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SYNTHESIZING JUDGEMENTS: A FUNCTIONAL EQUATIONS APPROACH

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Abstract—We discuss several conditions which are reasonable requirements for functions synthesizing either ratio or measure judgements (or both) and determine *all* synthesizing functions satisfying either shorter or longer lists of such assumptions (yielding more general or more specific synthesizing procedures, respectively).

1. INTRODUCTION

Synthesizing judgements is often an important part of the Analytic Hierarchy Process (AHP) (and of other disciplines, e.g. statistical or economic measurement). The typical situation is the following. Several, say n, individuals form quantifiable judgements either about a *measure* of an object (weight, length, area, height, volume, importance or other attributes, for instance in the framework of a hierarchy) or about a *ratio* of two such measures (how much heavier, longer, larger, taller, more important, preferable, more meritorious etc. one object is than another). After all information is taken into account and all efforts at changing each others opinions are exhausted, either a consensus has been reached or there are still different judgements which have to be *synthesized*, either by a further systematic procedure bringing consensus among the individuals or by an external decision maker who consults the group or by a combination of both. This is the subject with which we deal here. Saaty [1], for instance, has used the simple geometric mean as a synthesizing function and asked for a *motivation* of this usage, i.e. for natural, reasonable assumptions which *determine* this synthesizing function or a *family* of functions which contains the geometric mean. The research about which we report here has this objective. We do not intend to include all mathematical details but we give reference for those which we omit.

2. SEPARABILITY AND UNANIMITY CONDITIONS

It makes sense here to suppose that the numerical judgements $x_1, x_2, \ldots, x_n$ given by n individuals lie in a continuum (interval) P of positive numbers so that P may contain $x_1, x_2, \ldots, x_n$ as well as their powers, or reciprocals, and their geometric mean etc. The *synthesizing function* f will map P^n into a proper interval J and $f(x_1, x_2, \ldots, x_n)$ will be called the *result of the synthesis* for the judgements $x_1, x_2, \ldots, x_n$. Our main problem is to study which conditions must be satisfied by f in order to obtain reasonable results.

Saaty suggested [2], as the first assumption to be considered, the *separability condition*:

$$f(x_1, x_2, \ldots, x_n) = g(x_1) \circ g(x_2) \circ \cdots \circ g(x_n), \qquad \forall\, x_1, \ldots, x_n \text{ in } P. \tag{S}$$

Here g is a function which maps P onto a proper interval J (this is satisfied, for instance, if g is continuous and non-constant) while $\circ$ is a continuous *associative* and *cancellative* operation mapping $J \times J$ into J, i.e.

$$(x \circ y) \circ z = x \circ (y \circ z), \qquad \forall\, x, y, z \text{ in } J, \tag{A}$$

and

$$\text{if } x_1 \circ y = x_2 \circ y, \text{ for any } y, \qquad \text{then } x_1 = x_2$$

and

$$\text{if } x \circ y_1 = x \circ y_2, \text{ for any } x, \qquad \text{then } y_1 = y_2. \tag{C}$$

It is known [3] that these conditions are already enough to guarantee that there exists a continuous and strictly monotonic function ψ, such that

$$u \circ v = \psi^{-1}(\psi(u) + \psi(v)), \qquad \forall\, u, v \text{ in } J, \tag{1}$$

where ψ^{-1} denotes the inverse function of ψ. Putting equation (1) into condition (S), we get

$$f(x_1,\ldots,x_n) = \psi^{-1}(\psi[g(x_1)] + \cdots + \psi[g(x_n)]), \qquad x_1,\ldots,x_n \in P. \tag{2}$$

A further natural requirement is the *unanimity condition*:

$$f(x,\ldots,x) = x, \qquad \forall\, x \text{ in } P, \tag{U}$$

i.e. if all individuals give the same judgement x, that should also be the synthesized judgement.

Thus, if conditions (S) and (U) hold, we have by equation (2) and condition (U):

$$x = f(x,\ldots,x) = \psi^{-1}(n\psi[g(x)]), \qquad \forall\, x \text{ in } P,$$

so $\psi(x) = n\psi[g(x)]$ and equation (2) goes over into

$$f(x_1,\ldots,x_n) = \psi^{-1}\left(\frac{\psi(x_1) + \cdots + \psi(x_n)}{n}\right), \qquad x_1,\ldots,x_n \in P. \tag{3}$$

So we have proved the following.

Lemma 1

Let P be an interval of real numbers. A synthesizing function $f: P^n \to P$ is separable (S) (with continuous non-constant g and continuous, cancellative and associative $\circ$) and has the unanimity property (U) iff f is of the form (3) with an arbitrary continuous and strictly monotonic ψ, i.e. f is a quasi-arithmetic mean.

The family (3) of functions is important in many parts of mathematics and its applications: these are the *quasi-arithmetic means.*

3. RECIPROCAL PROPERTY

In many situations, in particular concerning *ratio judgements*, it is reasonable to assume, in addition to conditions (S) and (U), the following *reciprocal property*:

$$f\left(\frac{1}{x_1},\ldots,\frac{1}{x_n}\right) = 1/f(x_1,\ldots,x_n), \qquad x_1,\ldots,x_n \in P. \tag{R}$$

Indeed, let A and B be the two objects about which the ratio judgements are made (for instance, how much heavier A is than B). If we interchange A and B, then reasonably the judgements change into their reciprocals (if A is judged to be twice as heavy as B, then B should be judged half as heavy as A). The assumption (R) is that in this case also the synthesized judgement turns into its reciprocal.

In order for assumption (R) to make sense, *the interval P should contain with every element x its reciprocal $1/x$ and, in particular, it has to contain 1* (because it is an interval and contains x and $1/x$).

Putting equation (3) into assumption (R), we get

$$\psi^{-1}\left(\frac{\psi(1/x_1)+\cdots+\psi(1/x_n)}{n}\right)=1/\psi^{-1}\left(\frac{\psi(x_1)+\cdots+\psi(x_n)}{n}\right),\qquad x_1,\ldots,x_n\in P. \tag{4}$$

We need here and at some other points in our deliberations a very useful auxiliary result: *if the continuous strictly monotonic* ψ *and* $\tilde{\psi}$ *map the same real interval* P *into (possibly different) real intervals* I *and* $\tilde{I}$, *so that*

$$\psi^{-1}\left(\frac{\psi(x_1)+\cdots+\psi(x_n)}{n}\right)=\tilde{\psi}^{-1}\left(\frac{\tilde{\psi}(x_1)+\cdots+\tilde{\psi}(x_n)}{n}\right),\qquad x_1,\ldots,x_n\in P, \tag{5}$$

then there exist constants a $\neq 0$ *and* b *such that*

$$\tilde{\psi}(x)=\mathrm{a}\psi(x)+\mathrm{b},\qquad x\in P. \tag{6}$$

(See, for example, Ref. [3].)

We see that equation (4) is the particular case $\tilde{\psi}(x)=\psi(1/x)$ of equation (5). So from equation (6) we get

$$\psi(1/x)=\mathrm{a}\psi(x)+\mathrm{b}. \tag{7}$$

Now we use a simple property of $1/x$, which turns out to be rather important in this case, that it is *involutory*: $1/(1/x)=x$. So we get, from equation (7),

$$\psi(x)=\psi\left(\frac{1}{1/x}\right)=\mathrm{a}\psi\left(\frac{1}{x}\right)+\mathrm{b}=\mathrm{a}^2\psi(x)+\mathrm{ab}+\mathrm{b}.$$

Since ψ is strictly monotonic, this is possible only if

$$\mathrm{a}^2=1\quad\text{and}\quad \mathrm{ab}+\mathrm{b}=0.$$

Therefore we have two cases. The first is a $=1$, b $=0$ so that equation (7) becomes $\psi(1/x)=\psi(x)$. This is not possible, since ψ is strictly monotonic. The second case is a $=-1$, b arbitrary. So equation (7) goes now over into

$$\psi(1/x)=-\psi(x)+\mathrm{b},\qquad x\in P. \tag{8}$$

We introduce a new function ω by

$$\omega(t)=\psi(e^t)-(b/2). \tag{9}$$

For this function ω, equation (8) becomes

$$\omega(-t)=\omega(t), \tag{10}$$

i.e. ω is an *odd* function. The domain of ψ was P; from equation (9) we see that the t's are the logarithms of the x's. The set of all $\log x$, where x is in P, is denoted by $\log P$. So equation (10) is now true $\forall\, t\in\log P$. If, as in our case, P contains with x also $1/x$, then $\log P$ has to contain with every t also $-t$, as it should be.

Going back with the aid of equation (9) to ψ, we have

$$\psi(x) = \omega(\log x) + \frac{b}{2}, \qquad x \in P.$$

Therefore,

$$\psi^{-1}(y) = \exp\left[\omega^{-1}\left(y - \frac{b}{2}\right)\right]$$

and, putting this into equation (3), we get

$$f(x_1,\dots,x_n) = \exp\left[\omega^{-1}\left(\frac{\omega(\log x_1) + \cdots + \omega(\log x_n)}{n}\right)\right], \qquad x_1,\dots,x_n \in P. \tag{11}$$

Conversely, every function (11), where ω is an arbitrary *odd* function on $\log P$, is separable (S) and has the unanimity (U) and reciprocal (R) properties.

We can formulate our results as follows [2].

Theorem 1

Let P be an interval of positive numbers which with every element contains also its reciprocal. A synthesizing function $f\colon P^n \to P$ is separable (S) (with continuous non-constant g and continuous, cancellative and associative $\circ$), has the unanimity (U) and the reciprocal (R) properties iff f is of the form (11), where ω is an arbitrary, continuous, strictly monotonic and odd function.

If, in particular, $\omega(x) = x$, then equation (11) reduces to

$$f(x_1,\dots,x_n) = \exp\left(\frac{\log x_1 + \cdots + \log x_n}{n}\right) = \sqrt[n]{x_1 x_2 \dots x_n},$$

the geometric mean.

4. HOMOGENEITY CONDITION

As we can see, our assumptions to date allow many more general synthesizing functions (11) than just the geometric mean. Another condition, which makes sense both for ratio and (even more) for measure judgements, restricts the permissible synthesizing functions considerably, even without assuming the reciprocal property (R), and with (R) gives exactly the geometric mean.

This is the *homogeneity* condition:

$$f(ux_1,\dots,ux_n) = uf(x_1,\dots,x_n)$$
$$\text{when } u > 0 \text{ and } x_k, ux_k \ (k = 1,\dots,n) \text{ are all in } P. \tag{H}$$

For measure judgements condition (H) simply means *invariance under change of scale* (of units). For ratio judgements there are no units of measure left. Here condition (H) means that, if all individuals judge a ratio u times as large as another ratio, then the synthesized judgement should also be u times as large.

Putting equation (3) into condition (H), we get

$$\psi^{-1}\left(\frac{\psi(ux_1) + \cdots + \psi(ux_n)}{n}\right) = u\psi^{-1}\left(\frac{\psi(x_1) + \cdots + \psi(x_n)}{n}\right), \qquad x_k, ux_k \in P; \quad k = 1,\dots,n. \tag{H'}$$

Holding u temporarily fixed, we obtain with

$$\tilde{\psi}(x) = \psi(ux) \qquad \left(\text{so } \tilde{\psi}^{-1}(x) = \frac{1}{u}\psi^{-1}(x)\right), \tag{12}$$

equation (5) exactly, so by our "auxiliary result" in Section 3 we have equation (6), i.e. $\tilde{\psi}(x) = \mathrm{a}\psi(x) + \mathrm{b}$. Taking equation (12) into account and letting u vary again, we have to recognize that now the "constants" a, b may also depend upon u and we thus obtain

$$\psi(ux) = \mathrm{a}(u)\psi(x) + \mathrm{b}(u), \qquad u > 0; \quad x, ux \in P. \tag{13}$$

This equation has been solved (see Ref. [4]). *There are just two kinds of strictly monotonic solutions of equation (13):*

$$\psi(x) = \alpha \log x + \beta \tag{14}$$

and

$$\psi(x) = \alpha x^{\gamma} + \beta, \tag{15}$$

where $\alpha \neq 0$, $\gamma \neq 0$ but otherwise α, β, γ are arbitrary real constants. The quasi-arithmetic means (3) corresponding to these ψ are exactly the *geometric mean*

$$f(x_1, \ldots, x_n) = \sqrt[n]{x_1 x_2 \ldots x_n} \tag{16}$$

and the *root-mean-power*

$$f(x_1, \ldots, x_n) = \sqrt[\gamma]{(x_1^{\gamma} + \cdots + x_n^{\gamma})/n}, \qquad \gamma \neq 0. \tag{17}$$

It is immediate that these are indeed homogeneous. But equation (17) satisfies assumption (R) only for $x_1 = \cdots = x_n$. So we have the following [2].

Theorem 2

The general homogeneous quasi-arithmetic means (3), i.e. the general separable (S) synthesizing functions satisfying the unanimity and homogeneity conditions (U) and (H) are the geometric mean (16) and the root-mean-power (17). If, moreover, the reciprocal property (R) is supposed even for a single n-tuple $(x_1, \ldots, x_n)$, where not all x_k are equal, then only the geometric mean satisfies all our conditions.

5. POWER CONDITIONS

For measure judgements the reciprocity condition is not a particularly natural assumption. But it is the particular case $p = -1$ of the *power condition*:

$$f(x_1^p, \ldots, x_n^p) = f(x_1, \ldots, x_n)^p. \tag{P_p}$$

For other p this may make good sense in synthesizing measure judgments. For instance, if the kth individual judges the length of a side of a square to be x_k, she or he will presumably judge the area of that square as x_k^2. It is reasonable to suppose that the same thing will hold for the synthesized judgement which is the requirement (P_2) (if the synthesizing function is the same for lengths and areas). A similar assumption about the edge and volume of a cube gives requirement (P_3). Since dimensional analysis teaches us (see, for example, Ref. [5]) that most laws of science consist of (products of) powers of quantities, (P_p) makes quite good sense.

In what follows we explore the consequences of the power condition (P_p) for one or more exponents p, with or without the homogeneity condition (H). The situation with condition (H) is quite analogous to Theorem 2 since equation (17) does not satisfy, for any $\gamma \neq 0, (x_1, \ldots, x_n) \neq (x, \ldots, x)$, the power condition ($P_p$), for any $p = -1, 0, 1$, either. [Note that condition (P_1) is an identity, condition (P_0) means only $f(1, \ldots, 1) = 1$, while we have already taken care of condition (P_{-1}) = assumption (R).]

Proposition 1

In (the last sentence of) Theorem 2, assumption (R) may be replaced by any condition (P_p) with $p \neq -1, 0, 1$.

Let us now see what condition (P_p) ($p \neq -1, 0, 1$) does to the quasi-arithmetic means (3) (i.e. to separable synthesizing functions with the unanimity condition) without condition (H). Putting equation (3) into condition (P_p), we get

$$\psi^{-1}\left(\frac{\psi(x_1^p) + \cdots + \psi(x_n^p)}{n}\right) = \left[\psi^{-1}\left(\frac{\psi(x_1) + \cdots + \psi(x_n)}{n}\right)\right]^p, \qquad x_1, \ldots, x_n \in P.$$

This is again of the form (5), this time with $\tilde{\psi}(x) = \psi(x^p)$, so we have

$$\psi(x^p) = \mathrm{a}\psi(x) + \mathrm{b}, \qquad \mathrm{a} = 0; \qquad p = -1, 0, 1. \tag{18}$$

Here (contrary to the case $p = -1$), a is hardly restricted at all. It has been proved by Aczél and Alsina [6] that equation (18) *always has continuous strictly monotonic solutions ψ except in the cases*

$$\mathrm{a} = 1 \text{ if } \mathrm{b} = 0; \qquad \mathrm{a} = 1 \text{ if } p < 0; \operatorname{sign} \mathrm{a} = -\operatorname{sign} p$$
$$(\operatorname{sign} t = 1 \text{ if } t > 0, \qquad \operatorname{sign} t = -1 \text{ if } t < 0, \qquad \operatorname{sign} 0 = 0)$$

and, if 1 is in P, then the following cases are exceptional too:

$$\mathrm{a} = -1; \qquad \mathrm{a} = 1 \text{ (no restriction on b or } p) \quad \text{and} \quad \operatorname{sign} \log|\mathrm{a}| = -\operatorname{sign} \log|p|.$$

In all other cases (18) has continuous strictly monotonic solutions which can be chosen arbitrarily, at least between e and e^p, as long as they are continuous, strictly monotonic and $\psi(e^p) = \mathrm{a}\psi(e) + \mathrm{b}$. *Then one extends the solution by repeated use of equation (18). With these we have the complete set of solutions of form (3) and condition (P_p) for a given* $p(\neq -1, 0, 1)$.

Now [6], we postulate the power property for *two* different exponents [as in our example with squares and cubes we had power conditions (P_2) and (P_3)]. Notice that

$$f(x_1^p, \ldots, x_n^p) = f(x_1, \ldots, x_n)^p, \qquad x_k \in P; \qquad k = 1, \ldots, n, \tag{P_p}$$

implies (with $y_k = x_k^p$)

$$f(y_1^{1/p}, \ldots, y_n^{1/p}) = f(y_1, \ldots, y_n)^{1/p} \tag{$P_{1/p}$}$$

and also [from conditions (P_p) and ($P_{1/p}$) by induction],

$$f(x_1^{p^k}, \ldots, x_n^{p^k}) = f(x_1, \ldots, x_n)^{p^k}, \qquad k = 0, \pm 1, \pm 2, \ldots . \tag{P_{p^k}}$$

The interval P has to be one of the following (since it should contain with every x also x^p and $x^{1/p}$):

$$(0,1), (0,1], [1,\infty), (1,\infty) \quad \text{or} \quad (-\infty,\infty) \qquad \text{for } p < 0 \text{ only } (-\infty,\infty). \tag{19}$$

But our main point is that condition (P_p) implies condition (P_{p^k}), $k = 0, \pm 1, \pm 2, \ldots$, therefore supposing, in addition to condition (P_p), condition (P_q) also, where $\log|p|/\log|q|$ is a rational number, does not add much novelty. (Note: we have excluded $p = 1, 0, -1$, but all other negative or positive p are permissible.) So we will suppose that

$$\frac{\log|p|}{\log|q|} \quad \text{is a (finite) irrational number} \tag{20}$$

and that conditions (P_p) and (P_q) hold.

By iteration of conditions (P_p) and (P_q), as above, and their combination we get

$$f(x_1^{p^{2k}q^{2l}}, \ldots, x_n^{p^{2k}q^{2l}}) = f(x_1, \ldots, x_n)^{p^{2k}q^{2l}}, \qquad k, l = 0, \pm 1, \pm 2, \ldots .$$

Now, it is known (see, for instance, Ref. [7]) that *the numbers* $p^{2k}q^{2l}$, $k, l = 0, \pm 1, \pm 2, \ldots$, *lie everywhere dense among the non-negative reals if*

$$\frac{\log(p^2)}{\log(q^2)} = \frac{\log|p|}{\log|q|} \quad \text{is irrational,}$$

which is true in our case [see assumption (20)]. Since f [for instance, in the form (3)] is continuous, *we have now* from conditions (P_p) and (P_q) and assumption (20) *that*

$$f(x_1^\lambda, \ldots, x_n^\lambda) = f(x_1, \ldots, x_n)^\lambda \tag{P_λ}$$

holds for all non-negative λ. We have seen that p or q may (but need not) be negative. If, for instance, $p < 0$ then condition (P_{-1}) = assumption (R) follows from conditions (P_p) and (P_q) if assumption (20) holds. Indeed then condition (P_λ) has followed $\forall\, \lambda > 0$. Choose $\lambda = -p > 0$:

$$\begin{aligned} f(x_1^{-1}, \ldots, x_n^{-1}) &= [f(x_1^{-1}, \ldots, x_n^{-1})^{-p}]^{-1/p} = f(x_1^p, \ldots, x_n^p)^{-1/p} \\ &= [f(x_1, \ldots, x_n)^p]^{-1/p} = f(x_1, \ldots, x_n)^{-1}. \end{aligned}$$

As a further consequence, *in this case (p or q negative), condition (P_λ) holds for all real* λ: take $\lambda < 0$. By conditions (P_λ) and (P_{-1}),

$$\begin{aligned} f(x_1^\lambda, \ldots, x_n^\lambda) &= f[(x_1^{-1})^{-\lambda}, \ldots, (x_n^{-1})^{-\lambda}] \\ &= f(x_1^{-1}, \ldots, x_n^{-1})^{-\lambda} = [f(x_1, \ldots, x_n)^{-1}]^{-\lambda} \\ &= f(x_1, \ldots, x_n)^\lambda. \end{aligned}$$

Notice that for our square- and cube-preserving properties (P_2) and (P_3), $\log 2/\log 3$ is irrational, so condition (P_λ) follows $\forall\, \lambda \geqslant 0$.

If we put condition (P_λ) (λ goes through all non-negative reals) into the quasi-arithmetic means (3), we get

$$\psi^{-1}\left(\frac{\psi(x_1^\lambda) + \cdots + \psi(x_n^\lambda)}{n}\right) = \left[\psi^{-1}\left(\frac{\psi(x_1) + \cdots + \psi(x_n)}{n}\right)\right]^\lambda, \qquad x_k, x_k^\lambda \in P; \quad k = 1, \ldots, n. \tag{21}$$

By introducing $x_k = \exp(t_k)$, $k = 1, \ldots, n$, and

$$\phi(t) = \psi(e^t) \qquad \text{so } \phi^{-1}(u) = \log \psi^{-1}(u) \tag{22}$$

we get, taking logarithms of equation (21)

$$\phi^{-1}\left(\frac{\phi(\lambda t_1)+\cdots+\phi(\lambda t_n)}{n}\right)=\lambda\phi^{-1}\left(\frac{\phi(t_1)+\cdots+\phi(t_n)}{n}\right),\qquad t_k,\lambda t_k\in\log P;\ k=1,\ldots,n, \tag{23}$$

which, apart from the notation, is exactly condition (H′). If we choose, among the intervals (19), $P=(1,\infty)$ then $\log P=(0,\infty)$ and even the domain of equation (23) will be a positive interval as in condition (H′). So, corresponding to equations (14) and (15), we get

$$\phi(t)=\alpha\log t+\beta,\quad \text{i.e. } \psi(x)=\alpha\log(\log x)+\beta$$
$$\phi(t)=\alpha t^{\gamma}+\beta,\quad \text{i.e. } \psi(x)=\alpha(\log x)^{\gamma}+\beta.$$

Accordingly, *for $P=(1,\infty)$ the only quasi-arithmetic means [i.e. separable synthesizing functions with the unanimity condition (U)], satisfying condition (P_λ) for all non-negative λ [or just conditions (P_p) and (P_q) with $p>0$, $q>0$, $\log p/\log q$ irrational], are*

$$f(x_1,\ldots,x_n)=\exp(\sqrt[n]{\log x_1\cdot\log x_2\cdot\ldots\cdot\log x_n}) \tag{24}$$

and

$$f(x_1,\ldots,x_n)=\exp\{\sqrt[\gamma]{[(\log x_1)^{\gamma}+\cdots+(\log x_n)^{\gamma}]/n}\},\qquad \gamma\neq 0. \tag{25}$$

Going over to others among the intervals (19) for P, already for $P=[1,\infty)$, since f, as given by equation (3), is continuous and strictly increasing, *the solution (24) will be eliminated and the solution (25) restricted to $\gamma>0$.* Indeed, for equation (24), $f(1,x_2,\ldots,x_n)=\exp(\sqrt[n]{0\cdot\log x_2\ldots\log x_n}=1)$ is not strictly increasing, say in x_2, and for equation (25), with $\gamma<0$, $f(1,x_2,\ldots,x_n)$ does not exist and $\lim_{x_1\to 1}f(x_1,x_2,\ldots,x_n)=1$ is again not strictly monotonic, say in x_2.

Similarly (see Refs [4,6]), *for $P=(0,1$ the only separable synthesizing functions with the properties (U), (P_p) and (P_q) ($p>0$, $q>0$, $\log p/\log q$ irrational) are*

$$f(x_1,\ldots,x_n)=\exp[-\sqrt[n]{(-\log x_1)\cdot(-\log x_2)\cdot\ldots\cdot(-\log x_n)}] \tag{26}$$

and

$$f(x_1,\ldots,x_n)=\exp\{-\sqrt[\gamma]{[(-\log x_1)^{\gamma}+\cdots+(-\log x_n)^{\gamma}]/n}\},\qquad \gamma\neq 0, \tag{27}$$

$x_1,\ldots,x_n\in P$, while *for $P=(0,1]$, the solution (26) is eliminated and the solution (27) is restricted to $\gamma>0$.*

Finally, *for $P=(0,\infty)$*, we have to fit equation (25) to equation (27) (with $\gamma>0$) and *we get equation (3) with*

$$\psi(x)=\begin{cases}A(\log x)^{\gamma}+C, & x>1,\\ C, & x=1,\\ B(-\log x)^{\gamma}, & x<1,\end{cases} \tag{28}$$

where $AB<0$, $\gamma\neq 0$ *but otherwise* A, B, C, γ *are arbitrary constants* [4,6].

Now, *if $p<0$ or $q<0$, or both*, then condition (P_λ) holds for *all real* λ, in particular for $\lambda=-1$ which is assumption (R), so $\psi(e^t)-C$ is odd, $B=-A$ and *we have*

$$\psi(x)=A\,|\log x|^{\gamma}\operatorname{sign}(\log x)+C,\qquad A\neq 0,\quad \gamma\neq 0.$$

Even in this very special case, only if $\gamma = 1$ do we retrieve $\psi(x) = Ax + C$ and the geometric mean from equation (3):

$$f(x_1, \ldots, x_n) = \sqrt[n]{x_1 x_2, \ldots, x_n}.$$

6. WEIGHTED SYNTHESIS AND FURTHER TOPICS

R. Sarin has asked in connection with Aczél and Saaty [2], how the results change if in condition (S) not all g's are supposed to be the same function (weighted synthesis). So, in place of condition (S) we will now suppose the weighted separability property

$$f(x_1, x_2, \ldots, x_n) = g_1(x_1) \circ g_2(x_2) \circ \cdots \circ g_n(x_n), \qquad x_1, \ldots, x_n \in P$$

(not all judging individuals have the same weight when the judgements are synthesized and these different influences are reflected in the different functions $g_1, g_2, \ldots, g_n$). This problem was solved in Ref. [4]. Many of the proofs and results are similar; for instance the geometric mean (16) will be replaced by the weighted geometric mean $f(x_1, \ldots, x_n) = x_1^{q_1} \cdot x_2^{q_2} \cdot \ldots \cdot x_n^{q_n}$ $(q_1 + \cdots + q_n = 1)$.

Another generalization, where the judgements are represented not by numbers but by probability distributions, has been given in Ref. [8].

There are many other variations to the theme. For instance if the individual judges the measures (length, area or weight etc.) of two objects and also their ratios, then logically the latter should be the quotient of the two former measures. We can require that the same should happen with the synthesized judgements of the measures and of the ratios, thus connecting the synthesis of measures with that of ratios.

This time we allow for different synthesizing functions. Let the two sets of measures be synthesized by f and g, and the ratios by Q. Then our "quotient requirement" is

$$Q\left(\frac{y_1}{x_1}, \ldots, \frac{y_n}{x_n}\right) = \frac{g(y_1, \ldots, y_n)}{f(x_1, \ldots, x_n)}. \tag{Q}$$

Without supposing any separability or other property, just that one of f, g and Q be somewhat regular, for instance strictly monotonic at one point in each variable, we will have as general solutions of requirement (Q) $f(x_1, \ldots, x_n) = a x_1^{q_1} x_2^{q_2} \ldots x_n^{q_n}$, $g(y_1, \ldots, y_n) = b y_1^{q_1} y_2^{q_2} \ldots y_n^{q_n}$, $Q(r_1, \ldots, r_n) = \frac{b}{a} r_1^{q_1} r_2^{q_2} \ldots r_n^{q_n}$, *where* $a, b, q_1, \ldots, q_n$ *are arbitrary non-zero constants (see Refs [3, 5]). If*, quite naturally, *f should be positive, then* $a > 0$ *and if the unanimity condition (U) is to hold for f then* $a = 1$, $q_1 + \cdots + q_n = 1$, thus leading this time to *an exclusive characterization of the weighted geometric mean.*

It is interesting to note that the same equation (Q) comes up in a seemingly different situation. Let x_k be the estimation of the kth individual for the present price level, y_k for the price level in a year from now and r_k his/her estimation for the inflation rate. Let f and g be the synthesizing function of price judgements in consecutive years and Q that of inflation rate judgements. Again it is reasonable to have $y_k = r_k x_k$, $k = 1, \ldots, n$, and to suppose

$$g(r_1 x_1, \ldots, r_n x_n) = Q(r_1, \ldots, r_n) f(x_1, \ldots, x_n). \tag{Q}$$

We denote this too by equation (Q) because *it is exactly the above equation* (with $r_k = y_k/x_k$, $k = 1, \ldots, n$). *The solutions are also the same.* This is also the fundamental equation of dimensional analysis (see, for instance, Ref. [5]).

With respect to the preservation of more complicated relations than in equations (R), (P_p) or (Q), compare, for instance, Aczél and Saaty [2] (where x^p is replaced by an arbitrary function) and Aczél *et al.* [9] (where the linear transformations $x \mapsto rx$ and/or $f \mapsto Qf$ in equation (Q)—with

$g = f$—are (partly) replaced by affine ones; this leads to a generalization of dimensional analysis). For further details we refer to Aczél and Alsina [10].

REFERENCES

1. T. L. Saaty, *The Analytic Hierarchy Process*. McGraw-Hill, New York (1980).
2. J. Aczél and T. L. Saaty, Procedures for synthesizing ratio judgements. *J. math. Psychol.* **27**, 93–102 (1983).
3. J. Aczél, *Lectures on Functional Equations and Their Applications*. Academic Press, New York (1966).
4. J. Aczél, On weighted synthesis of judgements. *Aequationes Math.* **27**, 288–307 (1984).
5. E. Mohr, Bemerkungen zur Dimensionsanalysis. *Math. Nachr.* **6**, 145–153 (1951).
6. J. Aczél and C. Alsina, Classification of some classes of quasilinear functions with applications to triangular norms and to synthesizing judgements. *Meth. opl Res.* **48**, 3–22 (1984).
7. J. Niven, Irrational numbers. *Carus. Math. Monog.* No. 11. M.A.A./Wiley, New York (1956).
8. C. Alsina, Synthesizing judgements given by probability distribution functions (to appear).
9. J. Aczél, F. S. Roberts and Z. Rosenbaum, On scientific laws without dimensional constants. *J. math. Analysis Applic.* **119**, 389–416 (1986).
10. J. Aczél and C. Alsina, On synthesis of judgements. *Socio-Econ. Plann. Sci.* **20**, 333–339 (1986).