Second order Standard Model

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Abstract

It is known, though not commonly, that one can describe fermions using a second order in derivatives Lagrangian instead of the first order Dirac one. In this description the propagator is scalar, and the complexity is shifted to the vertex, which contains a derivative operator. In this paper we rewrite the Lagrangian of the fermionic sector of the Standard Model in such second order form. The new Lagrangian is extremely compact, and is obtained from the usual first order Lagrangian by integrating out all primed (or dotted) 2-component spinors. It thus contains just half of the 2-component spinors that appear in the usual Lagrangian, which suggests a new perspective on unification. We sketch a natural in this framework $SU(2) \times SU(4) \subset SO(9)$ unified theory.

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1. Introduction

> Different views suggest different kinds of modifications which might be made and hence are not equivalent in the hypotheses one generates from them in one's attempt to understand what is not yet understood.
>
> [Richard Feynman, Nobel Lecture]

It is well known that physical theories can be described by first as well as second derivative Lagrangians. A useful analogy here is between the Hamiltonian and Lagrangian mechanics.

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The Hamiltonian formulation gives first order evolution equations, but contains twice more independent variables than the second order Lagrangian framework. Integrating out the momentum variables one arrives at the second order formulation. Thus, which formulation to use is always a matter of convenience.

Often, however, a theory is first written (discovered) in one of the two available formulations, and this predetermines what is used by the physics community. This point is well illustrated by the first and second order formulations of General Relativity. Physicists use the second order metric formulation as the one in which GR was originally proposed. However, some aspects of the theory become more transparent in the first order formalism. For example, in the first order formulation the Lagrangian of GR is polynomial (cubic) in the fields [1], which is not at all the case in its second order version. Availability of a simple polynomial Lagrangian, even though containing more fields, is sometimes important, as in the case of the proof of the GR uniqueness in [1]. Another slightly less familiar example is the first order formalism for QCD [2]. Thus, one can rewrite the Yang–Mills Lagrangian in the BF form, plus a term quadratic in the B-field. Again, the first order Lagrangian is cubic in the fields, and some properties of the theory are much easier to see in this formulation.

The two examples show that integrating out fields and passing to the second order formulation typically leads to having vertices of higher valency. Thus, the trade-off is a smaller number of fields and thus simpler propagators vs. more involved propagators but simple (e.g. only cubic) interaction vertices. So, apart from historical reasons, the other justification for using the second order formulations in the case of gravity and YM theories is the fact that it is often convenient to work with a smaller number of fields. Indeed, in doing perturbative Feynman diagram computations it is convenient to have only one type of lines, even if this necessitates introduction of higher valent vertices.

The subject of this paper is the fermionic Lagrangian of the Standard Model. In its usual form, this is a version of the first order Dirac Lagrangian. The above discussion suggests that there is also an associated second order formulation that can be obtained by integrating out the "momenta" fields of the first order formalism. And indeed, such a second order formulation exists and has been studied by many authors. The list of references that we are aware of is [3–13], plus a few more works listed in [8]. Our approach uses 2-component spinors and is thus closest to that in [7]. As far as we are aware, the second order formalism has only been studied for the case of simple models (like QED), and the potentially more interesting case of the full SM has not been considered. One aim of this paper is to fill this gap.

When the Dirac Lagrangian is written in terms of 2-component spinors one realizes that the "momenta" canonically conjugate to, say, unprimed spinors, are the primed spinors. These spinors are anti-commuting variables treated in the path integral as independent (from the unprimed spinors), and thus one can integrate out the primed spinors, arriving at the second order Lagrangian for unprimed 2-component spinors. As is common to all second order Lagrangians, the propagator becomes very simple, being essentially that for scalars. The complexity of the first order formalism that to a large extent resided in the propagator $(k_\mu \gamma^\mu - m)^{-1}$ gets now shifted to the vertex, which in the second order formalism contains a derivative operator. There is also now a new quartic vertex, absent in the first order formulation. Thus, one obtains a formalism for fermions with Feynman rules very similar to those in QCD, with the familiar $(\partial A)A^2$ and A^4 vertices.

Because the propagator of the second order formulation is a multiple of the identity matrix, the algebra of γ-matrices that one has to do in computing Feynman diagrams is much simpler than is the case in the usual formalism. The second order formalism is also very ideally suited

for computations using the spinor helicity methods, see also [7] for an emphasis of this point. Thus, as in the case of QCD, it is the second-order formalism that is more efficient for practical calculations.

The reason why we don't learn about fermions directly in their more computationally superior second order version is that, unlike the case of QCD, there is a price to pay for going to the second order formulation. Thus, having integrated out the primed spinors, which in the Dirac Lagrangian are Hermitian conjugates of the unprimed, we have lost manifest unitarity. The theory is still unitary, but showing this now requires more work, as this is no longer true diagram by diagram, as for the case of a Hermitian Lagrangian. We will give more comments on the unitarity below.

At the time of writing this paper the theoretical physics community is excited about a new approach [14] to $\mathcal{N} = 4$ SYM theory that makes the Yangian symmetry manifest, at the expense of concealing locality and unitarity. Thus, the advocated here computationally more superior approach to the Standard Model that parts with the manifest unitarity may now be more psychologically acceptable than it would have been the case before. Our own main reasons to be interested in the second order Standard Model is that it is described by a very compact Lagrangian containing a (twice) smaller number of fermion fields, and thus points in the unusual direction in terms of Grand Unification, see more on this below.

Parting with this motivational discussion, our main result is an extremely simple second order Lagrangian for the fermionic sector of the SM. It is worth exhibiting the simplicity of this Lagrangian already in the Introduction. It reads

$$\mathcal{L}_{ferm} = -\frac{2}{\rho} D\bar{Q}^i DQ_i - \frac{2}{\rho} D\bar{L}^i DL_i - \rho \left(\Lambda\bar{Q}\right)^i Q_i - \rho \left(\Lambda\bar{L}\right)^i L_i. \tag{1}$$

This should be compared to (13). The notations are as follows:

$$Q_i = \begin{pmatrix} u_i \\ d_i \end{pmatrix}, \qquad L_i = \begin{pmatrix} v_i \\ e_i \end{pmatrix}, \qquad \left(\Lambda\bar{Q}\right)^i := \left(\Lambda_u^{ij}\bar{u}_j, \Lambda_d^{ij}\bar{d}_j\right),$$

$$\left(\Lambda\bar{L}\right)^i := \left(\Lambda_v^{ij}\bar{v}_j, \Lambda_e^{ij}\bar{e}_j\right), \tag{2}$$

all fermionic fields $u_i, d_i, v_i, e_i, \bar{u}_i, \bar{d}_i, \bar{v}_i, \bar{e}_i$ are 2-component unprimed spinors, with u_i, d_i, $\bar{u}_i, \bar{d}_i$ needed to describe quarks, and $v_i, e_i, \bar{v}_i, \bar{e}_i$ describing leptons. The strong SU(3) indices are suppressed, as are the fermionic ones. $\Lambda_{u,d,v,e}^{ij}$ are Hermitian 3×3 mass matrices with $i, j = 1, 2, 3$ being the generation index. As we explain in more details below, the doublets Q_i, L_i are SU(2) invariant ones constructed by absorbing the Higgs field into the fermions. The covariant derivatives D contain appropriate for each field gauge field, and in the case of Q_i, L_i contain SU(2) invariant $W^\pm, Z$ fields constructed from the SU(2) gauge field and the Higgs. Note that the right-handed fermions that are uncharged under the weak SU(2) got combined into doublets. We have also added the right-handed neutrinos as well as the corresponding mass matrix to account for the neutrino oscillations. This Lagrangian does not contain the Majorana mass terms for the right-handed neutrinos that would be necessary to achieve the seesaw mechanism. We give the corresponding expression in the main text as it is somewhat less elegant.

The Higgs field ϕ enters the second order Lagrangian only via the combination $\rho^2 := |\phi|^2$. The other 3 components of the Higgs doublet have been absorbed into the fields $Q_i, L_i, W^\pm, Z$ to make them SU(2) gauge-invariant objects. The idea to use gauge-invariant field combinations is not new, in the context of spontaneous symmetry breaking it has been considered in the literature several times, see [15–18]. We found that the second order formalism naturally leads to this gauge invariant formulation.

One of its most interesting features of (1) is that the Higgs field enters the new Lagrangian non-polynomially, here as a factor of $1/\rho$ in front of the kinetic term. In the main text we shall also give another version of the same Lagrangian where we absorb a factor of $1/\sqrt{\rho}$ into each fermionic field to render the kinetic terms canonical. This keeps the non-polynomial character of the Lagrangian, but now the Higgs enters via a contribution to the covariant derivatives, as well as in the mass terms.

Due to its non-polynomiality the Lagrangian (1) is not manifestly renormalizable, unlike its first order version that is just a ϕ^4 theory and the renormalizability is evident. The non-polynomiality of the second order Lagrangian in the Higgs field is in a striking resemblance to what one encounters in General Relativity. What is also similar with gravity is the fact that the non-polynomiality disappears if one passes to the first order formulation. We do not know whether all these similarities are just superficial or there is something deep, but they seem to be worth pointing out. We will further comment on the gravitational analogies in the last section.

The Lagrangian (1), together with the bosonic part that is given in terms of SU(2) invariant fields in the main text, can be used as the starting point for computations of the SM scattering amplitudes. We would like to emphasize that this results in a set of rules that are more computationally efficient than the usual first-order rules, and are completely equivalent.

The computation would proceed in two steps. In the first step one would compute correlation functions of all the fields, and in the second the field expansion into modes would be used to extract the scattering amplitudes. It is in this last step where the reality conditions for the free fermionic fields need to be added to the story, but these cause no difficulties. It would be interesting to develop this method of computing the amplitudes further, and possibly even implement it numerically, as it leads to computational advantages as compared to the rules based on the first order SM Lagrangian. This alternative method of computing the SM scattering amplitudes is another result of the present work.

Further, we found it hard to resist to think along the Grand Unification lines, and try to combine the terms in (1) into an even more compact expression. Given that the quark and leptons enter the Lagrangian so similarly, it is natural to think of the leptons as the fourth color of quarks, as is done in the Pati–Salam GUT [19]. Then the 4 terms in (1) combine into two, which we can write as

$$\mathcal{L}_{ferm} = -\frac{2}{\rho} D\bar{S}^i D S_i - \rho (\Lambda \bar{S})^i S_i, \tag{3}$$

where S_i, $\bar{S}^i$ are doublets with an additional index whose first 3 values enumerate the color and the fourth corresponds to leptons. The would be SU(4) symmetry is broken by the mass matrix (with a complicated structure, see below) in the last term. It is notable that in this second order formulation all fermions of the SM got combined into an 8-dimensional 2-component spinor multiplet S_i, plus a multiplet that transforms under a complex conjugate representation of the unbroken gauge group U(1) × SU(3).

A related point that is worth emphasizing at the outset is as follows. As we already mentioned, the procedure of integrating out the primed spinors from the Dirac Lagrangian naturally necessitates introduction of weak SU(2)-invariant fermionic doublets, in which fermions are dressed by the components of the Higgs field to give gauge-invariant objects. These Higgs-dressed fermionic fields transform non-trivially only under the unbroken SM gauge group U(1) × SU(3). Importantly, all fermions transform in a *real* representation. Thus, the SU(2)-invariant fermion configurations assemble into the already mentioned 8-dimensional set S_i and a complex conjugate set $\bar{S}^i$. Together, this forms a 16-dimensional real representation of the group U(1) × SU(3). This

is in contrast to what happens in the original first-order version of the SM, where the fermions transform in a *complex* representation of the group $U(1) \times SU(2) \times SU(3)$. The fact that we now have to account for a real, not complex fermionic representation allows for more possibilities as far as the grand unification is concerned. In particular, as we shall see, it is possible to put all the 16 fermions in (3) into a single 16-dimensional representation of SO(9), or into two fermionic representations of SO(8).

Finally, for readers that are not to be convinced by any beautiful rewritings of the Lagrangian, we give a computational complexity argument that favors our formulation to the usual first order one. Let us consider (for simplicity tree level) scattering amplitude in which a single fermion is scattered by n photons. In terms of Feynman graphs, all diagrams where a fermionic line is connected to n external photon lines should be considered. In the usual formalism there are only the 3-valent vertices, in our formalism one also needs to consider the possibility of having the 4-valent ones. However, these vertices can be taken care of at the step when one adds contributions of individual Feynman diagrams, i.e. considers the different ways that the external photon momenta can be attached to the fermionic line. At that step one groups the 4-valent vertex diagrams as contributions to the 3-valent vertex ones, see [8] for more details. So, the increase in the number of diagrams does not lead to an increase in the computational complexity, provided one properly groups the 4-valent diagrams with the relevant 3-valents ones.

Let us now compare the computation of each individual diagram in the second and first order formalisms. We keep the external momenta of n photons fixed and need to compute the amplitude as a function of these momenta. For definiteness, let us project the external photon lines on some polarizations, so that there are no free spacetime indices. We also assume that the fermionic momenta are fixed, and the fermionic lines are projected on some fixed external polarizations. The way this computation is carried out numerically in the usual formalism is as follows. One takes a product of n 4×4 matrices coming from the vertices, as well as $n - 1$ 4×4 matrices coming from the propagators. The operation of multiplication of two 4×4 matrices is worth of 2^7 operations (counting each multiplication and addition as a separate operation). Thus, overall, there are $(2^7)^{2n-2}$ operations to perform to compute the product of all the γ-matrices. Then there is finally the operation of projection on the external fermionic polarizations to be taken, which is 32 more operations. Overall, we get an order of 2^{14n} operations for this computation. In contrast, in the second order formalism, one is multiplying 2×2 matrices, and moreover just $n - 1$ of them, as the propagator is an identity in the field space. This results in $(2^4)^{n-1}$ operations, which is order of 2^{4n}. The result is that the second order computation is 2^{10n} times quicker than the usual one. Of course, there are more sophisticated methods of computing the scattering amplitudes than just computing and adding the contribution of all the diagrams. However, even for these other methods the simplification in the Feynman rules may be a bonus.

Overall, we hope to have convinced the reader that the second order formalism is a more computationally efficient way of treating fermions than the usual Dirac formulation. Thus, the second order formulation may be of particular relevance for the case of the Standard Model fermions, where one does have to perform extensive numerical investigations of various process occurring at LHC. Our paper may therefore be useful at least as giving this computationally more superior formulation.

The organization of this paper is as follows. We start in Section 2 by a quick reminder of how the usual Weyl, Majorana and Dirac actions look like in 2-component spinor notations. This fixes our conventions and notations. In this section we also carry the exercise of integrating out the primed spinor fields for the Dirac Lagrangian, thus arriving at the Lagrangian of second order QED. Section 3 reviews the fermionic sector of the SM in terms of 2-component spinors.

Section 4 is the main one where we obtain the second order Lagrangian (1). We also analyze the case with the Majorana mass term here. In Section 5 we write the bosonic sector Lagrangian in terms of the same gauge-invariant variables that we were led to introduce in the fermionic sector. The material here has appeared in the literature before, and we give it for completeness. Section 6 describes some simplest interaction vertices that follow from our Lagrangian, to get a better feel of how the second order perturbation theory works. We discuss unification in Section 7 and finish with a discussion.

2. Preliminaries: second order QED

We start by considering a much simpler setup of Quantum Electrodynamics (QED). To establish our conventions, and to prepare for integration over the primed spinors, let us write the familiar fermionic Lagrangians in terms of 2-component spinors. We follow the 2-component spinor conventions of [20].

2.1. Single massless Weyl fermion

The Lagrangian for a single massless Weyl fermion reads:

$$\mathcal{L}_{\text{Weyl}} = -i\sqrt{2}(\lambda^\dagger)_{A'}\theta^{\mu\,A'A}\partial_\mu\lambda_A \equiv -i\sqrt{2}\lambda^\dagger\partial\lambda. \tag{4}$$

Here λ_A is a 2-component spinor, $\lambda_A^\dagger$ is its Hermitian conjugate and $\theta_\mu^{A'A}$ is the soldering form

$$\theta_{\mu A'}^A\theta_{\nu A}{}^{A'} = \eta_{\mu\nu}, \tag{5}$$

where $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. We have also introduced the notation $\partial^{AA'} := \theta^\mu\partial_\mu^{AA'}$ and written the Lagrangian in an index-free way. Our index-free convention is that the unprimed fermions are always contracted as $\lambda\chi \equiv \lambda^A\chi_A$, while the primed fermions are contracted in the opposite way $\lambda^\dagger\chi^\dagger \equiv (\lambda^\dagger)_{A'}(\chi^\dagger)^{A'}$. The factor of $\sqrt{2}$ is introduced for convenience, and the minus in front of the kinetic term is convention dependent. With our conventions it is needed to get the positive-definite Hamiltonian. The Lagrangian is Hermitian, modulo a surface term.

2.2. Majorana mass term

When fermions are Grassmann valued we can have the Majorana mass term. Thus, consider

$$\mathcal{L}_{\text{Majorana}} = -i\sqrt{2}\lambda^\dagger\partial\lambda - (m/2)\lambda\lambda - (m/2)\lambda^\dagger\lambda^\dagger, \tag{6}$$

where we have used the index-free notation, and m is the parameter with dimensions of mass, later to be identified with the physical mass. Note that we need to add both terms in order for the Lagrangian to be Hermitian.

2.3. Dirac fermions

Dirac fermions are obtained by taking two massive Weyl fermions of equal mass. The system is then invariant under SO(2) rotations mixing the fermions. Since SO(2) $\sim$ U(1), complex linear combinations of fermions can be introduced and the Lagrangian rewritten in an explicitly U(1)-invariant way. This symmetry can be made local by introducing a U(1) gauge field and converting the usual derivative to the covariant one. Thus, we define

$$D_\mu \xi = (\partial_\mu - ieA_\mu)\xi, \qquad D_\mu \chi = (\partial_\mu + ieA_\mu)\chi, \tag{7}$$

where A_μ is the electromagnetic potential and e is the electron charge. Note that, since the fields ξ, χ are charged in the opposite way, the expressions for the covariant derivatives on these fields differ by a sign in front of A_μ. The gauge transformation rules are: for the fermions $\xi \to e^{i\phi}\xi$, $\chi \to e^{-i\phi}\chi$, and for the electromagnetic potential $A_\mu \to A_\mu - (1/e)\partial_\mu\varphi$. The Lagrangian becomes

$$\mathcal{L}_{\text{Dirac}} = -i\sqrt{2}\xi^\dagger D\xi - i\sqrt{2}\chi^\dagger D\chi - m\chi\xi - m\xi^\dagger\chi^\dagger, \tag{8}$$

where as before $D := \theta^\mu D_\mu$.

2.4. Chiral Dirac theory

At the level of the path integral we can integrate out the fermionic fields $\xi^\dagger$, $\chi^\dagger$ and obtain a chiral Lagrangian involving only unprimed spinors. From field equations for the primed spinors we get:

$$\xi^\dagger = -\frac{i\sqrt{2}}{m}D\chi, \qquad \chi^\dagger = -\frac{i\sqrt{2}}{m}D\xi. \tag{9}$$

Substituting this into the Lagrangian (8) we get:

$$\mathcal{L}_{\text{chiral}} = -\frac{2}{m}D\chi D\xi - m\chi\xi. \tag{10}$$

As the final step, the kinetic term here can be put into the canonical form by doing the rescaling

$$\xi \to \sqrt{m}\xi, \qquad \chi \to \sqrt{m}\chi. \tag{11}$$

The new spinor fields have mass dimension one, and the Lagrangian takes the following simple form

$$\mathcal{L}_{\text{chiral}} = -2D\chi D\xi - m^2\chi\xi \equiv -2D_{A'}{}^A \chi_A D^{A'B}\xi_B - m^2\chi^A\xi_A, \tag{12}$$

where we reintroduced the spinor indices to make everything explicit. Our aim is to write the SM fermionic sector in a similar fashion.

3. Preliminaries: SM Lagrangian via 2-component spinors

We follow [21], with some differences in conventions.

3.1. Standard Model fermions

The SM fermions can be put together in the following table

2-component fermions	SU(3)	SU(2)	Y	T_3	$Q = T_3 + Y$
$Q_i = \begin{pmatrix} u_i \\ d_i \end{pmatrix}$	triplet triplet	doublet	1/6 1/6	1/2 −1/2	2/3 −1/3
$\bar{u}_i$	triplet	singlet	−2/3	0	−2/3
$\bar{d}_i$	triplet	singlet	1/3	0	1/3
$L_i = \begin{pmatrix} v_i \\ e_i \end{pmatrix}$	singlet singlet	doublet	−1/2 −1/2	1/2 −1/2	0 −1
$\bar{e}_i$	singlet	singlet	1	0	1
$\bar{v}_i$	singlet	singlet	0	0	0

All fermionic fields here are unprimed 2-component spinors. Thus, $\bar{u}_i$ is *not* the Hermitian conjugate of u_i, with the latter being denoted by $u_i^\dagger$. The first 4 fermionic fields are the ones needed to describe quarks, the last 4 are those for describing leptons. Note that the last line is not a part of the Standard Model, but is now believed to be needed to account for neutrino masses. Also note that the strong SU(3) triplet fields are actually a collection of 3 different two-component spinors. Thus, e.g. u_i has two types of indices suppressed: the usual spinor index, as well as the strong SU(3) index. With no index suppression this field would be denoted by u_{iA}^α, where $A = 1, 2$ is the usual spinor index, and $\alpha = 1, 2, 3$ is the index on which SU(3) acts. The only index that is indicated explicitly is that enumerating the generations $i = 1, 2, 3$. It is an instructive exercise to count the number of 2-component fermionic fields in the above table. We have $4 \times 3 = 12$ for the quark sector and 4 for the lepton sector. Thus, we need 16 2-component spinors for each generation of the Standard Model.

3.2. Higgs field

This is the field that plays the central role in the Standard Model. It is a complex field of U(1) charge $Y = 1/2$. It is also a weak SU(2) doublet, i.e. it can be written as a column

Higgs	SU(3)	SU(2)	Y	T_3	$Q = T_3 + Y$
$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$	singlet	doublet	1/2 1/2	1/2 −1/2	1 0

Note that being an SU(2) doublet, it is really a collection of 2 complex fields ϕ^+ (the plus denotes the fact that the electric charge $Q = +1$) and ϕ^0 (with $Q = 0$). Alternatively, it is a collection of 4 real fields. We shall denote the weak SU(2) index by $a, b, \ldots = 1, 2$. Thus we can write the Higgs field as ϕ_a, with $\phi_1 = \phi^+$ and $\phi_2 = \phi^0$.

3.3. Fermionic sector of the Standard Model

Using index-free notations, the Lagrangian for the fermionic sector of the Standard Model reads:

$$
\begin{aligned}
\mathcal{L}_{ferm} = &-i\sqrt{2} Q^{\dagger i} D Q_i - i\sqrt{2} \bar{u}^{\dagger i} D \bar{u}_i - i\sqrt{2} \bar{d}^{\dagger i} D \bar{d}_i - i\sqrt{2} L^{\dagger i} D L_i - i\sqrt{2} \bar{e}^{\dagger i} D \bar{e}_i \\
&- i\sqrt{2} \bar{v}^{\dagger i} D \bar{v}_i + Y_u^{ij} \phi^T \varepsilon Q_i \bar{u}_j - Y_d^{ij} \phi^\dagger Q_i \bar{d}_j + Y_v^{ij} \phi^T \varepsilon L_i \bar{v}_j - Y_e^{ij} \phi^\dagger L_i \bar{e}_j \\
&- (Y_u^\dagger)^{ij} \bar{u}_i^\dagger Q_j^\dagger \varepsilon \phi^* - (Y_d^\dagger)^{ij} \bar{d}_i^\dagger Q_j^\dagger \phi - (Y_v^\dagger)^{ij} \bar{v}_i^\dagger L_j^\dagger \varepsilon \phi^* - (Y_e^\dagger)^{ij} \bar{e}_i^\dagger L_j^\dagger \phi \\
&- \frac{1}{2} M_{\bar{v}}^{ij} \bar{v}_i \bar{v}_j - \frac{1}{2} (M_{\bar{v}}^\dagger)^{ij} \bar{v}_i^\dagger \bar{v}_j^\dagger.
\end{aligned}
\tag{13}
$$

Here as before $D^{AA'} \equiv \theta^{\mu AA'} D_\mu$, where D_μ is an appropriate for each field covariant derivative. The quantities Y^{ij} are arbitrary complex 3×3 mass matrices. All fermions are Grassmann-valued.

We have given a formulation of the SM with the right-handed neutrinos added in. To get the usual SM one just has to erase all the terms containing the $\bar\nu$ field. Equivalently, it is obtained by setting to zero the mass matrices Y_ν^{ij}, M_ν^{ij}.

Only the SU(2) index structure in the above Lagrangian needs clarifications, all other suppressed indices are self-explanatory. Our SU(2) index conventions are as follows. As we have already mentioned, the Higgs field is considered to be an object ϕ_a with a single SU(2) index in the lower position. Its transpose is then an object $(\phi^T)^a$. The complex conjugate field $(\phi^*)_a$ still carries a lower position index, while the Hermitian conjugate is $(\phi^\dagger)^a$. Similarly, the quark doublet Q_a has a lower index position. Its Hermitian conjugate is an object $(Q^\dagger)^a$. The quantity $\epsilon \equiv \epsilon_a{}^b$ is the matrix

$$\epsilon_a{}^b = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{14}$$

Then the object $\phi^T \epsilon Q \equiv (\phi^T)^a \epsilon_a{}^b Q_a$ is invariant under the action of $SL(2,\mathbb{C})$ via $Q \to gQ$, $\phi \to g\phi$ since $g^T \epsilon g = \epsilon$. In particular, $\phi^T \epsilon Q$ is SU(2) invariant. It is then clear that all the terms in the second line of (13) are SU(2) invariant. The U(1) invariance is also easily checked by noting the Y-charges of the objects. The third line in (13) is then the Hermitian conjugate of the second. The last line is the Majorana mass terms for the right-handed neutrinos. It is also worth noting that the position of the generation indices (upper or lower) does not matter, as all these indices are explicitly indicated. We put them wherever convenient so as not to clatter the formulas.

4. Second order formulation of the Standard Model

We now carry out the procedure of integrating out the primed 2-component spinors from the significantly more involved SM Lagrangian.

4.1. Quark sector

We start with the quark sector as there is no Majorana mass term in this case. The equations of motion for the unprimed spinors are:

$$\boxed{\begin{aligned} Q_i^\dagger &: i\sqrt{2} D Q^i = -(\epsilon\phi^*)\,\bar u_j^\dagger (Y_u^\dagger)^{ji} - \phi\,\bar d_j^\dagger (Y_d^\dagger)^{ji} \\ \bar u_i^\dagger &: i\sqrt{2} D\bar u^i = -(Y_u^\dagger)^{ij} Q_j^\dagger (\epsilon\phi^*) \\ \bar d_i^\dagger &: i\sqrt{2} D\bar d^i = -(Y_d^\dagger)^{ij} Q_j^\dagger \phi \end{aligned}} \tag{15}$$

We now notice that some structure is making itself explicit in the equations of motion. Thus, let us combine the components of the Higgs field into the following 2×2 matrix:

$$\rho\Phi^\dagger := \left(\epsilon\phi^*, \phi\right) \equiv \begin{pmatrix} (\phi^0)^* & \phi^+ \\ -\phi^- & \phi^0 \end{pmatrix}. \tag{16}$$

Under the weak SU(2) the matrix $\Phi^\dagger$ transforms as:

$$\Phi^\dagger \mapsto \Omega\Phi^\dagger, \tag{17}$$

while the field ρ remains invariant. It is clear that ρ^2 is just the modulus squared of the Higgs field.

To make the above equations look more transparent, we define the new quark singlets as linear combinations of the old ones:

$$\bar{u}^i \ \rightarrow \ (Y_u^\dagger)^{ij}\bar{u}_j, \quad \bar{d}^i \ \rightarrow \ (Y_d^\dagger)^{ij}\bar{d}_j. \tag{18}$$

It is clear that this is designed to eliminate the mass matrices from the last pair of equations. Note that this is a constant reparameterization of the fields, so it does not lead to any change in the derivative operators. We further combine the new quark singlets into a row

$$\bar{Q}_i := \left(\bar{u}_i \ , \ \bar{d}_i\right) \tag{19}$$

In terms of the new quark singlets the equations of motion become:

$$\boxed{\begin{matrix} Q_i^\dagger : & i\sqrt{2}DQ_i = -\rho \ \Phi^\dagger \left(\bar{Q}^\dagger \Lambda\right)_i \\ \bar{Q}_i^\dagger : & i\sqrt{2}D\bar{Q}_i = -\rho \ Q_i^\dagger \Phi^\dagger \end{matrix}} \tag{20}$$

Here we introduced new Hermitian mass matrices

$$\Lambda_q^{ij} := Y_q^{ik}(Y_q^\dagger)^{kj}, \tag{21}$$

as well as a new column

$$\left(\bar{Q}^\dagger \Lambda\right)^i \equiv \begin{pmatrix} \bar{u}_j^\dagger \Lambda_u^{ji} \\ \bar{d}_j^\dagger \Lambda_d^{ji} \end{pmatrix}. \tag{22}$$

While introducing the new doublet $\bar{Q}_i$ has made the equations look more symmetric, there is no complete symmetry. Indeed, the doublet Q_i transforms under the weak SU(2), and so does the Higgs matrix $\Phi^\dagger$, while $\bar{Q}_i$ does not transform. This suggests that we define a new set of SU(2)-invariant quark variables ΦQ_i

$$\Phi Q_i := Q_i^{inv}. \tag{23}$$

This is a Higgs-field dependent SU(2) gauge rotation of the original quark doublet. As such, it can be pulled through the derivative as long as one takes into account that the gauge field also transforms. As we will work out in details below, the new gauge field will be an SU(2)-invariant object. Keeping in mind this change in the derivative operator we can write the field equations as:

$$\boxed{\begin{matrix} Q_i^\dagger : & i\sqrt{2}DQ_i = -\rho \ \left(\bar{Q}^\dagger \Lambda\right)_i \\ \bar{Q}_i^\dagger : & i\sqrt{2}D\bar{Q}_i = -\rho \ Q_i^\dagger \end{matrix}} \tag{24}$$

We have dropped the superscript *inv* from the Q_i to avoid clattering of the notations. We see that the equations become much simpler than in terms of the original fields.

We now substitute the primed spinors obtained from the above field equations into the Lagrangian (13) and obtain the following second-order Lagrangian:

$$\mathcal{L}_{quarks} = -\frac{2}{\rho}D\bar{Q}^i DQ_i - \rho \left(\Lambda \bar{Q}\right)^i Q_i, \tag{25}$$

where we have introduced a new row

$$\left(\Lambda \bar{Q}\right)^i := \left(\Lambda_u^{ij}\bar{u}_j, \ \Lambda_d^{ij}\bar{d}_j\right), \tag{26}$$

which is the Hermitian conjugate of (22) $\left(\Lambda \bar{Q}\right)^{\dagger} = \bar{Q}^{\dagger}\Lambda$. The calculation that leads to (25) simplifies significantly by noting that half of the kinetic terms cancels the mass terms for the primed spinors, while the other half survives. Then the kinetic term in (25) is easily obtained from the kinetic term $Q^{\dagger i} D Q_i$ by substituting the expression for $Q^{\dagger i}$. The mass term in (25) is easily obtained by combining the mass terms for the unprimed spinors in (13), and taking into account the definitions (18), (23) of the new fermionic variables. The derivative operator acting on Q_i in (25) takes into account the field redefinition (23).

The Lagrangian (25) is much simpler than the terms in (13) from which it was obtained, containing just half the terms from the quark sector. However, it is clearly non-polynomial in the Higgs scalar field ρ, because of the presence of $1/\rho$ in the kinetic term. This is exactly the same phenomenon as one sees in (10). In the case of Dirac theory we have rescaled the fermionic fields to give them mass dimension one, and brought the kinetic term into the canonical form. This suggests that we should do the same in (25). However, ρ is now a dynamical field. Absorbing it into the fermion fields thus changes the derivative operators acting on both $\bar{Q}_i, Q_i$. Denoting the new Higgs-containing derivative operators by the curly $\mathcal{D}$ we finally write:

$$\mathcal{L}^{(2)}_{quarks} = -2\mathcal{D}\bar{Q}^i \mathcal{D}Q_i - \rho^2 \left(\Lambda \bar{Q}\right)^i Q_i \tag{27}$$

where $1/\sqrt{\rho}$ was absorbed into each spinor field. The new covariant derivative $\mathcal{D}$ contains non-polynomial Higgs-quarks interactions as well as the physical SU(2)-frozen gauge fields when acting on the unbarred doublet. Expanding (27) around the Higgs VEV $\rho \rightarrow v + \rho$ one gets the free massive quarks with masses being multiples of the eigenvalues of the Hermitian mass matrices Λ_q^{ij}, together with quark interactions with the gauge fields as well as the Higgs. We will spell out the simplest interactions below. It is clear that interaction vertices with the Higgs can be of arbitrarily high valency (due to non-polynomiality in ρ).

The field equations (24) for the new fermionic fields of mass dimension one read

$$i\sqrt{2}\mathcal{D}Q_i = -\rho \left(\bar{Q}^{\dagger}\Lambda\right)_i, \qquad i\sqrt{2}\mathcal{D}\bar{Q}_i = -\rho\, Q_i^{\dagger}. \tag{28}$$

These are now to be interpreted as the reality conditions, whose linearized versions are to be imposed on the external lines.

4.2. Leptonic sector without the Majorana mass terms

Let us first set all the Majorana mass terms to zero. In this case the analysis is exactly the same as in the previous subsection. Thus, introducing the new barred fermion fields

$$\bar{v}^i \rightarrow (Y_v^{\dagger})^{ij}\bar{v}_j, \quad \bar{e}^i \rightarrow (Y_e^{\dagger})^{ij}\bar{e}_j, \tag{29}$$

we put the new fields together in a row

$$\bar{L}_i := \left(\bar{v}_i \,,\, \bar{e}_i\right), \tag{30}$$

and define another row

$$\left(\Lambda \bar{L}\right)^i := \left(\Lambda_v^{ij}\bar{v}_j, \Lambda_e^{ij}\bar{e}_j\right), \tag{31}$$

where $\Lambda_l = Y_l Y_l^{\dagger}$ are the Hermitian mass matrices. We also define the physical SU(2)-invariant unbarred leptonic doublet $L_i^{inv} = \Phi L_i$. Writing everything in terms of these quantities we get the following Lagrangian

$$\mathcal{L}_{leptons} = -i\sqrt{2}L^{\dagger i}DL_i - i\sqrt{2}\left(D\bar{L}^i\right)\left(\bar{L}^{\dagger}\Lambda\right)_i - \rho\left(\Lambda\bar{L}\right)^i L_i - \rho L^{\dagger i}\left(\bar{L}^{\dagger}\Lambda\right)_i. \quad (32)$$

The resulting equations for the primed spinors are

$$\boxed{\begin{array}{ll} L_i^{\dagger}: & i\sqrt{2}DL_i = -\rho\ (\bar{L}^{\dagger}\Lambda)_i \\ \bar{\nu}_i^{\dagger}: & i\sqrt{2}D\bar{L}_i = -\rho\ L_i^{\dagger} \end{array}} \quad (33)$$

Substituting the resulting primed spinors into the Lagrangian we get

$$\mathcal{L}_{leptons} = -\frac{2}{\rho}D\bar{L}^i DL_i - \rho\left(\Lambda\bar{L}\right)^i L_i. \quad (34)$$

One can now rescale the lepton fields to give them mass dimension one and convert the kinetic terms into a standard form. One obtains a Lagrangian as in (27). The sum of (34) and (25) is the Lagrangian quoted in the Introduction.

4.3. Majorana mass terms

We now reinstate the Majorana mass terms. This leads to a more complicated analysis and not so simple final result. Performing the same redefinitions of the fermionic variables as was done above, we can write the original Lagrangian in terms of the new spinor fields:

$$\begin{aligned} \mathcal{L}_{leptons} = &-i\sqrt{2}L^{\dagger i}DL_i - i\sqrt{2}\left(D\bar{L}^i\right)\left(\bar{L}^{\dagger}\Lambda\right)_i - \rho\left(\Lambda\bar{L}\right)^i L_i - \rho L^{\dagger i}\left(\bar{L}^{\dagger}\Lambda\right)_i \\ &-\frac{1}{2}(Y_{\nu}^{\dagger})^{ik}(Y_{\nu}^{\dagger})^{jl}M_{\bar{\nu}}^{ij}\bar{\nu}_k\bar{\nu}_l - \frac{1}{2}Y_{\nu}^{ki}Y_{\nu}^{lj}(M_{\bar{\nu}}^{\dagger})^{ij}\bar{\nu}_k^{\dagger}\bar{\nu}_l^{\dagger}. \end{aligned} \quad (35)$$

The structure of the last two terms suggests the following redefinition of the right-handed neutrino mass matrix

$$(Y_{\nu}^{\dagger})^{ik}(Y_{\nu}^{\dagger})^{jl}M_{\bar{\nu}}^{ij} \rightarrow M_{\bar{\nu}}^{kl}. \quad (36)$$

The new mass matrix is still symmetric. The Lagrangian becomes

$$\begin{aligned} \mathcal{L}_{leptons} = &-i\sqrt{2}L^{\dagger i}DL_i - i\sqrt{2}\left(D\bar{L}^i\right)\left(\bar{L}^{\dagger}\Lambda\right)_i - \rho\left(\Lambda\bar{L}\right)^i L_i - \rho L^{\dagger i}\left(\bar{L}^{\dagger}\Lambda\right)_i \\ &-\frac{1}{2}M_{\bar{\nu}}^{ij}\bar{\nu}_i\bar{\nu}_j - \frac{1}{2}(M_{\bar{\nu}}^{\dagger})^{ij}\bar{\nu}_i^{\dagger}\bar{\nu}_j^{\dagger}. \end{aligned} \quad (37)$$

The resulting equations of motion for the primed spinors are as follows

$$\boxed{\begin{array}{ll} L_i^{\dagger}: & i\sqrt{2}DL_i = -\rho\ (\bar{L}^{\dagger}\Lambda)_i \\ \bar{\nu}_i^{\dagger}: & i\sqrt{2}D\bar{\nu}^i = -\rho\ \nu^{\dagger i} - \bar{\nu}^{\dagger j}(M_{\bar{\nu}}^{\dagger})_{jk}(\Lambda^{-1})^{ki} \\ \bar{e}_i^{\dagger}: & i\sqrt{2}D\bar{e}_i = -\rho\ e_i^{\dagger} \end{array}} \quad (38)$$

We can now solve for the barred primed spinors using the first equation. From this we in particular get $\bar{\nu}^{\dagger}$, which can be substituted into the second equation. The last pair is then solved for $L^{\dagger}$ fermions.

After the solutions are substituted into the Lagrangian, simplifications result. Thus, it is easy to note that the first and the last terms in the first line of (37) cancel each other in view of the first equation in (38). Indeed, we can combine these two terms as

$$\mathcal{L}_{leptons} \supset L_i^{\dagger}\left(-i\sqrt{2}DL_i - \rho\left(\bar{L}^{\dagger}\Lambda\right)_i\right). \quad (39)$$

It is clear that this combination gives zero on the first equation in (38). To eliminate the remaining primed spinors we again need just the first field equation that gives us $\bar{L}^\dagger$. Overall, this gives:

$$
\mathcal{L}^{(2)}_{leptons} = -\frac{2}{\rho} D\bar{L}^i DL_i - \rho \left(\Lambda \bar{L}\right)^i L_i
$$
$$
- \frac{1}{2} M^{ij}_{\bar{\nu}} \bar{\nu}_i \bar{\nu}_j + \frac{1}{\rho^2} (\Lambda_\nu^{-1} M^\dagger_{\bar{\nu}} \Lambda_\nu^{-1})^{ij} (DL_i)^\nu (DL_j)^\nu,
\tag{40}
$$

where $(DL_i)^\nu$ stands for the first ν-component of the doublet DL_i.

This is a complicated Lagrangian. To understand what is going on here it is useful to return to the second field equation in (38). Substituting the solution for $\bar{\nu}^\dagger_i$ from the first equation one gets:

$$
i\sqrt{2} \left(D\bar{\nu}^i - \frac{1}{\rho}(\Lambda_\nu^{-1} M^\dagger_{\bar{\nu}} \Lambda_\nu^{-1})^{ij} (DL_j)^\nu \right) = -\rho \, \nu^{\dagger i}.
\tag{41}
$$

If one expands all terms in this equation around the Higgs VEV $\langle \rho \rangle = v$, the terms linear in the fields are

$$
i\sqrt{2} \, \partial \left(\bar{\nu}^i - \frac{1}{v}(\Lambda_\nu^{-1} M^\dagger_{\bar{\nu}} \Lambda_\nu^{-1})^{ij} \nu_j \right) = -v \, \nu^{\dagger i}.
\tag{42}
$$

This equation suggests that we should introduce a new barred neutrino field

$$
\bar{\nu}^{new}_i := \bar{\nu}_i - \frac{1}{v}(\Lambda_\nu^{-1} M^\dagger_{\bar{\nu}} \Lambda_\nu^{-1})_{ij} \nu^j,
\tag{43}
$$

as it is this field that satisfies reality conditions similar to those for all other 2-component fermions present. One should then rewrite the linearization of the Lagrangian (40), as well as the interaction vertices in terms of $\bar{\nu}^{new}_i$, and proceed with the usual perturbation theory calculations. It is clear that the result is complicated, as it relies on the field redefinition (43) that in turn relies on the Higgs assuming its VEV. Thus, it seems that no elegant second order Lagrangian is possible when the Majorana mass terms are present. Of course, one can still work with the Lagrangian that is not elegant, but then one may as well work with the usual first order Lagrangian (13) where the Majorana mass terms are not more unnatural than any other terms present.

Thus, if just for a moment one takes the viewpoint that the second order Lagrangian is more fundamental than the usual first order one, one seems to be led to the conclusion that the simple Lagrangian (1) is all one has. This Lagrangian explains neutrino oscillations by giving the neutrinos masses, but does not by itself explain why these masses are so small as compared to the other ones. Without the Majorana mass terms there is also not enough room to explain other SM mysteries. Indeed, with the Majorana mass terms the lepton number is no longer conserved, and this plays an important role in many beyond the SM scenarios. In particular, the Majorana mass terms are of importance in the νMSM explanation [22] of the baryon asymmetry of the Universe.

One way out of this could be to add to the Lagrangian (1) terms that are natural from the second order point of view, but spoil the renormalizability of the theory. This can provide new couplings which can then in turn be useful for the beyond the Standard Model scenarios. We will not attempt any of this in the present paper, and continue to explore the second order Lagrangian without the Majorana terms.

5. Bosonic sector in terms of gauge-invariant variables

In this section we rewrite the bosonic sector of the Standard Model in terms of the same gauge-invariant fields that were used in our fermionic Lagrangian. The idea that this is possible and, in fact, gives a much more clear perspective on the "symmetry breaking" mechanism is not new, as far as we know was first used in [15] in the context of the Hamiltonian formulation and more recently in [16–18]. Our analysis here basically repeats [17] with minor differences in conventions. We give it here for completeness, as well as to be able to spell out some simple interaction vertices that follow from our formalism in the next section.

5.1. Higgs sector

We denote the gauge fields associated to the Standard Model group $SU(2) \times U(1)$[1] by B_μ and Y_μ respectively and their coupling constants by g_2, g_1.

We first repeat the construction of an $SU(2)$ element out of the Higgs field. This is the same construction that already appeared in the previous sections. We start with the Higgs field that is an $SU(2)$ doublet with the Y-charge of $1/2$. The covariant derivative of the Higgs reads

$$D_\mu\phi = \partial_\mu\phi + ig_2 B_\mu\phi + \frac{ig_1}{2}Y_\mu\phi, \tag{44}$$

where $B_\mu = T^a B_\mu^a$ and $T^a = (1/2)\sigma^a$, where σ^a are the usual Pauli matrices. The $SU(2)$ connection B_μ transforms as $B_\mu \to \Omega^\dagger B_\mu \Omega + (1/ig_2)\Omega^\dagger(\partial_\mu\Omega)$, and the $U(1)$ connection as $Y_\mu \to Y_\mu + (1/g_1)\partial_\mu\xi$.

We now parametrize this doublet as

$$\phi \equiv \rho\chi, \qquad \rho \in \mathbb{R}^+, \ \chi \in \mathbb{C}^2 \text{ with } |\chi|^2 = 1. \tag{45}$$

Using the spinor metric ε, we can construct

$$\Phi \equiv \begin{pmatrix} (\varepsilon\chi)^T \\ \chi^\dagger \end{pmatrix} \in SU(2). \tag{46}$$

This $SU(2)$-valued object transforms under the Standard Model gauge group as follows:

$$U(1) : \Phi \mapsto e^{i\xi T^3}\Phi, \tag{47}$$

$$SU(2) : \Phi \mapsto \Phi\Omega^\dagger. \tag{48}$$

We can then define a covariant derivative operator such that $D_\mu\Phi$ transforms covariantly. This derivative operator is given by:

$$D_\mu\Phi := \partial_\mu\Phi - ig_2\Phi B_\mu + ig_1 Y_\mu T^3\Phi = \begin{pmatrix} (\varepsilon D_\mu\chi)^T \\ (D_\mu\chi)^\dagger \end{pmatrix}, \tag{49}$$

where to obtain the last expression we have used $\epsilon B_\mu = B_\mu^T \epsilon^T$ which can be checked to hold for all 3 generators T^a.

We can now use the object Φ, as well as its covariant derivative (49) to rewrite the Higgs kinetic term $|D_\mu\phi|^2$ as

---

[1] We omit the $SU(3)$ part as it does not affect the discussion that follows.

$$|D_\mu \phi|^2 = (\partial_\mu \rho)^2 + \rho^2 |D_\mu \chi|^2 = (\partial_\mu \rho)^2 + \frac{\rho^2}{2} \operatorname{Tr} |D_\mu \Phi|^2, \tag{50}$$

where we have defined

$$|\phi|^2 = \rho^2. \tag{51}$$

Now let us recall the construction of SU(2)-invariant doublets. In (23) we have defined Q_i^{new} so that $Q_i = \Phi^\dagger Q_i^{new}$. We then rewrote the Lagrangian in terms of Q_i^{new} and the gauge-transformed SU(2) connection

$$W_\mu := \Phi B_\mu \Phi^\dagger + \frac{1}{ig_2} \Phi \partial_\mu \Phi^\dagger = \Phi B_\mu \Phi^\dagger - \frac{1}{ig_2} \partial_\mu \Phi \Phi^\dagger. \tag{52}$$

This connection is SU(2)-invariant. It however transforms under the U(1) transformations

$$W_\mu \mapsto e^{i\xi T_3} W_\mu e^{-i\xi T_3} - \frac{1}{g_2} (\partial_\mu \xi) T_3. \tag{53}$$

We can recognize in the SU(2)-invariant connection W_μ a part of the quantity $D_\mu \Phi$. Indeed, we have

$$i(D_\mu \Phi)\Phi^\dagger = g_2 W_\mu - g_1 Y_\mu T^3. \tag{54}$$

Therefore we have

$$\operatorname{Tr} |D_\mu \Phi|^2 = \frac{1}{2} \left(g_2^2 \left(W_\mu^1 W^{1\,\mu} + W_\mu^2 W^{2\,\mu} \right) + \left(g_2 W_\mu^3 - g_1 Y_\mu \right)^2 \right), \tag{55}$$

where we decomposed $W_\mu = W_\mu^a T^a$. These are the mass terms for the W, Z bosons obtained from the kinetic term for the Higgs. Usually one obtains them by choosing a VEV for the Higgs and breaking the symmetry. Here we instead defined SU(2) invariant connection W_μ, which appears in the covariant derivative acting on the SU(2) invariant doublets. The kinetic term for the Higgs then gives the mass terms for W_μ without any symmetry breaking.

We can now define the usual linear combinations

$$W_\mu^\pm := \frac{1}{\sqrt{2}} \left(W_\mu^1 \mp i W_\mu^2 \right), \qquad Z_\mu := \frac{g_2 W_\mu^3 - g_1 Y_\mu}{\sqrt{g_1^2 + g_2^2}}, \tag{56}$$

where the normalization is a convention chosen for later convenience. By construction these fields are invariant under SU(2) and transform under U(1) as

$$W_\mu^\pm \mapsto e^{\pm i\xi(x)} W_\mu^\pm, \qquad Z_\mu \mapsto Z_\mu. \tag{57}$$

These three gauge fields are identified with the physical SU(2) bosons which one can measure in an experiment. We can further define the Weinberg angle θ_W so that

$$Z_\mu := \cos(\theta_W) W_\mu^3 - \sin(\theta_W) Y_\mu. \tag{58}$$

From this equation one can deduce (we will further motivate this choice later on) that the photon gauge field will be given by:

$$A_\mu := \sin(\theta_W) W_\mu^3 + \cos(\theta_W) Y_\mu. \tag{59}$$

Notice that the field redefinition from W_μ^3, Y_μ to Z^μ, A_μ is an SO(2) transformation. All in all,

the Higgs sector Lagrangian can be rewritten in terms of physical quantities as follows:

$$\mathcal{L}_{Higgs} = -|D_\mu \phi|^2 - V\left(|\phi|^2\right)$$

$$= -\left(\partial_\mu \rho\right)^2 - \frac{(g_2\rho)^2}{2}\left(W^+ W^- + \frac{1}{2\cos^2(\theta_W)} Z_\mu Z^\mu\right) - V(\rho^2). \tag{60}$$

It is worth emphasizing once again that to extract the mass terms for the W, Z bosons no symmetry breaking was needed. The Higgs sector was merely reformulated in terms of the physical SU(2)-invariant degrees of freedom of the theory. More details on this reparametrization of the Higgs field can be found in [17].

5.2. Yang–Mills sector

We now perform the same change of variables in the Yang–Mills sector. One starts with the following Lagrangian:

$$\mathcal{L}_{YM} = -\frac{1}{8}\text{Tr}\left(B_{\mu\nu}B^{\mu\nu}\right) - \frac{1}{4}Y_{\mu\nu}Y^{\mu\nu}, \tag{61}$$

where the curvature tensors are defined according to (95). Since the field redefinition (52) is a gauge transformation, we can immediately write

$$\mathcal{L}_{YM} = -\frac{1}{8}\text{Tr}\left(W_{\mu\nu}W^{\mu\nu}\right) - \frac{1}{4}Y_{\mu\nu}Y^{\mu\nu}. \tag{62}$$

It is now convenient to define the following curvature combinations:

$$W^\pm_{\mu\nu} := \frac{1}{\sqrt{2}}\left(W^1_{\mu\nu} \mp i W^2_{\mu\nu}\right) \equiv D_\mu W^\pm_\nu - D_\nu W^\pm_\mu, \tag{63}$$

where the covariant derivatives are

$$D_\mu W^\pm_\nu \equiv \left(\partial_\mu \pm i g_2 W^3_\mu\right) W^\pm_\nu. \tag{64}$$

We then have:

$$\text{Tr}\left(W_{\mu\nu}W^{\mu\nu}\right) = 2W^3_{\mu\nu}W^{3\,\mu\nu} + 4W^+_{\mu\nu}W^{-\,\mu\nu}. \tag{65}$$

Recall now that the W^3, Y connections can be expressed in terms of the physical Z, A connections as

$$\begin{pmatrix} W^3 \\ Y \end{pmatrix} = \begin{pmatrix} \cos(\theta_W) & \sin(\theta_W) \\ -\sin(\theta_W) & \cos(\theta_W) \end{pmatrix} \begin{pmatrix} Z \\ A \end{pmatrix}. \tag{66}$$

Hence,

$$D_\mu W^\pm_\nu \equiv \left(\partial_\mu \pm i e A_\mu \pm i g_2 \cos(\theta_W) Z_\mu\right) W^\pm_\nu, \tag{67}$$

where the electric charge e is given by

$$e := g_2 \sin(\theta_W) = g_1 \cos(\theta_W). \tag{68}$$

Another expression that we need is

$$W^3_{\mu\nu} = \cos(\theta_W)Z_{\mu\nu} + \sin(\theta_W)F_{\mu\nu} + i g_2\left(W^+_\mu W^-_\nu - W^-_\mu W^+_\nu\right), \tag{69}$$

where $F_{\mu\nu}$ and $Z_{\mu\nu}$ are the curvature tensors of the photon and Z boson. This gives the following final expression for the Yang–Mills sector Lagrangian:

$$
\begin{aligned}
\mathcal{L}_{YM} = &-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} Z_{\mu\nu} Z^{\mu\nu} - \frac{1}{2} W_{\mu\nu}^+ W^{-\mu\nu} \\
&+ \frac{g_2^2}{2} W_\mu^+ W_\nu^- \left(W^{+\mu} W^{-\nu} - W^{-\mu} W^{+\nu} \right) \\
&- ie \left(F_{\mu\nu} + \cot(\theta_W) Z_{\mu\nu} \right) W^{+\mu} W^{-\nu}.
\end{aligned}
\tag{70}
$$

6. Interactions

In the previous section we have discussed what the Higgs and Yang–Mills sector Lagrangians become when written in terms of the SU(2) invariant variables. We have seen that the components $W_\mu^\pm$, Z_μ of the SU(2) invariant connection W_μ are massive fields, with the mass determined by the VEV of the scalar field ρ. We have also expressed the gauge sector Lagrangian in terms of the physical fields $W_\mu^\pm$, Z_μ, A_μ. We can now put all this to use and discuss how physical SU(2) invariant gauge fields interact with the physical SU(2) invariant fermions. The interaction vertices are different in the second order formulation. We only consider the quark sector, as leptons are analogous.

6.1. Weak interactions

We first consider Higgsless interactions arising when the Higgs is taken to sit on its VEV $\rho = v$. We take the Lagrangian in the form (27), where the Higgs field is absorbed into the fermionic fields. Since we assume here that Higgs is a constant, the covariant derivative $\mathcal{D}$ contains just the weak and electromagnetic connections. We have:

$$
\mathcal{D}_\mu Q_i = \left(\partial_\mu + ig_2 W_\mu + \frac{ig_1}{6} Y_\mu \right) Q_i, \quad \mathcal{D}_\mu \bar{Q}_i = \left(\partial_\mu + ig_1 \mathcal{Q} Y_\mu \right) \bar{Q}_i,
\tag{71}
$$

where $\mathcal{Q}$ is the matrix of electric charges, which is in this case

$$
\mathcal{Q} \bar{Q}_i \equiv \begin{pmatrix} -2/3 & 0 \\ 0 & 1/3 \end{pmatrix} \begin{pmatrix} \bar{u}_i \\ \bar{d}_i \end{pmatrix}.
\tag{72}
$$

Note that the gauge fields appearing here are the gauge-invariant ones, i.e. the ones in which the pure gauge part of the Higgs field was absorbed.

Another remark is that the kinetic term contains the product $\mathcal{D}^\mu \bar{Q}^i \mathcal{D}_\mu Q_i$, where the covariant derivatives acting on $\bar{Q}_i$, Q_i contain a different set of gauge fields. Thus, the usage of the same symbol $\mathcal{D}_\mu$ to denote the covariant derivative is only justified if one always keeps in mind the representation of the gauge group under which the corresponding fermionic field transforms. It is clear that $\bar{Q}_i$, Q_i transform under two different representations. Moreover, the application of $\mathcal{D}_\mu$ to e.g. Q_i maps this field into a different representation, namely the complex conjugate representation to the one describing $\bar{Q}_i$, as is clear from e.g. the reality conditions (28). Keeping this in mind one can integrate in terms such as $\mathcal{D}^\mu \bar{Q}^i \mathcal{D}_\mu Q_i$ by parts without any inconsistencies.

We now rewrite everything in terms of the physical gauge-invariant fields. We have:

$$
ig_2 W_\mu + \frac{ig_1}{6} Y_\mu = \frac{ig_2}{\sqrt{2}} \begin{pmatrix} 0 & W_\mu^+ \\ W_\mu^- & 0 \end{pmatrix} + ie\mathcal{Q} A_\mu + \frac{ie}{s_W c_W} Z_\mu \left(\frac{1}{2} T^3 - s_W^2 \mathcal{Q} \right),
\tag{73}
$$

where, as before, $e := g_1 \cos(\theta_W)$ and $s_W \equiv \sin(\theta_W)$, $c_W \equiv \cos(\theta_W)$. For the barred quarks we have:

$$i g_1 Q Y_\mu = i e Q A_\mu - i e Q t_W Z_\mu, \tag{74}$$

where $t_W \equiv \tan(\theta_W)$. Note that the electric charges of the unbarred quarks are opposite of those of the barred ones. Thus, the quark fields interact with the electromagnetic field in the usual way.

Let us now consider the interactions with the W-bosons. The relevant part of the Lagrangian (27) becomes

$$-i\sqrt{2} g_2 \left(K^{ij} (\partial^A{}_{A'} \bar{u}_{iA}) W^{+A'B} d_{jB} + (K^\dagger)^{ij} (\partial^A{}_{A'} \bar{d}_{iA}) W^{-A'B} u_{jB} \right), \tag{75}$$

where we reinstated the suppressed before spinor indices for clarity. The unitary matrix K^{ij} is the Cabibbo–Kobayashi–Maskawa matrix that enters when the mass terms are diagonalized. Terms in (75) give the interactions responsible for the β-decay, as well as for the mixing between the generations, thus making the more massive generations unstable. What is interesting is that in the second order formalism there is a derivative present in the interaction vertex. Of course, this can be seen to be the standard vertex with no derivative present if one uses the "reality condition" (28) to express the derivative of the barred spinors in terms of the Hermitian conjugates of the unbarred. However, there is no need to introduce the primed spinors, and one can work with the Feynman rules that follow directly from (75).

The second order formalism also introduces new vertices quadratic in the gauge field. Indeed, we see that such vertices are present for both A and Z fields (but not for $W^\pm$). However, because the second order Lagrangian was obtained by a procedure of integrating out the primed 2-component spinors one is certain that the end results for the correlation functions of the unprimed fields are correctly reproduced.

6.2. Interactions with the Higgs

A much more interesting story is with the Higgs field interactions, which are rather unusual in the second order formalism. Thus, we now consider fluctuations around the Higgs VEV $\rho = v + h(x)$.

Polynomial interactions with the bosons are the usual ones and can be read off from (60). The self-interactions of the Higgs are also as usual. Here we are specifically concerned with the fermionic sector that exhibits non-polynomiality in the Higgs. Let us again consider only the quark sector; for leptons everything is analogous.

We recall that in the form of the Lagrangian (27) the covariant derivative was defined so that:

$$\frac{1}{\sqrt{\rho}} D \ \leftrightarrow \ \mathcal{D} \frac{1}{\sqrt{\rho}}. \tag{76}$$

Therefore:

$$\mathcal{D} = D + \frac{1}{2} \partial \ln \rho. \tag{77}$$

This logarithmic non-polynomiality suggests that we should parametrize the Higgs field in a different way:

$$\rho \equiv v e^{\phi(x)}. \tag{78}$$

This shifts the non-polynomiality from the covariant derivative to the mass terms, schematically

$$\rho^2(\Lambda \bar{Q})Q \; \rightarrow \; m^2 e^{2\phi(x)} \bar{Q} Q, \tag{79}$$

where m is the quark mass. At the same time, the covariant derivative is now simple:

$$\mathcal{D}Q \equiv \left(D + \frac{1}{2}\partial\phi(x) \right) Q. \tag{80}$$

The exponential non-polynomiality also enters into the Higgs with gauge fields interaction vertices, as well as in the kinetic term for the Higgs that now becomes:

$$(\partial_\mu \rho)^2 = v^2 (\partial_\mu \phi)^2 e^{2\phi}. \tag{81}$$

We do not know which of these equivalent non-polynomial forms is better suited for practical calculations. At the same time, in practice one is only interested in terms involving just a few external Higgs lines. For calculations of this type it should matter little which parameterization of the Higgs field is used. We will further return to the non-polynomiality issues in the last section.

7. Unification

In this section we discuss the unification in the second order formalism. The new feature here is that the fermions are now Higgs-dressed, and therefore invariant with respect to the action of the weak SU(2). The 16 fermions of one SM generation belong to two groups, transforming in complex conjugate representations of the unbroken gauge group U(1) × SU(3). Thus, overall, all 16 fermions form a *real* representation of U(1) × SU(3). So, the GUT models can be found by searching for GUT groups containing U(1) × SU(3), so that some appropriate *real* representation transforms correctly with respect to the embedded U(1) × SU(3). The fact that one can now use real representations allows for more possibilities as compared to the usual story, where one looks for an embedding of U(1) × SU(2) × SU(3) and a *complex* representation.

7.1. SU(2) × SU(4) *unification*

The simplest possibility is a Pati–Salam-like unification. Consider the second order Lagrangian in the form (1). Here the quarks' kinetic term contains a sum over the 3 color indices. Spelling this out we have the following kinetic term:

$$-\frac{2}{\rho}(D\bar{Q}^i)^r (DQ_i)^r - \frac{2}{\rho}(D\bar{Q}^i)^g (DQ_i)^g - \frac{2}{\rho}(D\bar{Q}^i)^b (DQ_i)^b - \frac{2}{\rho} D\bar{L}^i DL_i, \tag{82}$$

where r, g, b are the 3 colors and $(DQ_i)^*$ denotes the projection of a triplet on a particular color index. This form of the kinetic term cries for the interpretation of leptons as the fourth color of quarks. This suggests that we put all of the SM fermions into two multiplets:

$$S_i := \begin{pmatrix} u_i^r & u_i^g & u_i^b & v_i \\ d_i^r & d_i^g & d_i^b & e_i \end{pmatrix}, \qquad \bar{S}_i := \begin{pmatrix} \bar{u}_i^r & \bar{d}_i^r \\ \bar{u}_i^g & \bar{d}_i^g \\ \bar{u}_i^b & \bar{d}_i^b \\ \bar{v}_i & \bar{e}_i \end{pmatrix}. \tag{83}$$

We can then rewrite the Lagrangian in terms of S_i, $\bar{S}_i$, using appropriate covariant derivatives for each field. However, what seems to spoil this picture is the different electric charges of the quarks and leptons. The same problem arises in the usual Pati–Salam treatment, where it is solved by

using a non-trivial embedding of the SM gauge group into a larger group, see e.g. [23] for a nice exposition.

Thus, to understand what is happening with the electric charges, we need to understand how the SM symmetry group sits inside some larger gauge group. Note that the symmetry group that is unbroken in the Lagrangian (1) is U(1) × SU(3). The weak SU(2) no longer acts on our fermions, as they are all SU(2)-invariant objects. However, there is a leftover from this gauge group in the form of the massive gauge field (also SU(2)-invariant) that acts on doublets Q_i, L_i and does not act on $\bar{Q}_i$, $\bar{L}_i$. So, the group SU(2) is broken, but the fact that the fermions come as doublets tells us that it was there. Similarly, now that we put in (83) leptons on the same footing as the quarks, it appears that there is an SU(4) behind this construction. So, we take SU(2) × SU(4) as the GUT gauge group that acts on multiplets S_i, $\bar{S}_i$, and look for an embedding of U(1) × SU(3) into it. The sought embedding is given by

$$\text{U}(1) \times \text{SU}(3) \ni \{\alpha, h\} \rightarrow \left\{ \begin{pmatrix} \alpha^3 & 0 \\ 0 & \alpha^{-3} \end{pmatrix}, \begin{pmatrix} \alpha h & 0 \\ 0 & \alpha^{-3} \end{pmatrix} \right\} \in \text{SU}(2) \times \text{SU}(4). \tag{84}$$

Let us check how this works out for the charges. According to this prescription the U(1) acts on the up quarks as $u_i \rightarrow \alpha^{1+3} u_i$, which corresponds to the correct electric charge of $4/6 = 2/3$. Similarly, for the down quarks we have $d_i \rightarrow \alpha^{1-3} d_i$, which gives the correct electric charge of $-2/6 = -1/3$. For the neutrino we have $\nu_i \rightarrow \alpha^{3-3} \nu_i$, which gives zero electric charge, and for the electrons $e_i \rightarrow \alpha^{-3-3} e_i$, which gives the electric charge -1. This gives all the correct quantum numbers of the unbarred fermions. For the barred ones it is clear that we simply have to use the Hermitian conjugate representation of SU(2) × SU(4) (but not of the Lorentz group, because the barred fermions are still unprimed 2-component spinors).

We can now write the kinetic terms for all the fermions in a very compact form

$$-\frac{2}{\rho} D \bar{S}^i D S_i. \tag{85}$$

Here D is the relevant covariant derivative for each multiplet. The SU(3) and U(1) connections are present in both DS_i and $D\bar{S}^i$ in a symmetric way, with the Hermitian conjugate connections appearing in $D\bar{S}^i$. However, the massive SU(2) gauge field appears asymmetrically in that $D\bar{S}^i$ is diagonal in the isospin indices, while DS_i is not. As we have already seen in the previous section, only DS_i contains the $W^{\pm}$ gauge fields, while both $D\bar{S}^i$ and DS_i contain the Z field, albeit in an asymmetric way. This different coupling of the massive gauge fields to $\bar{S}_i$, S_i is what causes the theory to be left-right asymmetric.

Let us now discuss the mass terms. These can again be written in terms of $\bar{S}_i$, S_i as

$$-\rho \, \bar{S}_i \Lambda^{ij} S_j. \tag{86}$$

The mass matrices Λ^{ij} appearing here are complicated objects. Each of them is an 8×8 block matrix that consists of 4 different entries Λ_u^{ij}, Λ_d^{ij}, Λ_ν^{ij}, Λ_e^{ij}. It thus breaks SU(2) symmetry completely, while the SU(4) is broken down to U(1) × SU(3).

Overall, the sum of two terms (85), (86) gives the Lagrangian (3) quoted in the Introduction. We note that the unification described here is different from the Pati–Salam model, as no second SU(2) has been used. This seems natural in the second order formalism in which the weak SU(2) has been frozen from the beginning by using the gauge invariant variables. There is then no sense in introducing an independent copy of SU(2) that would act on the right-handed doublets, as in our framework these doublets simply transform under the Hermitian conjugate representation of SU(2) × SU(4).

7.2. SO(9) unification

Let us note that $SU(2) \times SU(4) \sim SO(3) \times SO(6) \subset SO(9)$. The group SO(9) has a single real 16-dimensional fermionic representation. It is thus natural to expect that all our fermions can fit there. This is indeed the case. However, describing the embedding requires some Lie algebra machinery, and explaining this would take us too far from the main theme of this article. So, we refrain from doing it here.

7.3. Comparison with SO(10) unification

It is known that *all* left-handed fermions of the SM can be put into a single 16-dimensional *complex* irreducible representation of SO(10), see e.g. [24], Section 97. So, our unprimed fermions $\bar{S}^i$, S^i can be combined into a single multiplet of SO(10). This SO(10) is obtained from the already encountered $SU(2) \times SU(4)$ by adding another SU(2) that mixes $\bar{S}^i$ and S^i. Putting these groups together we have $SU(2) \times SU(2) \times SU(4) \sim SO(4) \times SO(6) \subset SO(10)$. The Lagrangian one obtains is of the same schematic form (3), now with a single 16-dimensional fermionic multiplet $\mathcal{F}^i$.

The main difference between the SO(10) unification and the SO(9) scheme that we described is that in our case a real representation is used instead of a complex one. This is possible because we do not have to embed also the weak SU(2) inside the GUT group, as this is frozen in the Lagrangian where all the fields are dressed by the Higgs field. The difference of dimensions $\dim(SO(10)) - \dim(SO(9)) = 45 - 36 = 9$, and so the SO(9) scheme introduces much fewer unobserved gauge field components. It may be interesting to explore all this in more details, but we leave this to future work.

8. Discussion

In this paper we have rewritten the SM Lagrangian in a chiral form. The new Lagrangian involves only unprimed 2-component spinors and was obtained from the usual first order Lagrangian by integrating out the primed spinors. It is second order in derivatives. In the process, we were led to combine the SU(2) singlets into doublets, as well as to define SU(2)-invariant combinations from the fermion doublets and the Higgs. Similarly, the angular part of the Higgs field was absorbed into the gauge fields to produce SU(2)-invariant massive gauge fields. The bosonic part of this story is not new and has been explored by several authors. It appears that such a treatment of the full fermionic sector of the Standard Model is new.

While the procedure of integrating out the primed spinors leads naturally to the introduction of the Higgs-dressed unprimed fermions, this dressing can also be done at the level of the first-order Lagrangian. So, if the reader is uncomfortable with the second-order formulation because of its non-manifest unitarity, there is still a lesson to be learned from this work: the procedure of dressing up the fermion doublets with the Higgs field decreases the complexity of the Lagrangian, and suggests new insights about the unification. We further comment on the unification aspects below.

The obtained second-order Lagrangian is quite compact, see (1) and can be used as the starting point for concrete computations of SM scattering amplitudes. As we already emphasized in the Introduction, it is likely to be worth converting the arising computation rules into a computer code, as the second order formalism leads to faster performance than the one based on the first order Lagrangian.

The second-order formulation is not manifestly unitary. However, perturbative unitarity can still be established order by order in perturbation theory. This will be spelled out elsewhere. But the fact that the theory is still unitary should not come as surprise: we have obtained the second-order formulation by doing legitimate at the level of the path integral manipulation with the Lagrangian of the theory. It is thus guaranteed that the same perturbation theory results, and the unitarity still holds. It is of course an interesting and non-trivial exercise to verify this directly, but this is not the subject of the present work.

An interesting aspect of the new Lagrangian (1) is that the Higgs field $\rho^2 = |\phi|^2$ enters it non-polynomially. The analogy between the Higgs field (ρ in our notation) and the conformal factor of the metric was already emphasized in [17]. Our rewrite of the SM Lagrangian strengthens this interpretation. Indeed, consider a conformal transformation $g_{\mu\nu} \to \rho^2 g_{\mu\nu}$. Under such a conformal rescaling the spinor metric ϵ_{AB} transforms as $\epsilon_{AB} \to \rho\epsilon_{AB}, \epsilon^{AB} \to \rho^{-1}\epsilon^{AB}$. The Dirac operator changes as $\nabla_{AA'}\chi_B \to \nabla_{AA'}\chi_B - (\partial_{BA'} \log\rho)\chi_A$, see e.g. [25], formula (5.6.15). Then, if we define the transformation rule for the spinors to be $S_A \to \rho^{-1}S_A$, similarly for $\bar{S}_A$, the quantity $\epsilon^{AB}D_{AA'}S_B$ transforms homogeneously $\epsilon^{AB}D_{AA'}S_B \to \rho^{-2}\epsilon^{AB}D_{AA'}S_B$, with the covariant derivative remaining unchanged. This implies that under such a transformation

$$\sqrt{g}\,D\bar{S}DS \to \frac{1}{\rho}\sqrt{g}\,D\bar{S}DS, \tag{87}$$

where we have taken into account that $\sqrt{g} \to \rho^4\sqrt{g}$, and there is an extra factor of $1/\rho$ coming from the contraction of the primed spinor indices. Similarly,

$$\sqrt{g}\,\bar{S}S \to \rho\sqrt{g}\,\bar{S}S. \tag{88}$$

Thus, we see that, as already observed in [17] for the bosonic sector, the Higgs field ρ enters the fermionic Lagrangian (1) as the conformal factor of a transformation $g_{\mu\nu} \to \rho^2 g_{\mu\nu}$.

Finally, we believe that one of the most interesting outcomes of this work is the realization that the procedure of dressing the fermion doublets with the Higgs field leads to so many simplifications in the structure of the Lagrangian. As we already said, this can already be done at the level of the first order Dirac Lagrangian. This procedure exactly parallels what is known to be possible in the bosonic sector, and clearly exhibits the structure of the interactions of the fermions with the massive gauge bosons. We are not aware of any previous work containing such treatment of the SM fermions.

The introduction of the Higgs-dressed fermions freezes the week SU(2), and leaves just the group U(1) × SU(3) as the unbroken SM gauge group. This suggests that one should seek to embed just this gauge group inside the GUT group. Moreover, the fermions now form a real representation of U(1) × SU(3), which suggests that real representations of GUT can be considered. As an example we have briefly described the Pati–Salam-like SU(2) × SU(4) theory, as well as a more advanced SO(9) GUT that puts all one generation fermions into a single spinor representation. It would be interesting to explore this GUT model in more details.

Acknowledgements

Appendix A. Curvature and covariant derivative conventions

In this paper we dealing exclusively with unitary groups so that the inverse of a group object is its Hermitian conjugate. Let $A_\mu := A_\mu^s T^s$ be a connection gauge field. T^s ($s = 1, \ldots, \dim(G)$) are the generators of the Lie group, which we take to be Hermitian, that satisfy:

$$[T^s, T^r] = i f^{srt} T^t \tag{89}$$

A vector ϕ in the fundamental representation transforms as:

$$\phi \mapsto \phi^\Omega \equiv \Omega \phi, \tag{90}$$

where

$$\Omega \equiv \exp\left(i q \xi^s(x) T^s\right), \quad \xi^s(x) \in \mathbb{R} \tag{91}$$

and q stands for the charge of the field while $\xi^s(x)$ are coordinates that parametrize the transformation.

The covariant derivative is constructed as follows

$$D_\mu \phi := \left(\partial_\mu + i g q A_\mu\right) \phi. \tag{92}$$

If we require that this transforms covariantly under the gauge transformations

$$D_\mu \phi \mapsto \Omega D_\mu \phi \tag{93}$$

we deduce the transformation rule for the connection:

$$A_\mu \mapsto A_\mu^\Omega \equiv \Omega^\dagger A_\mu \Omega + \frac{1}{igq} \Omega^\dagger \left(\partial_\mu \Omega\right), \quad \Omega \in G \tag{94}$$

where g denotes the coupling constant of the group. The field strength tensor or Yang–Mills curvature tensor is defined as:

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + i g q \left[A_\mu, A_\nu\right] \tag{95}$$

It transforms in the adjoint representation of the Lie group:

$$F_{\mu\nu} \mapsto F_{\mu\nu}^\Omega \equiv \Omega^\dagger F_{\mu\nu} \Omega. \tag{96}$$

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