to generalize the methods used for bosonic EFTs with masses to the nucleon case. As mentioned earlier, it is not only interesting from the theoretical side, but also necessary for the evaluation of nucleon parton distributions at $x \sim m_\pi/M_N$ [3,16]. In the paper we present the extension of the RG method of [5] to nucleon–pion ChPT. With its help, we calculate the LLog coefficients for the chiral expansion of the nucleon mass in the heavy-baryon formulation of ChPT. The main results are presented in Sections 6.3 and 6.4. An earlier application of LLogs in the nucleon sector was the calculation of the two-loop LLog contribution to the axial nucleon coupling constant g_A [17].

The paper is organized as follows: In Section 2 we introduce the concept of renormalization group order (RGO). This is needed since in the nucleon sector chiral counting and loop counting are not identical. Section 3 shows how the RGO concept works in the meson sector and quotes some known results. Section 4 introduces the heavy baryon ChPT Lagrangian in its two most common variants and the different meson parametrizations we have used as a check on our result. Section 5 shows how the RGO can be used to prove the calculation of the leading logarithms using only one-loop diagrams also in the nucleon sector. This is then used to calculate the LLogs for the nucleon mass in Section 6. Some technicalities are discussed in Sections 6.1 and 6.2. We then calculate the LLogs for the nucleon mass as well as the odd-power next-to-leading logarithms (NLLogs) in Section 6.3 up to four respectively five loops. The observed regularity in the leading logarithm allows to also calculate the five loop result with a mild assumption.

The LLogs, then essentially known to five loops, show a remarkable regularity when rewritten in the physical pion mass. We conjecture that this regularity holds to all orders and in that case using the known results for the pion LLogs we have a result for the nucleon mass LLogs up to 7 loops. This is described in Section 6.4. A short numerical discussion of our results is given in Section 6.5. We summarize our conclusions in Section 7. The LLogs for a general heavy baryon one-loop integral are discussed in Appendix A.

2. Renormalization group and order

2.1. Renormalization group operator

In this section we present a short, hopefully self-contained, introduction to the renormalization group approach in EFTs. Our main goal is to present the method of obtaining the dependence of observables on the renormalization or subtraction scale (μ). The material is presented in a form transparent for the application at higher orders. More extensive discussions can be found in [5,12,18]. In particular, what we call LLogs is the contribution with the highest power of $\log\mu$ at a given order of the expansion.

To start with, we remind the reader that the Lagrangian of an EFT is the most general local Lagrangian satisfying given symmetry properties with a given set of degrees of freedom or fields. Such a Lagrangian contains an infinite number of terms. In the absence of additional restrictions, every independent operator is multiplied by an unknown coupling constant, usually called low-energy constant (LEC).

It is convenient to multiply every operator by the counting parameter $\hbar$ to the power which reflects the minimal order of the perturbative expansion the operator contributes to. In this way, the constant $\hbar$ resembles the coupling constant in a renormalizable field theory as a way to keep track of (loop) orders in the expansion. Therefore, an EFT Lagrangian takes the form

$$\mathcal{L}^{\mathrm{EFT}}_{\mathrm{bare}} = \sum_{n=0}^{\infty} \hbar^n \mathcal{L}^{(n)}_{\mathrm{bare}}. \tag{1}$$

The Lagrangian $\mathcal{L}^{(n)}$ we call the Lagrangian of nth $\hbar$-order[1] and its LECs are consequently called LECs of nth $\hbar$-order. Let us, following [12], denote LECs of nth $\hbar$-order as $c_i^{(n)}$, where the index i enumerates independent operators. In this way, the nth $\hbar$-order Lagrangian reads

$$\mathcal{L}^{(n)}_{\mathrm{bare}} = \sum_i c^{(n)}_{(\mathrm{bare})i} \mathcal{O}^{(n)}_i. \tag{2}$$

For some low-energy EFTs, like mesonic ChPT, the $\hbar$-ordering of operators is in one-to-one correspondence with the chiral ordering. However, the definition (1) is more general. It can be applied to any EFT, and, even, to renormalizable theories, some examples can be found in [5,18]. We should mention that there is no unique definition of the $\hbar$-ordering for a theory. The only constraint is that the $\hbar$-order of an operator should increase with increasing perturbative order. The choice made is for EFTs often referred to as the choice of power counting.

The bare Lagrangian is now split into a part with renormalized couplings $c_i^{(n)}$, which depend on μ, and the counterterms. The renormalization scale independence of the Lagrangian leads to the set of RG equations for the LECs $c_i^{(n)}$. These equations are of the form

[1] The Lagrangian which contains the propagator of fields, must be included in the zeroth $\hbar$-order.

$$\mu^2 \frac{d}{d\mu^2} c_i^{(n)}\left(\mu^2\right) = \beta_i^{(n)}\left(\left\{c_j^{(m)}\left(\mu^2\right)\right\}\right), \tag{3}$$

where the beta-function is a polynomial in the LECs and we have indicated explicitly the μ-dependence. An important point, used later, is that the right-hand side of (3) contains only combinations of LECs with total $\hbar$-order strictly less then n.

The general formal solution of the system of equations (3) is

$$c_i^{(n)}\left(\mu^2\right) = \hat{R}\left(\frac{\mu}{\mu_0}\right) c_i^{(n)}\left(\mu_0^2\right) = \exp\left(\log\left(\frac{\mu^2}{\mu_0^2}\right)\hat{H}\right) c_i^{(n)}\left(\mu_0^2\right). \tag{4}$$

This defines also $\hat{R}$. The operator $\hat{H}$ is defined as

$$\hat{H} = \int d\rho^2 \sum_{n,i} \beta_i^{(n)}\left(\left\{c_j^{(m)}\left(\rho^2\right)\right\}\right) \frac{\delta}{\delta c_i^{(n)}\left(\rho^2\right)}. \tag{5}$$

The derivative in (5) is defined by

$$\frac{\delta}{\delta c_i^{(n)}\left(\rho^2\right)} c_j^{(m)}\left(\mu^2\right) = \delta_{ij}\delta^{mn}\delta\left(\rho^2 - \mu^2\right) \tag{6}$$

such that

$$\hat{H} c_i^{(n)}\left(\mu^2\right) = \beta_i^{(n)}\left(\left\{c_j^{(m)}\left(\mu^2\right)\right\}\right). \tag{7}$$

With the help of $\hat{H}$ or $\hat{R}$, one can obtain the coefficients of the LLog for any observable, without actual calculation of loop diagrams, if the beta-functions are already known. We will demonstrate this explicitly in the next sections.

2.2. Renormalization group order

The crucial property of the operator $\hat{H}$ is that the repetitive action of $\hat{H}$ nullifies any given LEC (or products of LECs). This is the direct consequence of two features of $\hbar$-counting. The first one is that the lowest order couplings, with $\hbar$-order equal to zero, have zero beta-function, and therefore $\hat{H} c_i^{(0)} = 0$. The second one is that the β-function of LEC $c_i^{(n)}$, as defined in (3), contains only products of couplings with total $\hbar$-order lower then n. Thus, every application of the operator $\hat{H}$ onto a product of LECs lowers the total $\hbar$-order of that product, until it becomes zero.

For future convenience, we introduce the concept of renormalization group order (RGO). A product P_c of LECs has RGO g if

$$\hat{H}^g P_c \neq 0 \quad \text{and} \quad \hat{H}^{g+1} P_c = 0. \tag{8}$$

For a generic[2] quantity with a tree level contribution of $\hbar$-order n, the RGO is the same as the maximum loop order that can appear when calculating that quantity to $\hbar^n$.

In the bosonic EFTs treated in the earlier works, e.g. [7,12–14], there is a one-to-one correspondence between $\hbar$-order and RGO, namely $g = n$. Therefore, the notion of RGO is unnecessary and was not used in these works. However, such a relation does not hold in general,

[2] We will use this term below to indicate that there are exceptions where the beta-functions are zero "accidentally." An example of this is the constant L_7^r in three-flavor bosonic ChPT. This does not invalidate our later use of the RGO.

i.e. LECs of different $\hbar$-order can have the same RGO. For example, in the nucleon ChPT, the $\hbar$-counting as is related to RGO as $g=[n/2]$, where $[x]$ indicates the integer part of x (see detailed discussion in Section 5). In such a case the use of the RGO concept is convenient.

It is natural to split the beta-function into terms with the same RGO. For the beta-function of a coupling constant $c_i^{(n)}$ with RGO g we can write

$$\beta_i^{(n)} = \sum_{p=0}^{g-1} \beta_i^{(n,p)}(c). \tag{9}$$

Here $g=n$ or $[n/2]$ for the two cases mentioned above. One can show that the part of the beta function with the highest RGO, $\beta_i^{(n,g-1)}$, contains only contributions from one-loop. The next part, $\beta_i^{(n,g-2)}$, contains contributions from one and two-loop diagrams and so on. Thus, the expression for the operator $\hat{H}$ can be ordered by RGO as

$$\hat{H} = \sum_{p=1}^{\infty} \hat{H}_p. \tag{10}$$

$\hat{H}_p$ contains the beta-functions $\beta^{(n,g-p)}$ of the coupling constants $c_i^{(n)}$ of RGO g. As a consequence, acting with $\hat{H}_p$ on an expression reduces its RGO by p.

3. LLog in mesonic ChPT

In mesonic ChPT the choice of $\hbar$-counting versus chiral counting relates both as $\hbar^n \sim \mathcal{O}(p^{2n+2})$. The lowest order Lagrangian is of the second chiral order and reads[3]

$$\mathcal{L}_\pi^{(0)} = \frac{F^2}{4}\mathrm{tr}\big[u_\mu u^\mu + \chi_+\big], \tag{11}$$

where we use the standard notation

$$u_\mu = i\big(u^\dagger \partial_\mu u - u\partial_\mu u^\dagger\big), \qquad \chi_+ = u^\dagger \chi u^\dagger + u\chi^\dagger u = m^2\big(u^2 + u^{\dagger 2}\big). \tag{12}$$

Here and throughout the text, F is the bare pion decay constant and m is the bare pion mass, $m^2 = 2B\hat{m}$ in the notation of [19]. u contains the meson fields, a few examples of possible parametrizations are given in (25)–(27). The next order Lagrangian $\mathcal{L}^{(1)}$ is of fourth chiral order. The absence of odd chiral order Lagrangians is guaranteed by Lorentz invariance.

In mesonic ChPT, the generic RGO of an LEC $c^{(n)}$ is equal to n. The one-to-one correspondence between generic RGO and the $\hbar$-order is the result of the absence of odd-chiral-order Lagrangians. Any product of LECs is also in one-to-one correspondence with its generic RGO, which is equal to the sum of the LECs' $\hbar$-orders. That, in turn, results in the simple ordering of beta-functions: the beta-function $\beta^{(n,n-l)}$ contains only l-loop beta functions.

As an example of using the operator $\hat{H}$ of (5) to obtain the LLog, we look at the physical pion mass. In order to obtain the physical pion mass one should solve the equation $m_\pi^2 - m^2 + \Sigma_\pi(m_\pi^2, m^2) = 0$, where $\Sigma_\pi(p^2, m^2)$ is a series of perturbative corrections to the pion propagator. The expression for Σ_π has the general form

[3] We write here only the terms relevant for the mass and neglect external fields.

$$\Sigma_\pi\left(p^2=m_\pi^2,m^2\right)=m^2\sum_{n=1}^{\infty}\left(\frac{m^2}{(4\pi F)^2}\right)^n\Sigma_\pi^{(n)}\left(\frac{\mu^2}{m^2},c\left(\mu^2\right)\right), \tag{13}$$

where $\Sigma_\pi^{(n)}$ is a dimensionless expression of maximum $\hbar$-order n. The first argument of $\Sigma_\pi^{(n)}$ appears only as the argument of logarithms. We have suppressed the arguments p^2/m^2. Note, that p^2/m^2 can also enter the arguments of logarithms, moreover there can be logarithms of more complicated expressions of it. Such logarithms are not RG logarithms, and cannot in general be obtained by any procedure based on RG.

The expression for Σ_π is renormalization scale independent.[4] Moreover, it is renormalization scale invariant at every chiral order independently:

$$\begin{aligned}&\left[\mu^2\frac{\partial}{\partial\mu^2}+\sum_{i,n}\beta_i^{(n)}\frac{\partial}{\partial c_i^{(n)}(\mu^2)}\right]\Sigma_\pi^{(n)}\left(\frac{\mu^2}{m^2},c\left(\mu^2\right)\right)\\&\quad=\left[\mu^2\frac{\partial}{\partial\mu^2}+\hat H\right]\Sigma_\pi^{(n)}\left(\frac{\mu^2}{m^2},c\left(\mu^2\right)\right)=0.\end{aligned} \tag{14}$$

Therefore, we again have as solution, similar to (4),

$$\Sigma_\pi^{(n)}\left(\frac{\mu^2}{m^2},c\left(\mu^2\right)\right)=\hat R\left(\frac{\mu^2}{\mu_0^2}\right)\Sigma_\pi^{(n)}\left(\frac{\mu_0^2}{m^2},c\left(\mu_0^2\right)\right). \tag{15}$$

Choosing $\mu_0^2=m^2$, one neglects all the RG logarithms in $\Sigma_\pi^{(n)}$ on the right-hand-side. Thus, all RG logarithms are collected in the action of the operator $\hat R$.

The expression for $\Sigma_\pi^{(n)}$ has the following form

$$\Sigma_\pi^{(n)}=\sum_i\left\{c_i^{(n)}\right\}V_i^{(n)}+\text{terms with lower RGO}, \tag{16}$$

where $\{c_i^{(n)}\}V_i^{(n)}$ form the tree contribution to $\Sigma^{(n)}$. The symbol $\{c_i^{(n)}\}V_i^{(n)}$ denotes here the products of $c_j^{(m)}$ with the highest possible RGO for the product. The $V_i^{(n)}$ depend on p^2/m^2. The highest power of the RG logarithm $\log\mu^2$ in the expression (15) accompanies the highest power of $\hat H$, in $\hat R=\exp(\log(\mu^2/\mu_0)\hat H)$, which gives a non-zero result acting on $\Sigma^{(n)}$. Since every action of $\hat H$ reduces the RGO of expression, the coefficient of the LLog is

$$\Sigma_\pi^{(n)}\left(\frac{\mu^2}{m^2},c\left(\mu^2\right)\right)=\frac{1}{n!}\log^n\left(\frac{\mu^2}{m^2}\right)\hat H_1^n\sum_i c_i^{(n)}V_i^{(n)}+\mathcal{O}(\text{NLLog}), \tag{17}$$

where NLLog is the acronym for the next-to-leading logarithms, and hence $\mathcal{O}$(NLLog) denotes the part of expression without LLog.

Therefore, constructing the higher chiral order Lagrangians, and calculating the one-loop beta-functions of their LECs, one can obtain the LL coefficients without actual calculation of multi-loop diagrams. Moreover, the result is independent on the details of the higher order Lagrangians, as long as they are sufficiently general for the process at hand [12]. Practically it is

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[4] We use here a scheme where all one-particle irreducible diagrams are made finite, otherwise one should apply the argument to a well defined Green function of external currents.

convenient to use a non-minimal Lagrangian generated "on-the-fly" by the counterterms to one-loop diagrams only. This was the approach used in [12–15] for the mesonic theory for several processes.

The solution of the pole equation gives us the expression for the physical pion mass to LLog accuracy. It is known up to sixth power of logarithms [14] and reads

$$m^2_{\mathrm{phys}} = m^2\biggl(1 - \frac{1}{2}L + \frac{17}{8}L^2 - \frac{103}{24}L^3 + \frac{24\,367}{1152}L^4 - \frac{8821}{144}L^5 + \frac{1\,922\,964\,667}{6\,220\,800}L^6 + \cdots\biggr), \tag{18}$$

where

$$L = \frac{m^2}{(4\pi F)^2}\log\biggl(\frac{\mu^2}{m^2}\biggr). \tag{19}$$

4. Heavy baryon Lagrangian

The nucleon–meson ChPT in the naive form has the problem of the large nucleon mass M. There are several ways of dealing with the presence of this large scale, each with advantages and disadvantages. In this article, we use the heavy baryon approach to meson–nucleon ChPT since in this approach all scales that explicitly appear are soft and there are no divergences nor μ-dependence associated directly with the scale M.

For the LLog calculation, we have to determine the Lagrangians of zero RGO. For the pion–nucleon system, these are Lagrangians of the first and the second chiral orders. The first chiral order Lagrangian, neglecting terms with external fields, reads

$$\mathcal{L}^{(0)}_{N\pi} = \bar{N}\bigl(iv^\mu D_\mu + g_A S^\mu u_\mu\bigr)N, \tag{20}$$

where S^μ is a spin vector. We use the standard notation for the field combinations (see also the definitions in (12)):

$$D_\mu = \partial_\mu + \Gamma_\mu, \qquad u^2 = U, \qquad \Gamma_\mu = \frac{1}{2}\bigl(u^\dagger\partial_\mu u + u\partial_\mu u^\dagger\bigr). \tag{21}$$

The second order Lagrangian is sensitive to redefinitions of the nucleon field. The most standard form of the second chiral order heavy baryon Lagrangian reads [20,21]

$$\mathcal{L}^{(1)}_{\pi N} = \bar{N}_v\biggl[\frac{(v\cdot D)^2 - D\cdot D - ig_A\{S\cdot D, v\cdot u\}}{2M} + c_1\mathrm{tr}(\chi_+) + \biggl(c_2 - \frac{g_A^2}{8M}\biggr)(v\cdot u)^2 + c_3 u\cdot u + \biggl(c_4 + \frac{1}{4M}\biggr)i\epsilon^{\mu\nu\rho\sigma}u_\mu u_\nu v_\rho S_\sigma\biggr]N_v. \tag{22}$$

A different but equivalent version of the second order chiral Lagrangian is given in [22], and it reads

$$\mathcal{L}^{(1)}_{N\pi} = \frac{1}{M}\bar{N}\biggl[-\frac{1}{2}\bigl(D_\mu D^\mu + ig_A\bigl\{S_\mu D^\mu, v_\nu u^\nu\bigr\}\bigr) + A_1\mathrm{Tr}\bigl(u_\mu u^\mu\bigr) + A_2\mathrm{Tr}\bigl((v_\mu u^\mu)^2\bigr) + A_3\mathrm{Tr}(\chi_+) + A_5 i\epsilon^{\mu\nu\rho\sigma}v_\mu S_\nu u_\rho u_\sigma\biggr]N. \tag{23}$$

The relation between the LECs A_i and c_i is the following

Fig. 1. An example of diagrams which contribute to the renormalization of the nucleon mass at the same chiral order (here, the fifth chiral order), but which have different RGO. The left diagram has zero RGO, while the right diagram has RGO equal to one. Therefore, the left diagram does not contribute to the LLogs. The number in the box indicates the chiral order of the vertex. Thick arrowed lines indicate nucleon propagators, thin lines indicate pion propagators.

$$A_1 = \frac{Mc_3}{2} + \frac{g_A^2}{16}, \qquad A_2 = \frac{Mc_2}{2} - \frac{g_A^2}{8},$$
$$A_3 = Mc_1, \qquad A_5 = Mc_4 + \frac{1-g_A^2}{4}. \tag{24}$$

Although the S-matrix elements are independent of the parametrization of the nucleon field, the contributions of individual diagrams, and expressions for the beta-functions are dependent on the field parametrization. Therefore, the comparison of results for calculations performed in different parameterizations is a very strong check of a calculation. The calculations presented in the next sections have been done in both parametrizations of the nucleon field. Additionally for further cross-checks, we used different parameterizations of the pion field u. We have used

$$u = \exp\left(i\frac{\pi^a\tau^a}{2F}\right), \tag{25}$$

$$u = \sqrt{1-\frac{\vec{\pi}^2}{4F^2}} + i\frac{\pi^a\tau^a}{2F}, \tag{26}$$

and

$$u = \sqrt{\frac{Y}{\sqrt{2}}} + i\frac{\pi^a\tau^a}{F}\sqrt{\frac{1}{2Y}} \quad \text{with } Y = 1+\sqrt{1-\frac{\vec{\pi}^2}{F^2}}. \tag{27}$$

5. LLog in nucleon–meson ChPT: general comments

The consideration of the LLog behavior of nucleon–meson systems is similar to meson systems, but with some additional features. The main additional feature of meson–nucleon systems is the presence of operators with odd number of derivatives. Therefore, the relation between $\hbar$-order and chiral order is $\hbar^n \sim \mathcal{O}(p^{n+1})$ for the single-nucleon sector of the ChPT Lagrangian, and $\hbar^n \sim \mathcal{O}(p^{n+2})$ for the meson sector of EFT Lagrangian. At the same time, every loop increases the chiral order by at least two. Therefore, the RGO is not in one-to-one correspondence with $\hbar$-order. An LEC of nth $\hbar$-order $c^{(n)}$ has generically an RGO $[\frac{n}{2}]$. This has important consequences in the RG and LLog structure of the theory.

The first consequence is the contribution of diagrams with different RGO to the same chiral order. Indeed, a loop diagram with several vertices of even chiral order (i.e. odd $\hbar$ order) has an RGO less then a diagram with the same chiral order but with fewer even-chiral-order vertices. An example of diagrams with the same chiral order but different RGO is shown in Fig. 1. Using the relation between chiral order and RGO, one can see that every two even-chiral-order vertices reduce the RGO of a diagram by one, from the possible maximum. For example: for any diagram with two even-chiral-order vertices with certain RGOs, there exists diagrams of the same chiral order and of the same topology, but with these two vertices replaced by odd-chiral-order vertices, one with a chiral order one lower and one chiral-order one higher. The lower chiral order has the same generic RGO as the even vertex but the higher chiral order vertex has generic RGO one higher. The latter diagram has thus a higher RGO. We conclude that at a given chiral order the highest RGO contribution is given by the diagrams with *zero or one* vertex of even chiral order.

The second consequence is the ambiguity of the definition of a leading logarithm. The natural definition of LLogs, the logarithm of a maximum power of $\log\mu^2$ at a given chiral order, does not always coincides with the RG definition for the same observable. In Section 6 we will show that in the chiral expansion of the physical nucleon mass, the LLog terms of odd chiral order are actually of NLLog origin when seen from an RGO perspective.

6. Nucleon mass at LLog accuracy

6.1. Propagator at LLog accuracy

The physical mass of the nucleon is given by the position of the pole in the Dyson propagator. In the heavy baryon approach the inverse Dyson propagator reads (we remind the reader that superscripts n refer to the $\hbar$-order of quantities, and that the meson and meson–nucleon sectors of the action have different chiral counting)

$$S^{-1} = (rv) + \sum_{n=1}^{\infty} \Sigma^{(n)}\big((r\cdot v), r^2\big), \tag{28}$$

where $r^\mu = p^\mu - Mv^\mu$, p is the momentum of the nucleon, and v_μ is the reference four velocity of the heavy nucleon. There are some intricacies of defining the propagator and renormalization of the nucleon wave function in heavy baryon theory, see e.g. [22,23]. However for the determination of the nucleon mass the straightforward usage of (28) is sufficient.

The expressions for $\Sigma^{(n)}$ are the result of the calculation of one-particle irreducible diagrams with a nucleon line at the $(n+1)$th chiral order. The maximum power of the logarithm $\log\mu^2$ which can appear in $\Sigma^{(n)}$ is $[\frac{n}{2}]$.

The derivation of the LLog coefficient is the same as the derivation of expression (17). We collect all the RG-logarithms by the action of the normalization point rescaling operator $\hat{R}$, again suppressing the other arguments of $\Sigma^{(n)}$,

$$\Sigma^{(n)}\left(\frac{\mu^2}{m^2}, c\big(\mu^2\big)\right) = \hat{R}\left(\frac{\mu^2}{\mu_0^2}\right)\Sigma^{(n)}\left(\frac{\mu_0^2}{m^2}, c\big(\mu_0^2\big)\right). \tag{29}$$

The highest RGO in $\Sigma^{(n)}$ has the tree diagram with only a $c_i^{(n)}$ vertex, therefore, the LLog coefficient is given by

$$\text{LLog coef.} = \left(\left[\frac{n}{2}\right]!\right)^{-1} \hat{H}_1^{[n/2]} \sum_i c_i^{(n)} V_i^{(n)}, \tag{30}$$

where $V_i^{(n)}$ are expression for the $c_i^{(n)}$-vertices.

It is also interesting to look at the NLLog contribution. The NLLog coefficient comes from the $([n/2]-1)$ term of the exponent series in (29). There are several different parts of $\Sigma^{(n)}$ which survive after the action by $\hat{H}^{[n/2]-1}$. These are the terms with RGO $[\frac{n}{2}]$ and $[\frac{n-2}{2}]$. While the first are given by tree diagrams, the second are given by one-loop diagrams:

$$\Sigma^{(n)}\left(\frac{\mu^2}{m^2}, c(\mu^2)\right) = \sum_i c_i^{(n)}(\mu^2) V_i^{(n)} + \sum_{\text{1-loop diag.}} W\left(\frac{\mu^2}{m^2}, c^{(k)}(\mu^2)\right) + \cdots, \tag{31}$$

where the V_i are the expressions for the tree level diagrams and W indicates the expressions for the one-loop diagrams at a fixed renormalization scale. The dots represents the contributions with lower RGO. We note that W contains only one-loop diagrams with RGO equal to $[\frac{n-2}{2}]$, but not all possible one-loop diagrams. The NLLog coefficient is given by

$$\begin{aligned}\text{NLLog coef.} = {} & \frac{1}{([n/2]-1)!} \hat{H}_1^{[n/2]-1} \sum_i c_i^{(n)} V_i^{(n)} \\ & + \frac{1}{([n/2]-1)!} \left[\sum_{k=0}^{[n/2]-1} \hat{H}_1^k \hat{H}_2 \hat{H}_1^{[n/2]-1-k}\right] \sum_i c_i^{(n)} V_i^{(n)} \\ & + \frac{1}{([n/2]-1)!} \hat{H}_1^{[n/2]-1} \sum_{\text{1-loop diag.}} W\left(\frac{\mu_0^2}{m^2}, c^{(k)}\right),\end{aligned} \tag{32}$$

where $\hat{H}_2$ contains two-loop beta-functions in addition to one-loop beta-functions.

The expressions given by the different terms on the right-hand-side in (32) have significantly different properties. The first term of (32) gives the contribution to NLLogs with LECs from the next-to-leading chiral Lagrangian. These terms are LLog terms of $([n/2]-1)$-loop diagrams with insertion of higher order vertices. The second line of (32) gives the "true" NLLog contribution with LECs of the lowest order Lagrangian only. These are the NLLog terms of $[n/2]$-loop diagrams. The third line represents the non-analytical contribution of $[n/2]$-loop diagrams to the NLLog coefficient. We should mention that the part of NLLog coefficient given by the second line is renormalization-scheme-dependent, while the parts given by the first and the third lines are scheme-independent.

If the quantity has no tree-order contribution, the only non-zero part of (32) is the last line. In this case the NLLog can be calculated from one-loop diagrams only. An example of such behavior are the non-analytic in quark mass terms. These terms result only from the loops and therefore, their contribution to NLLog can be calculated with one-loop diagrams only. The methods of [12] can also be used to prove this. The absence of the tree level contribution allows the NLLog to be determined from the set of equations relating the different loop-order contributions.

6.2. *Pole equation at LLog accuracy*

In this section we discuss the properties of the solution of the pole equation at LLog accuracy. From the previous discussion it follows that it is also valid for the NLLog multiplied by a nonanalytic power of the quark mass.

The position of the pole in the propagator (28) is a Lorentz invariant quantity, when evaluated to all orders in the expansion. Therefore, one can choose[5] r^μ or p^μ such that $r^2 = (r \cdot v)^2$. Then, $\omega = (r \cdot v)$ gives the difference between the physical mass and the bare mass, $(r \cdot v) = \delta M = M_{\text{phys}} - M$. In this regime we can expand the expression for $\Sigma^{(n)}$ in powers of δM

$$\Sigma^{(n-1)}(\delta M) = \sum_{k=0}^{n} \sigma^{k,n-k} \delta M^k m^{n-k}, \tag{33}$$

[5] If one chooses $v^\mu = (1, 0, 0, 0)$ this corresponds to $\vec{r} = \vec{p} = 0$.

where the coefficients $\sigma^{(i,j)}$ contain logarithms. The coefficients $\sigma^{k,n-k}$ have mass-dimension $(1-n)$.

The solution of the pole equation $S^{-1}=0$, (28), can be found perturbatively in m

$$\delta M=\sum_{n=2}^{\infty} a_n m^n \tag{34}$$

where again the a_i contain logarithms. Inserting the expansions (34) and (33) into the Dyson propagator (28), and considering the pole equation for every power of m independently, we obtain a system of equations for the coefficients a_n,

$$a_n+\sum_{\{i\},j\leqslant n-2} a_{i_1}a_{i_2}\dots a_{i_j}\sigma^{j,n-\sum i}=0, \tag{35}$$

where summation runs over all possible sets of indices including empty set and permutations.

Let us consider the system of equations (35) in the LLog regime. We recall that the power of the LLog is $[\frac{n}{2}]$ for $\Sigma^{(n)}$. However, the coefficients a_n have different logarithm counting. The reason is the presence of terms non-analytical in quark masses. The Lagrangian of ChPT is necessarily analytical in the quark masses, i.e. it contains only even powers of m. The terms non-analytic in quark masses appear only through loop-integrals, and, therefore, they cannot appear in the expression (30) or in the first two lines of the expression (32). In this way, the number of logarithms in front of the pion mass in the odd power is suppressed by one (at least). Summarizing, we obtain the following LLog counting for the coefficients a and σ

$$a_n\sim \log^{[(n-2)/2]}(\mu),\qquad \sigma^{s,t}\sim\begin{cases}\log^{[(s+t-1)/2]}(\mu) & t\in\text{even}\\ \log^{[(s+t-3)/2]}(\mu) & t\in\text{odd}.\end{cases} \tag{36}$$

Using the counting (36), we neglect the NLLog terms in the equations (35) and obtain the system of equations in the LLog regime:

$$a_n+\sigma^{0,n}+\sum_{k=2,4,\dots}^{n-2} a_k\sigma^{1,n-k}=0,\quad n\in\text{even}, \tag{37}$$

$$a_n+\sigma^{0,n}+a_{n-1}\sigma^{1,1}+\sum_{k=3,5,\dots}^{n-2} a_k\sigma^{1,n-k}=0,\quad n\in\text{odd}. \tag{38}$$

This is a system of linear equations. The important result is that the even-n coefficients allow LLog evaluation only, because they involve only the LLog coefficient of analytical in quark mass terms. At the same time, the odd coefficients involve the terms non-analytical in quark masses. These coefficients are really NLLog. However, they can be obtained from a one-loop calculation as well, because they follow from the third line of (32).

One can see that the system (37)–(38) involves only the coefficients $\sigma^{0,n}$ and $\sigma^{1,n}$, which are the coefficients of the zeroth and the first powers of $(r\cdot v)$ in the propagator diagrams. It is a reflection of the fact that according to (36) the quantity $(r\cdot v)^2=\delta M^2$ is of NLLog order. Therefore, the powers of ω can be eliminated from the equation $S^{-1}=0$. The solution can be presented in the simple form

$$\delta M=\frac{-\Sigma(0)}{1+\Sigma'(0)}+\mathcal{O}(\text{NLLog}), \tag{39}$$

where $\Sigma(r\cdot v)=\sum_n \Sigma^{(n)}(r\cdot v)$ and Σ' is its derivative with respect to $(r\cdot v)$.

Table 1
The coefficients k_i defined in (40) of the LLog expansion of the nucleon mass.

k_2	$-4c_1M$
k_3	$-\frac{3}{2}g_A^2$
k_4	$\frac{3}{4}(g_A^2+(c_2+4c_3-4c_1)M)-3c_1M$
k_5	$\frac{3g_A^2}{8}(3-16g_A^2)$
k_6	$-\frac{3}{4}(g_A^2+(c_2+4c_3-4c_1)M)+\frac{3}{2}c_1M$
k_7	$g_A^2(-18g_A^4+\frac{35g_A^2}{4}-\frac{443}{64})$
k_8	$\frac{27}{8}(g_A^2+(c_2+4c_3-4c_1)M)-\frac{9}{2}c_1M$
k_9	$\frac{g_A^2}{3}(-116g_A^6+\frac{2537g_A^4}{20}-\frac{3569g_A^2}{24}+\frac{55\,609}{1280})$
k_{10}	$-\frac{257}{32}(g_A^2+(c_2+4c_3-4c_1)M)+\frac{257}{32}c_1M$
k_{11}	$\frac{g_A^2}{2}(-95g_A^8+\frac{5\,187\,407g_A^6}{20\,160}-\frac{449\,039g_A^4}{945}+\frac{16\,733\,923g_A^2}{60\,480}-\frac{298\,785\,521}{1\,935\,360})$

6.3. *Expression for the physical mass*

We have performed the calculation of the nucleon mass up to the fourth power of RG logarithms. We present the results in the form:

$$\begin{aligned} M_{\text{phys}} &= M + k_2\frac{m^2}{M} + k_3\frac{\pi m^3}{(4\pi F)^2} + k_4\frac{m^4}{(4\pi F)^2 M}\log\left(\frac{\mu^2}{m^2}\right) \\ &\quad + k_5\frac{\pi m^5}{(4\pi F)^4}\log\left(\frac{\mu^2}{m^2}\right) + \cdots \\ &= M + \frac{m^2}{M}\sum_{n=1}^{\infty}k_{2n}L^{n-1} + \pi m\frac{m^2}{(4\pi F)^2}\sum_{n=1}^{\infty}k_{2n+1}L^{n-1}, \end{aligned} \tag{40}$$

where L is defined in (19). The coefficients up to k_{11} are presented in Table 1. This corresponds to the four-loop calculation of LLog and five-loop calculation for the terms non-analytical in quark masses.

The presented results have been obtained via the different parametrizations of the Lagrangians (see Section 4), which gives a very strong check of calculation. Additionally, the coefficients up to k_6 agree with known results. The one-loop coefficients $k_{3,4}$ are well known, see e.g. [21]. The two-loop coefficient k_5 was first derived in [24]. The two-loop coefficients k_6 and k_5 are known from the full two-loop calculation for the nucleon mass performed in the EOMS scheme [25].

The generation of the higher order Lagrangians and the evaluation of one-loop beta-functions has been done automatically using the computer algebra system FORM [26]. The algorithm we used is similar to that used and described in [12,13]. The main integral needed for evaluation of beta-functions is presented in Appendix A. Although the calculation involves only one-loop diagrams, it is very demanding in machine time and memory. The most demanding factor is the length of the expression for the high order effective vertices and the number of diagrams to compute. These quantities grow rapidly with chiral order. For example, in order to calculate the k_{10} coefficient one needs to evaluate nearly 10^4 one-loop diagrams.

The calculation of the even coefficients k can be significantly simplified by using the conjectures discussed below in Section 6.4. So, by neglecting higher powers of g_A during the evaluation of the diagrams, we could also evaluate the five-loop coefficient k_{12}. Adding the further conjecture about the relation with the LLog in the pion mass, we can obtain the six and seven-loop coefficients k_{14} and k_{16}. However, these coefficients are the result of conjectures and, therefore, are presented in Table 3, separately from the results of the full calculation.

6.4. Properties of the result and conjectures

The straightforward calculation, limited by the available computer power, gives us the coefficients $k_1, \dots, k_{11}$ presented in Table 1. Considering the presented coefficients a number of regularities show up immediately. Some of the regularities we can explain easily, while some of them we cannot.

The first observation is that only even powers of g_A show up. This can be easily understood. The meson Lagrangian and the nucleon Lagrangian are invariant under the transformation $u \leftrightarrow u^\dagger$ and $g_A \leftrightarrow -g_A$. As a consequence only terms even in g_A can appear in the nucleon mass. This explains the pattern occurring in the odd coefficients k_{2n+1}.

The second observation is that the coefficients k_{2n} contains a very particular combination of LECs. The pattern appearing in k_{2n} is not well understood yet. Let us consider it in detail.

As shown in Section 5, one can have at most one insertion of the order p^2 Lagrangian. The expression for k_{2n} are thus at most linear in c_1, c_2, c_3 and the other terms in $\mathcal{L}^{(1)}_{\pi N}$. The coupling constant c_4 or A_5 cannot enter the nucleon mass at LLog since it produces an $\epsilon_{\mu\nu\alpha\beta}$. However, we found no simple argument why g_A only appears up to order g_A^2.

For the powers of g_A, there are two sources for factors of g_A in the loop diagrams, namely from the vertices $\mathcal{L}^{(0)}$ (20) and from the vertices $\mathcal{L}^{(1)}$ (22)–(23). While the number of vertices from $\mathcal{L}^{(1)}$ is restricted to one, the number of vertices from $\mathcal{L}^{(0)}$ is naturally unrestricted. Moreover, the expression for Σ contains all allowed powers of g_A. These powers cancels within the solution (39). We have checked that if one introduces new LECs for the terms proportional to g_A in $\mathcal{L}^{(1)}$ (say coefficients $B_{1,2}$ in front of the first two terms in (23)) the coefficients k_{2n} would contain higher powers of g_A. These induced higher powers are proportional to $(B_1 - B_2)$ and disappear when $B_1 = B_2$. Since these operators appear in $\mathcal{L}^{(1)}$ as the compensation of the non-relativistic nucleon reference frame, we conclude that absence of higher powers of g_A in coefficients k_{2n} is a consequence of Lorentz invariance.

Supposing that the cancellation of the higher powers of g_A takes place at all orders, one can neglect these powers during the computation of diagrams. This procedure significantly reduces the demands for computer time and allows us to calculate the coefficient k_{12}, which is presented in Table 3.

Considering Eq. (37) one can see that the coefficients k_{2n} consist from the terms proportional to exactly the first power of $\sigma^{0,k}$ where k is even. We remind that $\sigma^{0,\text{even}}$ are the result of diagrams with even chiral order and hence proportional to a single vertex from $\mathcal{L}^{(1)}$ (which is also checked by explicit calculation with coefficients $B_{1,2}$). Therefore, the term g_A^2 which appears in the coefficients k_{2n} resulted solely from $\mathcal{L}^{(1)}$. In its own turn, it implies that there is no contribution from the diagrams with vertices proportional to g_A only from $\mathcal{L}^{(0)}$. All such vertices have an odd number of pions. Absence of such vertices implies that *diagrams with more than two odd-number-of-pion vertices do not contribute to the LLog coefficient of nucleon mass*. Undoubtedly such a structure is a consequence of the additional subtractions of infrared (heavy mass) singularities into renormalization counterterms within heavy baryon theory, but we have not been able to prove this.

Table 2
The coefficients r of LLog expansion of the nucleon mass using the physical pion mass as defined in (42).

r_2	$-4c_1M$
r_3	$-\frac{3}{2}g_A^2$
r_4	$\frac{3}{4}(g_A^2+(c_2+4c_3-4c_1)M)-5c_1M$
r_5	$-6g_A^4$
r_6	$5c_1M$
r_7	$\frac{g_A^2}{4}(-8+5g_A^2-72g_A^4)$
r_8	$\frac{25}{3}c_1M$
r_9	$\frac{g_A^2}{3}(-116g_A^6+\frac{647g_A^4}{20}-\frac{457g_A^2}{12}+\frac{17}{40})$
r_{10}	$\frac{725}{36}c_1M$
r_{11}	$\frac{g_A^2}{2}(-95g_A^8+\frac{1\,679\,567g_A^6}{20\,160}-\frac{451\,799g_A^4}{3780}+\frac{320\,557g_A^2}{15\,120}-\frac{896\,467}{60\,480})$

Considering the first six coefficients k_{2n} one can observe that they have the pattern

$$k_{2n}=b_n\left(\frac{-3c_1M}{n-1}+\frac{3}{4}\left(g_A^2+(c_2+4c_3-4c_1)M\right)\right), \tag{41}$$

where b_n are some rational numbers. The coefficients b_n can be obtained from the calculation of the physical pion mass as we demonstrate below.

The nucleon mass LLog coefficient in terms of the physical pion mass m_{phys} has the form

$$M_{\text{phys}}=M+\frac{m_{\text{phys}}^2}{M}\sum_{n=1}^{\infty}r_{2n}L_\pi^{n-1}+\pi m_{\text{phys}}\frac{m_{\text{phys}}^2}{(4\pi F)^2}\sum_{n=1}^{\infty}r_{2n+1}L_\pi^{n-1}, \tag{42}$$

where

$$L_\pi=\frac{m_{\text{phys}}^2}{(4\pi F)^2}\log\left(\frac{\mu^2}{m_{\text{phys}}^2}\right).$$

The coefficients r_n of this expansion are presented in Table 2.

One can see that the non-analytical in quark mass terms r_{odd} do not simplify in this form of expansion, while the expressions for the coefficient r_{even} are significantly simplified. Moreover the combination of the LECs proportional to b_n in (41) completely disappears from the higher order terms. We conclude that the coefficients b_n are the coefficients of the LLog expansion of m^4 in the terms of physical pion mass. Thus, assuming that the pattern (41) holds for all orders we conjecture the LLog part of the expression for the nucleon bare mass via the physical masses[6] *at all orders* to be

$$M=M_{\text{phys}}+\frac{3}{4}m_{\text{phys}}^4\frac{\log(\frac{\mu^2}{m_{\text{phys}}^2})}{(4\pi F)^2}\left(\frac{g_A^2}{M_{\text{phys}}}-4c_1+c_2+4c_3\right)$$
$$-\frac{3c_1}{(4\pi F)^2}\int_{m_{\text{phys}}^2}^{\mu^2}m_{\text{phys}}^4(\mu')\,\frac{d\mu'^2}{\mu'^2}. \tag{43}$$

[6] This expression should be understood as not rewriting the term k_2m^2/M in the physical pion mass and the integral over μ^2 should be done after applying (18) to $m_{\text{phys}}^4(\mu')$.

Table 3
The coefficients k_i and r_i defined in (40) and (42), that are obtained by using the conjectures described in Section 6.4.

k_{12} *	$\frac{115}{3}(g_A^2 + (c_2 + 4c_3 - 4c_1)M) - \frac{92}{3}c_1 M$
k_{14} **	$-\frac{186\,515}{1536}(g_A^2 + (c_2 + 4c_3 - 4c_1)M) + \frac{186\,515}{2304}c_1 M$
k_{16} **	$\frac{153\,149\,887}{259\,200}(g_A^2 + (c_2 + 4c_3 - 4c_1)M) - \frac{153\,149\,887}{453\,600}c_1 M$
r_{12} *	$\frac{175}{4}c_1 M$
r_{14} **	$\frac{4\,153\,903}{24\,300}c_1 M$

* The coefficients k_{12} and r_{12} have been calculated within the simplified scheme by neglecting higher powers in g_A.
** The coefficients $k_{14,16}$ and r_{14} are the result suggested by the expression (43). r_{16} would require the knowledge of the L^7 term in the expression for the pion mass.

The expression for the physical pion mass is known up to 6-loop order, Eq. (18), therefore, we can guess two more LLog coefficients for the physical nucleon mass. These are presented in Table 3 and indicated by the double-star marks.

6.5. Numerical results

As mentioned in the introduction, the LLog are not necessarily dominant. They do however give an indication of the size of corrections to be expected. We use here one set of inputs to show an example. The input we use uses the c_i as determined in [21] and reasonable values for the other quantities. The actual values we use are:

$$\begin{aligned} &M = 938\ \text{MeV}, \quad c_1 = -0.87\ \text{GeV}^{-1}, \quad c_2 = 3.34\ \text{GeV}^{-1}, \quad \mu = 0.77\ \text{GeV}, \\ &F = 92.4\ \text{MeV}, \quad c_3 = -5.25\ \text{GeV}^{-1}, \quad g_A = 1.25. \end{aligned} \tag{44}$$

We plot in Fig. 2 the total correction $M_{\text{phys}} - M$ of (40) by loop order. We have included the results up to the k_{12} term since we do not have the odd powers higher than five loops. As can be seen there is a reasonable convergence for the range given.

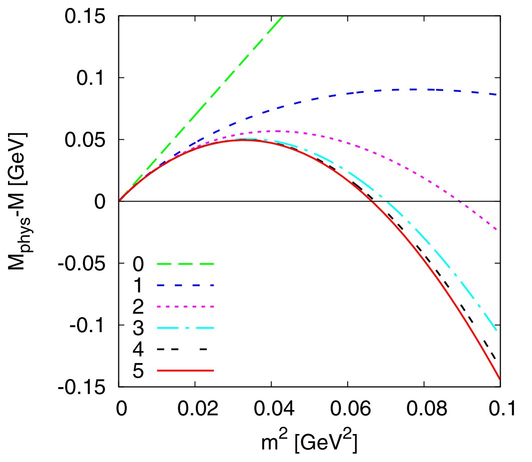

Fig. 2. The contribution of the terms in mass correction of (40) with the terms included up to a given loop-order.

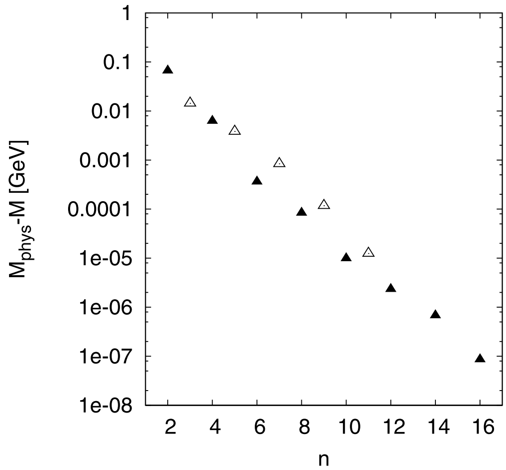

Fig. 3. The absolute value of the contribution of the individual terms ($\sim k_n$) in (40) at $m = 138$ MeV. Open symbols are the odd orders. Filled symbols are the even orders.

To see the convergence better, we have plotted in Fig. 3 the absolute value of the individual terms containing k_i of (40) for $m = 138$ MeV. Note the excellent convergence.

7. Conclusions

In this paper we have presented the application of the renormalization group method for nucleon–pion chiral perturbation theory. The theoretical basis of the method was developed in [5]. The method has been applied before only for bosonic theories, see [7,10,12–15]. In particular, we have calculated the physical mass of the nucleon within the heavy baryon formulation in the LLog approximation (analytical and non-analytical in quark mass terms) up to five-loop order. The results of the calculation are presented in (40), (42) and Tables 1, 2.

The theories with fermions (or more precisely, the theories involving Lagrangians of odd chiral order) have a more involved structure of RG equations. In contrast to the bosonic theories, where all one-loop beta functions contribute to LLog coefficients, in theories with fermions some one-loop beta functions do not contribute to the LLog coefficients. For resolving the RG hierarchy, we have introduced the concept of renormalization group order (RGO), see Section 2.2. Using the RGO allows us to extract the diagrams which contribute to the LLog approximation (or any other order of RG logarithms).

The calculation of the necessary one-loop beta-functions has been performed symbolically with the help of the FORM computer symbolic computation system. The results of our calculation agree with known one- and two-loop results, see e.g. [25]. The calculations were performed in several different parametrizations of the nucleon and pion fields with the same result, providing a very strong check for the computational algorithm. The analytical part of the computation, namely the expression for a basis one-loop-integral in the heavy baryon theory, is presented in Appendix A.

The obtained LLog coefficients show a number of regularities. Some of which can be easily understood, while the rest is more involved. The most intriguing regularity is the absence of higher powers of axial coupling constant g_A in the LLog coefficients k_{2n} (see (40) and Table 1). Moreover, the pattern of the LLog coefficients allows us to guess the *all-order* expression for the

LLog contribution to the nucleon mass in terms of *physical* pion mass (43). Although the latter is only a conjecture, we consider it as an exact result, most likely a consequence of the subtraction of heavy-mass-singularities within heavy baryon theory and Lorentz invariance.

We also showed some numerical results in Section 6.5.

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Appendix A. Loop integrals

The most resource demanding part of LLog evaluation is the calculation of the one-loop diagrams with generally a large number of external fields. Also at high chiral orders the loop-integrals involves a large number of open indices. In general in heavy baryon theory one faces loop-integrals of the very general form

$$\int \frac{d^d k}{(2\pi)^d} \frac{k^{\mu_1} \dots k^{\mu_n}}{(kv + \omega_1) \dots (kv + \omega_{N_N - 1})((k + p_1)^2 - m_\pi^2) \dots ((k + p_{N_\pi + 1})^2 - m_\pi^2)}, \tag{45}$$

where N_N is the number of vertices with nucleon, and N_π is the number of pure pionic vertices involved in the diagram.

The first step of evaluation of such loop-integrals is the joining of propagators of the same type into a single propagator with the help of Feynman variables, x_i for the nucleon propagators, y_i for the meson propagators. The subsequent shift of integration momentum allows one to remove the momenta p_i from the denominators (leaving them in the "mass"). The cost is a significant growth of the numerator, which is, however, a purely algebraic problem. The resulting sum integrals consist of simpler base integrals, the expressions for which we present below. Finally, the integrals over the Feynman parameters are done.

A.1. The base mesonic integral

The diagrams with a single or none nucleon vertex contain base integrals of the form

$$I_p^{\mu \dots \mu_n} = \int \frac{d^d k}{(2\pi)^d} \frac{k^{\mu_1} \dots k^{\mu_n}}{(k^2 - m^2)^p}, \tag{46}$$

where m^2 is a difficult combination of p_i, Feynman parameters and pion masses. The expression under the integral contains no intrinsic vectors, the result is thus proportional to metric tensors only. Therefore, the integral is zero for odd n. For even n it is completely symmetric in all the indices μ_i and reads

$$I_p^{\mu \dots \nu} = \frac{i(-1)^{\frac{n}{2}+p}}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(p - \frac{n}{2} - \frac{d}{2})}{2^{\frac{n}{2}} \Gamma(p)} g_s^{\mu_1 \dots \mu_n} \left(m^2\right)^{\frac{n}{2} - p + \frac{d}{2}}, \tag{47}$$

where $g_s^{\mu \dots \nu}$ is the totally symmetric combination of metric tensors. In our calculation we will need only the pole part of integral I, it reads

$$I_p^{\mu\ldots\nu} = \frac{i}{\epsilon(4\pi)^{\frac{d}{2}}} \frac{g_s^{\mu\ldots\nu}(m^2)^{2+\frac{n}{2}-p}}{2^{\frac{n}{2}}\Gamma(p)(2+\frac{n}{2}-p)!} + \mathcal{O}(\epsilon^0), \tag{48}$$

where we have used that $d = 4 - 2\epsilon$. This agrees with the expression in [12].

A.2. The base integral with nucleon propagators

The most general integral combines into the base integral of the form

$$I_{r,p}^{\mu\ldots\nu} = \int \frac{d^d k}{(2\pi)^d} \frac{k^\mu \ldots k^\nu}{(kv+\omega)^r (k^2 - m^2)^p}, \tag{49}$$

where $\omega = \sum_i \omega_i x_i$ (with x_i Feynman parameters), and m^2 is a combination of p_i, Feynman parameters y_i and pion mass. Using the pseudo-Feynman parameter z, we rewrite the integral as

$$I_{r,p}^{\mu\ldots\nu} = 2^r \frac{\Gamma(r+p)}{\Gamma(r)\Gamma(p)} \int \frac{d^d k}{(2\pi)^d} \int_0^\infty dz \frac{z^{r-1} k^\mu \ldots k^\nu}{(k^2 - m^2 + 2zkv + 2\omega z)^{r+p}}, \tag{50}$$

where z has mass dimension 1.

Performing the shift of the variable

$$k^\mu \to k^\mu - z v^\mu, \tag{51}$$

we represent the integral $I_{r,p}$ as a sum of (mesonic) base integrals which can be evaluated with (47). The resulting integral over z is of the form

$$\begin{aligned} \tilde{I} &= 2^r \frac{\Gamma(r+p)}{\Gamma(r)\Gamma(p)} \int_0^\infty dz \int \frac{d^d k}{(2\pi)^d} \frac{z^{r+l-1} k^{\mu_1} \ldots k^{\mu_n}}{(k^2 - m^2 - z^2 + 2\omega z)^{r+p}} \\ &= \frac{i(-1)^{\frac{n}{2}-r-p}}{(4\pi)^{\frac{d}{2}}} \frac{g_s^{\mu\ldots\nu}\Gamma(r+p-\frac{n}{2}-\frac{d}{2})}{2^{\frac{n}{2}-r}\Gamma(r)\Gamma(p)} \int_0^\infty dz\, z^{l+r-1}\left(m^2 + z^2 - 2\omega z\right)^{\frac{n}{2}+\frac{d}{2}-r-p}, \end{aligned} \tag{52}$$

where $l \geq 0$ follows from (51).

The parameters p, r, n, l are all integers and satisfy $p, r \geq 1$; $n, l \geq 0$. We introduce the special notation for the overall mass dimension of the integral:

$$A = n + l + 4 - r - 2p \in \mathbb{Z}.$$

In this notation the integral (52) reads

$$\tilde{I} = \frac{i(-1)^{\frac{A-l-r}{2}}}{(4\pi)^{\frac{d}{2}}} \frac{g_s^{\mu\ldots\nu}\Gamma(\frac{r+l-A}{2}+\epsilon)}{2^{\frac{n}{2}-r}\Gamma(r)\Gamma(p)} \int_0^\infty dz\, z^{l+r-1}\left(m^2 + z^2 - 2\omega z\right)^{\frac{A-l-r}{2}-\epsilon}. \tag{53}$$

We have two sources for the ϵ-pole, namely, the Γ-function and the integral over the pseudo-Feynman parameter z. Since $l + r > 0$, the only divergence in the z integral takes place at $z \to \infty$. We distinguish three cases:

(i) $A < 0$ the pseudo-Feynman integral is convergent. Since $r > 1$ and $l \geq 0$, we have that $r + k - A \geq 1$. There is no pole in ϵ.

(ii) $A \geq 0$ the pseudo-Feynman integral is divergent, and $r + l - A \geq 1$, such that the Γ-function has no pole.
(iii) $A \geq 0$ the pseudo-Feynman integral is divergent, but with $r + l - A \leq 0$ such that the Γ-function also has a pole.

The important point is that all divergent cases are for non-negative A. Expanding around $m = 0$, and taking the integral for every term we obtain

$$\tilde{I} = \frac{i(-1)^{\frac{3A-l-r}{2}-4\epsilon}}{(4\pi)^{\frac{d}{2}}} \frac{g_s^{\mu\ldots\nu} 2^{A+r-\frac{n}{2}-2\epsilon}}{\Gamma(r)\Gamma(p)} \times \sum_{j=0}^{\infty} \left(-\frac{m^2}{4}\right)^j \frac{\omega^{A-2j-2\epsilon}\Gamma(2j - A + 2\epsilon)\Gamma(\frac{A+l+r}{2} - j - \epsilon)}{j!}. \tag{54}$$

The gamma-functions cannot have poles simultaneously, due to $l \geq 1$. We have two series of poles

$$0 \leq j \leq \frac{A}{2}, \quad \text{and} \quad j \geq \frac{A + l + r}{2}. \tag{55}$$

The second series of poles appears at $z = 0$. Therefore, these poles are artifacts of the small-mass expansion. The first series represents the UV-poles we want. Expanding the integral around these poles, we obtain

$$\tilde{I} = \frac{i(-1)^{\frac{A-l-r}{2}}}{\epsilon(4\pi)^{\frac{d}{2}}} \frac{g_s^{\mu\ldots\nu} 2^{A+r-\frac{n}{2}-1}}{\Gamma(r)\Gamma(p)} \sum_{j=0}^{A/2} \left(-\frac{m^2}{4}\right)^j \frac{\omega^{A-2j}\Gamma(\frac{A+l+r}{2} - j)}{j!(A - 2j)!}. \tag{56}$$

We note that this also implies that $A \geq 0$, and it is also the dimension of the initial integral $I_{r,p}$. The argument of the last gamma-function is always integer because n is even.

A.3. Non-analytical in quark mass part

As shown in Section 6, we could also consider the terms non-analytical in the quark mass. They can be obtained from (53). The integral over z can be done exactly with the help of Jacobi polynomials and then expanded in ω. Or it can be done in the following way. Expanding the integrand of (53) in ω we obtain

$$\tilde{I} = \sum_{j=0}^{\infty} \frac{i(-1)^{\frac{A-l-r}{2}+j}}{(4\pi)^{\frac{d}{2}}} \frac{\omega^j g_s^{\mu\ldots\nu}\Gamma(\frac{r+l-A}{2} + \epsilon)}{2^{\frac{n}{2}-r-j}\Gamma(r)\Gamma(p)} \frac{\Gamma(\frac{A-l-r}{2} - \epsilon + 1)}{j!\Gamma(\frac{A-l-r}{2} - j - \epsilon + 1)} \times \int_0^{\infty} dz\, z^{l+r+j-1}\left(m^2 + z^2\right)^{\frac{A-l-r}{2}-\epsilon-j}. \tag{57}$$

The integral over z can be reduced to an Euler integral of the second kind. After simplifications the result reads

$$\tilde{I} = \frac{i(-1)^{\frac{A-l-r}{2}}}{(4\pi)^{\frac{d}{2}}} \frac{2^{r-\frac{n}{2}-1} g_s^{\mu\ldots\nu}}{\Gamma(r)\Gamma(p)} \sum_{j=0}^{\infty} (2\omega)^j m^{A-j-2\epsilon} \frac{\Gamma(\frac{j+l+r}{2})\Gamma(\frac{j-A}{2} + \epsilon)}{j!}, \tag{58}$$

which is equivalent to (54). Here we have three interesting cases:

1. $j \le A$ (only for $A \ge 0$), and $j - A$ even: the expression contains an ϵ pole, and is analytical in the quark mass. The ϵ-pole coefficient coincides with the one calculated before, (56).
2. $j > A$, and $j - A$ even: the expression is finite, and analytical in quark mass. It is of no interest for this work.
3. $j - A$ odd: the expression is finite, but non-analytical in quark mass. The leading term of the ϵ-expansion reads

$$\frac{i(-1)^{\frac{n}{2}-r-p}}{(4\pi)^{\frac{d}{2}}} \frac{2^{r-\frac{n}{2}-1} g_s^{\mu\dots\nu}}{\Gamma(r)\Gamma(p)} J^{(l+r)}_{n+l+4-r-2p}(\omega), \tag{59}$$

where

$$J_A^{(s)}(\omega) = \sum_{j=0\ (j+A\ \text{odd})}^{\infty} (2\omega)^j m^{A-j} \frac{\Gamma(\frac{j+s}{2})\Gamma(\frac{j-A}{2})}{j!}.$$

The expressions for $A = 2a$ (even) and $A = 2a + 1$ (odd) are:

$$J^s_{2a} = 2(-1)^a \sqrt{\pi} \frac{\Gamma(\frac{1+s}{2})}{(\frac{1}{2})_a} \frac{\omega}{m} \left(m^2\right)^a {}_2F_1\left(\frac{1+s}{2}, \frac{1}{2} - a; \frac{3}{2}; \frac{\omega^2}{m^2}\right),$$

$$J^s_{2a+1} = 2(-1)^{a+1} \sqrt{\pi} \frac{\Gamma(\frac{s}{2})}{(\frac{3}{2})_a} m \left(m^2\right)^a {}_2F_1\left(\frac{s}{2}, -\frac{1}{2} - a; \frac{1}{2}; \frac{\omega^2}{m^2}\right).$$

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