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How to build a brick

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Abstract

A graph is *matching covered* if it connected, has at least two vertices and each of its edges is contained in a perfect matching. A 3-connected graph G is a *brick* if, for any two vertices u and v of G, the graph $G - \{u, v\}$ has a perfect matching. As shown by Lovász [Matching structure and the matching lattice, J. Combin. Theory Ser. B 43 (1987) 187–222] every matching covered graph G may be decomposed, in an essentially unique manner, into bricks and bipartite graphs known as braces. The number of bricks resulting from this decomposition is denoted by $b(G)$.

The object of this paper is to present a recursive procedure for generating bricks. We define four simple operations that can be used to construct new bricks from given bricks. We show that all bricks may be generated from three basic bricks K_4, $\overline{C}_6$ and the Petersen graph by means of these four operations. In order to establish this, it turns out to be necessary to show that every brick G distinct from the three basic bricks has a *thin* edge, that is, an edge e such that (i) $G - e$ is a matching covered graph with $b(G - e) = 1$ and (ii) for each barrier B of $G - e$, the graph $G - e - B$ has precisely $|B| - 1$ isolated vertices, each of which has degree two in $G - e$. Improving upon a theorem proved in [M.H. de Carvalho, C.L. Lucchesi, U.S.R. Murty, On a conjecture of Lovász concerning bricks, I, The characteristic of a matching covered graph, J. Combin. Theory Ser. B 85 (2002) 94–136; M.H. de Carvalho, C.L. Lucchesi, U.S.R. Murty, On a conjecture of Lovász concerning bricks, II, Bricks of finite characteristic, J. Combin. Theory Ser. B 85 (2002) 137–180] we show here that every brick different from the three basic bricks has an edge that is thin.

A cut of a matching covered graph G is *separating* if each of the two graphs obtained from G by shrinking the shores of the cut to single vertices is also matching covered. A brick is *solid* if it does not have any nontrivial separating cuts. Solid bricks have many interesting properties, but the complexity status of deciding whether a given brick is solid is not known. Here, by using our theorem on the existence of thin edges, we show that every simple planar solid brick is an odd wheel.

Keywords: Matching covered graphs; Brick generation

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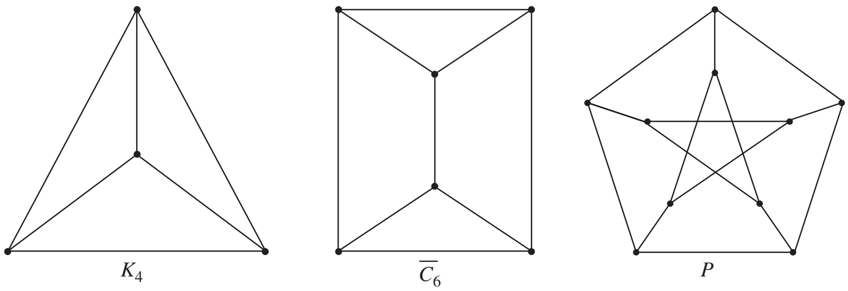


Fig. 1. The three basic bricks.

1. Introduction

All graphs considered in this paper are loopless. For any 3-connected graph H, clearly any graph G that is obtained from H by adding an edge is also 3-connected. Now suppose that H is a 3-connected graph with a vertex x of degree greater than three and let H' be a graph obtained from H by splitting x into two vertices b_1 and b_2 such that each b_i has at least two neighbours in H'. Then the graph $G := H' + b_1b_2$ is said to be obtained from H by an *expansion of the vertex* x. It is not difficult to see that G is 3-connected. The following theorem is one of the basic facts concerning 3-connected graphs.

Theorem 1 (*See Tutte [15], Thomassen [14]*). *Every 3-connected graph may be obtained from K_4 by means of edge additions and vertex expansions.*

A nontrivial graph G is *bicritical* if $G - \{u, v\}$ has a perfect matching for any two vertices u and v of G. Bricks, which are 3-connected bicritical graphs, play an important role in matching theory. The main objective of this paper is to prove an analogue of Theorem 1 for bricks.

The graph obtained from a brick by the addition of an edge is also a brick. But a vertex expansion of a brick does not even preserve the parity of the number of vertices. However, suppose that H is a brick and let H' be a graph obtained from H by splitting a vertex x as described earlier. Now, let G be a graph obtained from H' by adding a new vertex a and joining it to b_1 and b_2 and to some vertex w in $V(H - x)$. Then it can be shown that G is also a brick. We refer to this operation as an *expansion of the vertex x by a barrier of size two*. (The reason for the choice of this terminology will be made clear later on.) Apart from edge additions and expansions of vertices by barriers of size two, there are two other operations of expansions that may also be used to obtain new bricks from given bricks. (These are somewhat more complicated to describe and will be presented in Section 6.) We shall prove that every brick other than three bricks called the basic bricks (see Fig. 1) can be obtained from one of them by a sequence of applications of the four operations mentioned above.

This work relies heavily on the theory of matching covered graphs. We begin with a brief account of the salient concepts and theorems from that theory.

2. Matching covered graphs

An edge e of a graph G is *admissible* if there is some perfect matching of G that contains e. A graph is *matching covered* if it has at least two vertices, is connected and each of its edges is admissible. There is an extensive theory of matching covered graphs, see [8,9]. We shall assume that the reader is familiar with the basic concepts and results concerning matchings and matching covered graphs. However, we shall review a few basic definitions and results that are relevant to this work. For notation and terminology not defined here, see [1,2,8].

2.1. Bricks and braces

For a subset X of the vertex set V of a graph G, we denote the coboundary of X by $\nabla(X)$. A *cut* of G is a subset of the edge set E of the form $\nabla(X)$ for some subset X of V. If G is connected and $C := \nabla(X)$, then X and $\overline{X}$ are the only two subsets of V whose coboundaries are equal to C. In this case, we refer to X and $\overline{X}$ as the *shores* of C and refer to properties of the subgraphs $G[X]$ and $G[\overline{X}]$ as the properties of X and $\overline{X}$, respectively. For example, if $C := \nabla(X)$ and $G[X]$ is bipartite and $G[\overline{X}]$ is nonbipartite, then we shall refer to X as the *bipartite shore* of C and $\overline{X}$ as the *nonbipartite shore* of C. A cut C is *trivial* if one of its shores has cardinality one.

For a cut $C := \nabla(X)$ of G the graph obtained from G by shrinking X to a single vertex x is denoted by $G\{X \to x\}$. Similarly, the graph obtained from G by shrinking $\overline{X}$ to a single vertex $\overline{x}$ is denoted by $G\{\overline{X} \to \overline{x}\}$. These two graphs $G\{X \to x\}$ and $G\{\overline{X} \to \overline{x}\}$ are called the *C-contractions* of G.

A cut C of a matching covered graph G is a *separating cut* of G if the two C-contractions of G are also matching covered. A cut C of G is a *tight cut* of G if $|C \cap M| = 1$ for every perfect matching M of G. It is easy to see that every tight cut of G is a separating cut of G. Thus, if C is a nontrivial tight cut of G, the two C-contractions of G are matching covered graphs that have fewer vertices than G. If either of the C-contractions has a nontrivial tight cut, that graph can be further decomposed into even smaller matching covered graphs. This procedure can be repeated until one obtains a list of graphs each of which is a matching covered graph that has no nontrivial tight cuts. This procedure is known as a *tight cut decomposition* of G.

A matching covered graph that is bipartite and has no nontrivial tight cuts is a *brace*, and one that is nonbipartite and has no nontrivial tight cuts is a *brick*. (This definition of a brick is equivalent to the one given in the Abstract—see Theorem 8.) Fig. 1 shows three bricks (the complete graph K_4, the complement $\overline{C}_6$ of the 6-circuit and the Petersen graph P) which have played important roles in our work. We shall refer to these three bricks as the *basic bricks*.

Two cuts $\nabla(X)$ and $\nabla(Y)$ of a graph G are said to *cross* each other if $X \cap Y$, $X \cap \overline{Y}$, $Y \cap \overline{X}$ and $\overline{X} \cap \overline{Y}$ are all nonempty. Cuts C and D that do not cross are *laminar* to each other; in this case, one of the shores of C includes a shore of D. We shall need the following property of tight cuts:

Proposition 2 (*See Lovász [8]*). *Let G be a matching covered graph and let $\nabla(X)$ and $\nabla(Y)$ be two tight cuts such that $|X \cap Y|$ is odd. Then $\nabla(X \cap Y)$ and $\nabla(\overline{X} \cap \overline{Y})$ are also tight. Furthermore, no edge connects $X \cap \overline{Y}$ to $Y \cap \overline{X}$.*

Using the above proposition, Lovász proved the following fundamental theorem:

Theorem 3 (*See Lovász [8]*). *Any two tight cut decompositions of a matching covered graph yield the same list of bricks and braces (except possibly for multiplicities of edges).*

In particular, any two tight cut decompositions of a matching covered graph G yield the same number of bricks; this number is denoted by $b(G)$. A *near-brick* is a matching covered graph G with $b(G) = 1$.

The properties of a matching covered graph can often be analysed by analysing its bricks and braces separately. (For example, a matching covered graph has a Pfaffian orientation if and only if each of its bricks and braces has a Pfaffian orientation, see [7].) For this reason, it is important to understand the structure of bricks and braces and to have inductive procedures that can be used to generate them. In [11] McCuaig described a procedure for generating braces and used it (in an unpublished paper)[4] to provide a constructive characterization of bipartite matching covered graphs that have a Pfaffian orientation. In this paper we show how every brick may be built from the three basic bricks (Fig. 1) by means of four simple operations. A detailed statement of this will be given in Section 2.4.

2.2. Removable edges

An edge e of a matching covered graph G is said to be *removable* if the graph $G - e$ is also matching covered. The function $b(G)$ satisfies the following monotonicity property:

Proposition 4 (*See de Carvalho et al. [1]*). *For any removable edge e of a matching covered graph G, $b(G-e) \geqslant b(G)$.*

A removable edge e of G is *b-invariant* if $b(G - e) = b(G)$. In particular, if G is a brick, an edge e of G is b-invariant if $G - e$ is a matching covered graph with $b(G - e) = b(G) = 1$; in other words, e is b-invariant if $G - e$ is a near-brick. (In [1,2], we referred to a removable b-invariant edge as a *b-removable* edge.) The two bricks K_4 and $\overline{C}_6$ have no

[4] Added in proof: W. McCuaig, Pólya's Permanent Problem. The Electronic J. of Combinatorics 11, 2004.

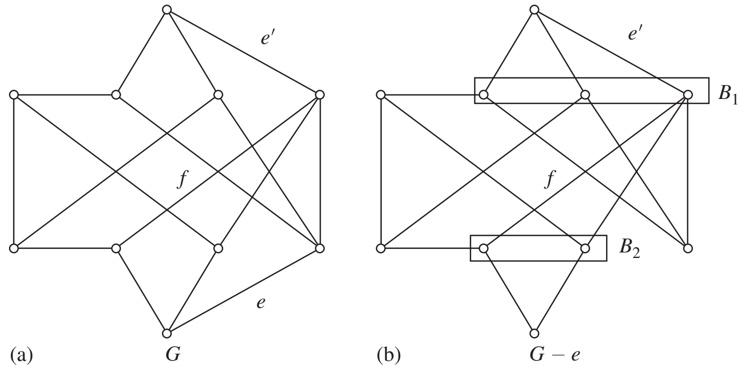


Fig. 2. The maximal nontrivial barriers.

removable edges at all. Every edge of the Petersen graph is removable but is not b-invariant. We proved the following theorem in [1,2].

Theorem 5. *Every brick distinct from K_4, $\overline{C_6}$ and P has a b-invariant edge.*

A *Petersen brick* is a brick whose underlying simple graph is the Petersen graph. The following strengthening of the above theorem was also proved in [1,2].

Theorem 6. *Every brick that is not a Petersen brick and is distinct from K_4 and $\overline{C_6}$ has a b-invariant edge e such that the brick of $G-e$ is not a Petersen brick.*

In fact, the main theorem in [1,2] is stronger than even Theorem 6. We used that theorem in [3] to find optimal ear decompositions of matching covered graphs.

2.3. Barriers

Let G be a graph that has a perfect matching. A nonempty subset B of $V(G)$ is a *barrier* of G if $\mathcal{O}(G-B)=|B|$, where $\mathcal{O}(G-B)$ denotes the number of odd components of $G-B$. All singletons are barriers; these are *trivial*. The graph in Fig. 2(a) shows a brick G; it has no nontrivial barriers. The edges e, e' and f are b-invariant in G. Fig. 2(b) shows the graph $G-e$; it has two nontrivial maximal barriers B_1 and B_2 that are indicated. The reader will easily be able to check that $G-e'$ has two nontrivial maximal barriers and that $G-f$ has just one nontrivial maximal barrier. We remark that a matching covered graph is bicritical if and only if it is free of nontrivial barriers.

The following theorem plays a crucial role in arguments involving barriers.

Theorem 7 (*See Lovász and Plummer [9]*). *Let G be a matching covered graph, and let $\sim$ denote the binary relation on V where $u\sim v$ if $G-\{u,v\}$ has no perfect matching. Then, the relation $\sim$ is an equivalence relation on V and the equivalence classes are precisely the maximal barriers of G.*

It can be deduced from the above theorem that the maximal barriers of any matching covered graph G partition $V(G)$. This partition is called the *canonical partition of G*.

Let B be a barrier of G and let K be a component of $G-B$. As G is matching covered, component K must be odd. It is easy to see that $\nabla(V(K))$ is a tight cut of G. We shall say that $\nabla(V(K))$ is a tight cut *associated* with B. Such tight cuts are known as *barrier cuts*. A matching covered graph may have tight cuts that are not barrier cuts but, as we shall see (Lemma 13), every tight cut in a near-brick is a barrier cut.

Edmonds et al. [6] established the following characterization of bricks.

Theorem 8 (*See Edmonds et al. [6], Lovász [8], Szigeti [13]*). *A graph G is a brick if and only if it is bicritical and 3-connected.*

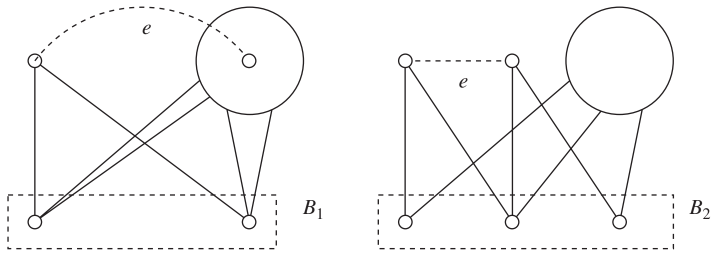


Fig. 3. Two types of thin barriers.

Corollary 9. *Every graph that has a brick as a spanning subgraph is itself a brick.*

2.4. The Main Theorem

Let G be a matching covered graph. For each barrier B of G, let $I_G(B)$ denote the set of vertices of $G-B$ that are isolated in $G-B$. A barrier B of G is *special* if $G-B$ has precisely one nontrivial component. Thus, a barrier B of G is special if and only if $|I_G(B)|=|B|-1$. A barrier B of G is *thin* if it is special and each vertex of $I_G(B)$ has degree two in G. Let e be a removable edge of G. Edge e is *special* if each barrier of $G-e$ is special and is *thin* if each barrier of $G-e$ is thin. In the brick G shown in Fig. 2, the edge e is not thin but the edge f is. Fig. 3 depicts the two types of thin barriers of graph $G-e$, obtained by the deletion of an edge e from a bicritical G.

Theorem 10 (*Main Theorem*). *Let G be a brick distinct from K_4, $\overline{C_6}$ and the Petersen graph. Then, G has a b-invariant edge that is thin.*

As a corollary of the above theorem, we have the following interesting property of bricks.

Corollary 11. *Every brick G of minimum degree greater than three has an edge e such that $G-e$ is a brick.*

2.5. An analogue for braces

A nontrivial tight cut of a bipartite matching covered graph is *thin* if one of its shores has cardinality three. The following theorem, which may be regarded as an analogue of Theorem 10 for braces, may be deduced from a theorem of McCuaig [11].

Theorem 12. *Every brace G different from K_2 and C_4 has a removable edge e such that* (i) *$G-e$ has at most two nontrivial tight cuts and* (ii) *each nontrivial tight cut of $G-e$ is thin.*

2.6. Outline of the proof of the Main Theorem

For any b-invariant edge e of a brick G, the *index* of e is the number of nontrivial maximal barriers of $G-e$. We proved in [1] that the index of a b-invariant edge in a brick is at most two. If e is an edge of index zero, then $G-e$ is a brick and hence e is thin. If a b-invariant edge has index greater than zero and is not thin, we shall need to show that there is some other edge that is thin.

Let e be a b-invariant edge of a brick G that is not thin. Then we show that there is some other b-invariant edge e' of G such that the size of the brick of $G-e'$ is at least as large as the size of the brick of $G-e$. This leads us to define the rank of e: we denote the size of the brick of $G-e$ by $r(G-e)$ and refer to $r(G-e)$ as the *rank* of e. (For example, if e is of index zero, then $r(G-e)=|V|$.) The idea behind this notion is that the larger the rank of a b-invariant edge is, the closer it is to being thin. We shall indeed show that there is a b-invariant edge of maximum rank that is thin.

In the next section we shall discuss properties of barriers and tight cuts in near-bricks obtained from bricks by the deletion of b-invariant edges. These properties are useful for computing and comparing the ranks of b-invariant edges of a brick.

As noted above, if a brick has a b-invariant edge that is not thin, then there is some other b-invariant edge of the brick of equal or higher rank. In Section 4, we shall establish the tools necessary for finding such alternatives.

We shall present a proof of the Main Theorem in Section 5 and in Section 6 describe the four operations by means of which all bricks may be built starting from the three basic bricks. Finally, in Section 7 we shall apply our Main Theorem to show that odd wheels are the only simple planar solid bricks.

3. Barriers and tight cuts in near-bricks

In this section we shall establish the relationship between barriers and tight cuts in near-bricks. We start by proving the following basic fact concerning tight cuts in near-bricks.

Lemma 13. *For any tight cut $C := \nabla(X)$ in a near-brick G, precisely one of the shores of C is bipartite. Furthermore, C is a barrier cut associated with a special barrier of G.*

Proof. Let $G_1 := G\{X \to x\}$ and $G_2 := G\{\overline{X} \to \overline{x}\}$ denote the two C-contractions of G. Then, $b(G_1) + b(G_2) = b(G) = 1$, whence one of $b(G_1)$ and $b(G_2)$ is equal to zero, the other to one. Thus, one of G_1 and G_2 is bipartite, the other is nonbipartite.

Adjust notation so that $G[X]$, the subgraph of G induced by X, is bipartite. Each C-contraction of G is matching covered, therefore 2-connected, whence each shore of C spans a connected subgraph of G. In particular, $G[X]$ is connected. The bipartition of $G[X]$ is unique. Let $\{B, I\}$ denote this bipartition and adjust notation so that $|B| > |I|$. Then, it can be verified that B is a special barrier of G, I is the set of isolated vertices of $G - B$ and that $G[\overline{X}]$ is the nontrivial component of $G - B$. □

Let G be a near-brick, C a tight cut of G, X the bipartite shore of C (see 2.1). The majority part B of $G[X]$ is called the *external part* of X, the minority part I the *internal part*. See Fig. 4 (the vertices of the barrier associated with the cut C are indicated by solid dots).

Lemma 13 shows that given any tight cut of a near-brick G, there is a unique special barrier that corresponds to it. Conversely, given any special barrier of G, there is a unique tight cut that corresponds to it. We shall use the following notation for nontrivial tight cuts associated with special barriers of near-bricks.

Notation 14. *Let C be a tight cut of a near-brick G. Then, $X_G(C)$ denotes the bipartite shore of C, $B_G(C)$ its external part, $I_G(C)$ its internal part. Whenever G and C are understood, we shall write simply X, B and I, instead of $X_G(C)$, $B_G(C)$ and $I_G(C)$, respectively. We shall adopt the same notational conventions for tight cuts C_1, C' etc., writing simply X_1, B_1, I_1, X', B' and I' etc., respectively.*

Correspondingly, let B denote a special barrier of a graph G. Then, $I_G(B)$ denotes the set of isolated vertices of $G - B$, $X_G(B)$ denotes set $B \cup I_G(B)$ and $C_G(B)$ denotes cut $\nabla_G(X_G(B))$. Whenever G and B are understood, we shall write simply I, X and C, instead of $I_G(B)$, $X_G(B)$ and $C_G(B)$, respectively. We shall adopt the same notational conventions for special barriers B_1 and B' etc., writing simply I_1, X_1, C_1, I', X' and C' etc., respectively.

In general, not every barrier in a near-brick is special. (For example, the near-brick in Fig. 4 has a barrier whose deletion gives two nontrivial components, one of which is bipartite.)

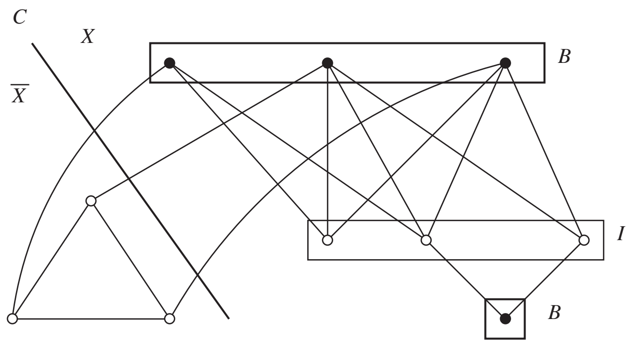


Fig. 4. Cuts and their shores in near-bricks.

Proposition 15. *Every maximal barrier in a near-brick is special.*

Proof. Let G be a near-brick, B a maximal barrier of G. Assume, to the contrary, that B is not special. Let K_1 and K_2 denote two nontrivial components of $G - B$. For $i = 1, 2$, cut $C_i := \nabla(V(K_i))$ is tight in G. As G is a near-brick, precisely one of the C_i-contractions of G is bipartite. Let $G_i := G\{\overline{V(K_i)} \to x_i\}$.

We assert that at least one of G_1 and G_2 is bipartite. If G_1 is bipartite then we are done. Assume thus that $H := G\{V(K_1) \to v\}$ is bipartite. Then, C_2 is a tight cut of H, a matching covered bipartite graph, whence both C_2-contractions of H are bipartite; in particular, G_2 is bipartite. As asserted, at least one of G_1 and G_2 is bipartite. Adjust notation so that G_1 is bipartite. Let (A_1, B_1) be a partition of G_1 so that x_1 lies in B_1. Then, $B \cup (B_1 - x_1)$ is a barrier of G. As K_1 is nontrivial, it follows that B_1 is not a singleton, whence $B_1 - x_1$ is nonnull. This contradicts the maximality of B. □

A tight cut C of a matching covered graph G is *extremal* if it has at least one bipartite shore that is maximal among all the bipartite shores of tight cuts of G.

Lemma 16. *Let G be a near-brick, C an extremal tight cut of G. Then, the associated barrier B of G is maximal.*

Proof. Certainly B is a barrier of G. To prove that it is maximal, let B' denote the maximal barrier of G that includes B. By hypothesis, G is a near-brick. Thus, B' is special. No vertex of I lies in B', otherwise B' would span all the edges of G incident with the vertex, and that would imply that G has nonadmissible edges. Every vertex of I thus lies in $G - B'$ and is isolated in $G - B'$, because B' includes B. Thus, I is a part of I'. By definition, B is a part of B'. It follows that the bipartite shore of C' includes that of C. But C is extremal, whence $C' = C$. By the uniqueness of the bipartition of $G[X]$, we conclude that $B = B'$, whence B is maximal. □

The converse of the preceding result is not true in general. But it is true for near-bricks that are the result of the removal of a b-invariant edge from a brick. More generally, it is true for every graph such that every barrier is special.

Lemma 17. *Let G be a matching covered graph such that every barrier is special. Then, for every (special) maximal barrier of B of G, the associated cut C is extremal.*

Proof. By hypothesis, every barrier of G is special. Therefore, cut C is well defined. Let C' denote an extremal special cut of G whose bipartite shore includes that of C. The bipartition $\{B, I\}$ of the bipartite shore X of C is unique. The bipartition $\{B', I'\}$ of the bipartite shore X' of C' is also unique. Moreover, $G[X]$ is an induced subgraph of $G[X']$. Therefore, $\{B' \cap X, I' \cap X\} = \{B, I\}$. One of B' and I' is a barrier of G. Adjust notation so that B' is a barrier of G.

Assume, to the contrary, that $I' \cap X = B$, whereupon $B' \cap X = I$. In that case, $B' - X$ is a barrier of G, with precisely two nontrivial components: $G[\overline{X'}]$ and $G[X]$. This is a contradiction to the hypothesis that every barrier of G is special. Thus, $B' \cap X = B$ and $I' \cap X = I$. By the maximality of B, it follows that $B = B'$, whence $|I|' = |I|$, and therefore $X = X'$ and $C = C'$. Thus, C is extremal. □

3.1. Near-bricks inherited from bricks

A near-brick obtained from the deletion of a b-invariant edge from a brick is said to be *inherited* from that brick. Many properties that do not hold in general for all near-bricks hold for near-bricks inherited from bricks. The purpose of this subsection is to describe some of those properties.

Lemma 18 (*See de Carvalho et al. [1]*). *Let G be a brick and let e be a b-invariant edge in G. Then all barriers of $G - e$ are special.*

Lemma 19 (*See de Carvalho et al. [1]*). *Let G be a brick, e a b-invariant edge of G, B a (special) nontrivial maximal barrier of $G - e$. Then, graph $G' := G\{X \to x\}$ is a brick.*

Proof. Let $C := \nabla(X)$. Graph $G' - e$ is the nonbipartite $(C - e)$-contraction of $G - e$, whence it is a near-brick. Therefore, G' has perfect matchings. Thus, for every set B' of vertices of G', the number of odd components of $G' - B'$ is at most $|B'|$. Call *extremal* those sets B' for which equality holds. We assert that every extremal set is trivial. For this, assume the contrary, let B' denote a nontrivial extremal set. Then, B' is a barrier of $G' - e$. If B' contains x, the contraction vertex, then $B \cup (B' - x)$ is a barrier of $G - e$ that properly includes B, a contradiction to the maximality of B. If B' does not contain x then x lies in some component of $G' - B'$. Graph $G[X] - e$ is connected, it is the bipartite shore of tight cut $C - e$ of $G - e$. Therefore, the component of $G' - B'$ that contains x is the contraction of set X to vertex x in a component of $G - B'$. Then, the number of odd components of $G - B'$ is also $|B'|$, whence B' is a nontrivial barrier of G, a contradiction. As asserted, every extremal set is trivial.

This implies that edge e is admissible in G', otherwise its ends would lie in a nontrivial extremal set. Since $G' - e$ is matching covered, so too is G'. Since G' is free of nontrivial extremal sets, then G' is bicritical. Since $G' - e$ is a near-brick, then so too is G', by Proposition 4. In sum, G' is a bicritical near-brick. By Lemma 13, near-brick G' is free of nontrivial tight cuts. We conclude that G' is a brick. $\square$

The following lemma is of fundamental importance. It will play a crucial role in the rank computations that we shall encounter in the proof of the Main Theorem.

Lemma 20 (*The three case lemma, see de Carvalho et al. [1]*). *Let G be a brick, e a b-invariant edge of G. Then, one of the following alternatives holds*:

(1) *graph $G - e$ is a brick, or*
(2) *graph $G - e$ has precisely one maximal nontrivial barrier, B, and the graph $(G - e)\{X \to x\}$ is a brick, or*
(3) *graph $G - e$ has precisely two maximal nontrivial barriers, B_1 and B_2, and*
   - *edge e has one end in $I_1 - X_2$ and the other in $I_2 - X_1$,*
   - *the graph $(G - e)\{X_1 \to x_1\}$ is a near-brick, $B_2 - X_1$ is its unique nontrivial maximal barrier and $X_2 - X_1$ is the bipartite shore of its unique nontrivial extremal tight cut,*
   - *the graph $(G - e)\{X_1 \to x_1\}\{X_2 - X_1 \to x_2\}$ is a brick.*

Recall that, for a b-invariant edge e of a brick G, we refer to the number of nontrivial maximal barriers of $G - e$ as the *index* of e. By Lemma 20, the index of any b-invariant edge is either zero, one or two. If an edge e is of index zero, then $G - e$ is a brick. In this case, $r(G - e) = |V(G)|$. We describe below the procedure for finding the rank of $G - e$ when the index of e is positive.

Suppose that a b-invariant edge e of a brick G has index one and suppose that B is the only nontrivial maximal barrier of $G - e$. Then the edge e might have both its ends in I or one end in I and one end in $\overline{X}$. In either case, the graph obtained from $G - e$ by contracting X to a single vertex is the brick of $G - e$. Thus, in this case, $r(G - e) = |V(G)| - |X| + 1$.

Now suppose that a b-invariant edge e of a brick G has index two and suppose that B_1 and B_2 are the two nontrivial maximal barriers of $G - e$. In this case, $B_2' := B_2 - X_1$ is the only maximal nontrivial barrier of $G' := G - e\{X_1 \to x_1\}$ and $G'\{X_2 - X_1 \to x_2\}$ is the brick of $G - e$. Thus, in this case $r(G - e) = (|V(G)| - |X_1| + 1) - |X_2 - X_1| + 1 = |V(G)| - (|X_1| + |X_2 - X_1|) + 2$. Fig. 5 illustrates a possible interaction of the maximal nontrivial barriers in a near-brick $G - e$ obtained by deleting a b-invariant edge e of index two from a brick G. In this example, both the nontrivial maximal barriers B_1 and B_2 have cardinality three. They, as they are expected to be, are disjoint. But B_2 contains a vertex of I_1 and B_1 contains a vertex of I_2, and $r(G - e) = |V(G)| - (|X_1| + |X_2 - X_1|) + 2 = 10 - (5 + 3) + 2 = 4$.

4. Removable edges in near-bricks

Suppose that G is a brick and e is a b-invariant edge of G that is not thin. Our objective then is to show that there is some other b-invariant edge of G that is thin. The first step is to show that there exist b-invariant edges (thin or not) other than e. This section is devoted to the development of tools that are necessary for finding such edges. If B is a nontrivial maximal barrier of $G - e$, it turns out that one can always find a b-invariant edge of G incident with a vertex in I that is of degree greater than two. We begin our discussion with some general results concerning removable edges in matching covered graphs.

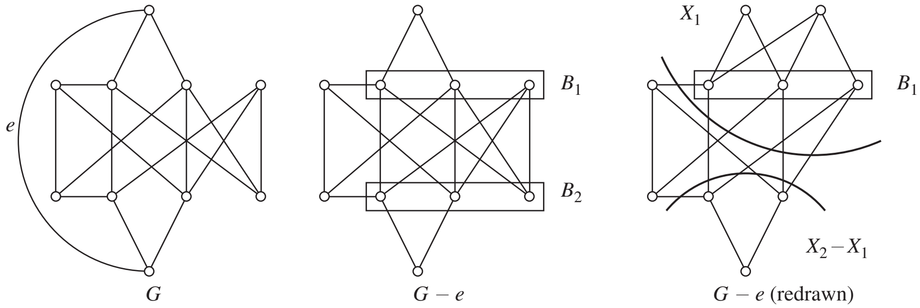


Fig. 5. A b-invariant edge of index two in a brick.

The following elementary proposition gives conditions under which a removable edge of $G-e$ is also removable in G.

Proposition 21. *Let G be a matching covered graph, e a removable edge of G, e' an edge of $E(G)-e$ that is removable in $G-e$. Edge e' is removable in G if and only if G has a perfect matching that contains edge e but does not contain edge e'.*

Proof. By hypothesis, edge e' is removable in $G-e$. Thus, graph $G-e-e'$ is matching covered. That is, $G-e-e'$ is connected and each edge of $G-e-e'$ lies in some perfect matching of $G-e-e'$. Thus, $G-e'$ is connected and each edge of $G-e'$ distinct from e lies in some perfect matching of $G-e'$. As asserted, edge e' is removable in G if and only if G has a perfect matching that contains e but does not contain e'. □

The next result, which can be easily derived using Hall's Theorem, gives conditions under which an edge of a bipartite graph is not admissible.

Lemma 22 (*See de Carvalho et al. [1, Theorem 2.3]*). *An edge e of a bipartite graph H with bipartition $\{A, B\}$ that has a perfect matching is not admissible if, and only if, there exists a partition (A', A'') of A and a partition (B', B'') of B such that $|A'|=|B'|$, edge e joins a vertex of A'' to a vertex of B' but no edge of H joins some vertex of A' to some vertex of B''.*

We shall now establish a general result concerning bicritical graphs which provides an essential tool in achieving the objective stated at the beginning of this section. Towards this end, let G be a bicritical graph, e a removable edge of G, B a special (possibly trivial) barrier of $G-e$. Let K denote the nontrivial component of $G-e-B$. Let H be the graph $(G-e)\{V(K)\to v_K\}$. Then, graph H is bipartite, with bipartition $\{A, B\}$, where $A = I \cup \{v_K\}$.

Lemma 23. *Let v be a vertex of H in A.* (i) *If $v = v_K$, then every edge of H incident with v is removable in H and* (ii) *if $v \neq v_K$ and the degree of v is greater than two, then at most one edge of H incident with v is not removable in H.*

Proof. By induction on $|B|$. If $|B|=1$, then v_K is the only vertex in A and every edge of H is a multiple edge. Therefore, the assertion is trivially valid in this case. So, suppose that $|B|>1$.

If every edge of H incident with v is removable in H then the assertion holds immediately. We may thus assume that there is an edge e' incident with v that is not removable in H. Let w be its end in B. Let x denote the end of e' in G that does not lie in B. Thus, if $v \neq v_K$ then $x = v$, whereas if $v = v_K$ then x lies in $V(K)$.

By Lemma 22 there exists a partition (B', B'') of B and a partition (A', A'') of A such that $|A'|=|B'|$ and e' is the only edge of H that joins a vertex of A' to a vertex of B''. If v_K does not lie in A'' then set $A''+x$ is a nontrivial barrier of G, a contradiction. Thus, v_K lies in A''. We conclude that edge e' is not incident with vertex v_K in H. Thus, $v \neq v_K$. This conclusion holds for every edge e' incident with v that is not removable in H. This proves the first part of the assertion.

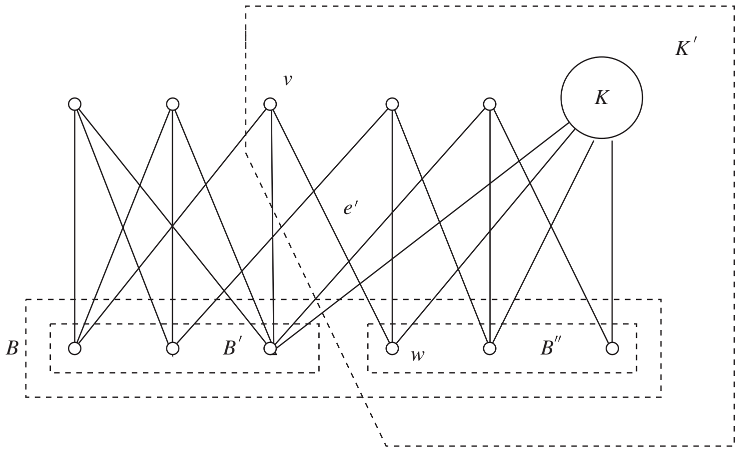


Fig. 6. An illustration for the proof of Lemma 23.

To prove the second part, assume that $v \neq v_K$ and that the degree of v in H is at least three. Observe that B' is a (possibly trivial) special barrier of $G-e$. Let K' denote the nontrivial component of $G-e-B'$. (Note that $V(K')=V(K)\cup(A''-v_K+v)\cup B''$. See Fig. 6.) Let H' denote the bipartite graph obtained from $G-e$ by shrinking $V(K')$ to a single vertex $v_{K'}$.

Since $|B'|<|B|$, it follows by induction that every edge of H' incident with $v_{K'}$ is removable in H'. To complete the proof, note that $\nabla(V(K'))$ is a tight cut of the bipartite graph H. The graph H' is one of the two $\nabla(V(K'))$-contractions of H. In the other $\nabla(V(K'))$-contraction of H, all edges in $\nabla_H(v)-e'$ are multiple edges and hence, are removable in that graph. Moreover, all such edges are incident with $v_{K'}$ in H'. It follows that all edges in $\nabla_H(v)-e'$ are removable in H. □

Based on the above results, we are now in a position to establish the required result concerning b-invariant edges in $G-e$.

4.1. Existence of b-invariant edges in $G-e$

Theorem 24. *Let G be a brick, e a b-invariant edge of G, B a maximal nontrivial (special) barrier of $G-e$. Let v be a vertex of I that has degree strictly greater than two in $G-e$. Then, the following properties hold:*

(1) *At most one edge of $\nabla(v)-e$ is not b-invariant in $G-e$.*
(2) *At most one edge of $\nabla(v)-e$ is b-invariant in $G-e$ but not b-invariant in G.*
(3) *If e is of index two, then every edge of $\nabla(v)-e$ that is b-invariant in $G-e$ is also b-invariant in G.*

Proof. Let H denote the bipartite graph defined before the statement of Lemma 23. Let e' be any edge that is removable in H and lies in $\nabla(v)$. Recall that we denote by C the cut associated with special barrier B. Cut $C-e$ is tight in $G-e$. One of the $(C-e)$-contractions of $G-e$ is H, the other is $G' := (G-e)\{X\to x\}$. Edge e is b-invariant in G, whence G' is a near-brick.

One of the $(C-e)$-contractions of $G-e-e'$ is G', the near-brick seen above. The other $(C-e)$-contraction is $H-e'$, a bipartite graph. We conclude that any edge of H that does not lie in C and is removable in H is b-invariant in $G-e$. By Lemma 23, at most one edge of $\nabla(v)-e$ is not b-invariant in $G-e$. This completes the proof of the first item of the assertion.

Let e' be an edge of $\nabla(v)-e$ that is b-invariant in $G-e$. Let M denote a perfect matching of G that contains edge e. If e' does not lie in M then, by Proposition 21, edge e' is removable in G. Moreover,

$$b(G-e')\leqslant b(G-e-e')=b(G-e)=b(G),$$

where the inequality follows from the monotonicity of function b, and the equalities from the fact that edge e' is b-invariant in $G-e$ and edge e is b-invariant in G. Thus, edge e' is b-invariant in G. This conclusion holds for every edge e' of $\nabla(v)-e$ that is b-invariant in $G-e$ and does not lie in some perfect matching of G that contains edge e. Thus, at most one edge of $\nabla(v)-e$ is b-invariant in $G-e$ but not b-invariant in G. If B is the only nontrivial maximal barrier of $G-e$ then the proof of the assertion is complete.

Now suppose that e is of index two. Let B' denote the second nontrivial maximal barrier of $G-e$ and let z denote the end of e in $\overline{X}$. It follows from Lemma 20 that vertex z lies in $I'-X$ and thus z is not adjacent to any vertex of X in $G-e$.

Assume, to the contrary, that $\nabla(v)-e$ contains an edge, e_1, that is b-invariant in $G-e$ but not removable in G. Then, edge e depends on e_1, that is, every perfect matching of G that contains edge e contains also edge e_1. Therefore, graph $G-e-e_1$ has barriers that contain both ends of edge e. Let B_1 be a maximal barrier of $G-e-e_1$ that contains both ends of e. Graph $G-e-e_1$ is matching covered, therefore e is the only edge of G that depends on e_1. We conclude that in G the set B_1 spans edge e and no other edge of G.

Proposition 25. *The end z of e in $\overline{X}$ is the only vertex of B_1 in $\overline{X}$. Moreover, $G[\overline{X}]-z$ is connected. Consequently all the vertices of $\overline{X}-z$ lie in the same component of $G-e-e_1-B_1$.*

Proof. Certainly, z, an end of e, lies in B_1. Let $G_1 := G\{X \to x\}$. By Lemma 19, G_1 is a brick. Bricks are 3-connected. Therefore, G_1-x-z is connected. That is, $G[\overline{X}]-z$ is connected. Let z' denote any vertex of $\overline{X}-z$. Graph G_1, a brick, is bicritical, thereforeG_1-z-z' has a perfect matching, say, M'. Let f be the edge of C in M'. Edge e is incident with z in G_1, therefore edge e does not lie in M'. Thus, f lies in $C-e$.

Edge e_1 is an edge of G that is removable in $G-e$ and does not lie in C. Thus, f and e_1 are distinct, whence, f lies in a perfect matching, say, M'', of $G-e-e_1$. Then, $M' \cup (M'' \cap E(G[X]))$ is a perfect matching of $G-e-e_1-z-z'$. Therefore, vertex z' does not lie in B_1. This conclusion holds for each vertex z' of $\overline{X}-z$. □

Consider set $B_2 := B_1 - z$. For $i=1,2$, let $\mathscr{K}_i$ denote the set of components of $G-e-e_1-B_i$. For every matching covered graph L and every barrier Z of L, each component of $L-Z$ is odd. In particular, $G-e-e_1$ is matching covered and B_1 is a barrier of $G-e-e_1$, whence the components of $\mathscr{K}_1$ are odd.

Let K denote the component in $\mathscr{K}_1$ that contains the vertices of $\overline{X}-z$. We have seen that all the vertices that are adjacent to z in $G-e$ lie in $\overline{X}$. Therefore, all the vertices that are adjacent to z in $G-e-e_1$ lie in $V(K)$. Consequently, $\mathscr{K}_2 = (\mathscr{K}_1 - K) \cup \{G[V(K)+z]\}$. Thus, $G-e-e_1-B_2$ has $|B_2|$ odd components and one even component. Thus, B_2 is a barrier of matching covered graph $G-e-e_1$ such that $G-e-e_1-B_2$ has an even component. This is a contradiction. As asserted, under the hypothesis that $G-e$ has two nontrivial maximal barriers, we conclude that every edge of $\nabla(v)-e$ that is b-invariant in $G-e$ is also b-invariant in G. □

5. Proof of the Main Theorem

Let G be a brick and let e be a b-invariant edge of G that is not thin. Then $G-e$ has at least one nontrivial maximal barrier. Let B be one such barrier of $G-e$ and let v_I be a vertex of I that has degree greater than two. We have seen in the last section that at least one edge of $G-e$ incident with v_I is b-invariant in both G and $G-e$. Let e' be such an edge.

A fact that should be kept in mind is that the edge e' has both its ends in X, whereas the edge e has either one (Fig. 7a) or two ends in I (Fig. 7b). (The former case must occur when e is of index two.) Another important fact to remember is that B, a barrier of $G-e$, is also a barrier of $G-e-e'$. Moreover, any barrier of $G-e-e'$ that includes B is also a barrier of $G-e$, because e' has an end in B. Since B is a maximal barrier of $G-e$, it follows that B is a maximal barrier of $G-e-e'$.

We begin our analysis by showing that the rank of e' is at least as big as that of e.

Lemma 26. $r(G-e') \geqslant r(G-e-e') = r(G-e)$.

Proof. Recall that edge e is b-invariant in G and e' is b-invariant in each of $G-e$ and G, whence $G-e$, $G-e'$ and $G-e-e'$ are near-bricks. As B is a barrier of $G-e-e'$, cut $C=\nabla(X)$ is also a tight cut of $G-e-e'$. Since e' has

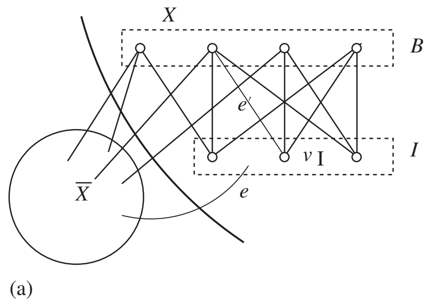


Fig. 7. The two b-invariant edges e and e'.

both its ends in X, the nonbipartite shore $\overline{X}$ of C in $G - e - e'$ is the same as it is in $G - e$. Therefore:

$$r(G - e - e') = r(G - e).$$

On the other hand, note that if $\nabla_{G-e'}(S)$ is any tight cut of $G - e'$, then $\nabla_{G-e-e'}(S)$ is a tight cut of $G - e - e'$. Thus, clearly,

$$r(G - e') \geqslant r(G - e - e').$$

It follows that $r(G - e') \geqslant r(G - e - e') = r(G - e)$. $\square$

The crucial part of the proof of the Main Theorem involves the study of conditions under which the ranks of e and e' can be equal. We now prove two basic lemmas that are needed for this purpose.

Lemma 27. *Let C' be a nontrivial tight cut of $G - e'$. If C' does not cross C, then $X' \subset X$.*

Proof. Clearly, $C' - e$ is a tight cut of $G - e - e'$. As $G - e - e'$ is a near-brick, one shore of $C' - e$ is bipartite and it must be X', with B' and I' as its external and internal parts, respectively.

The cut C' is tight in $G - e'$ but C is not. Thus, $C' \neq C$. Thus, as C' does not cross C, the possible containment relations between X and X' are: (i) $X' \subset \overline{X}$, (ii) $\overline{X} \subset X'$, (iii) $X \subset X'$, or (iv) $X' \subset X$. Our task is to show that (iv) holds. We do this by showing that the other possibilities cannot occur.

Since C' is a nontrivial tight cut of $G - e'$, the edge e' must have at least one end in I'. Otherwise, C' would be a tight cut of G itself, which is not possible. Now, since e' has both its ends in X, it follows that X' cannot be a subset of $\overline{X}$. That is, (i) cannot hold.

The shore $\overline{X}$ of $C - e$ is nonbipartite whereas the shore X' of $C' - e$ is bipartite. Hence (ii) cannot hold.

Note that $\{B, I\}$ is the bipartition of X and $\{B', I'\}$ is the bipartition of X'. Assume, to the contrary, that X is a subset of X'. Then, $\{B' \cap X, I' \cap X\} = \{B, I\}$. Thus, one of B and I is a subset of B'. But edge e' has one end in B, the other in I, therefore in either case edge e' has an end in B'. Thus, B', a nontrivial barrier of $G - e'$, is also a nontrivial barrier of brick G, a contradiction. Thus, (iii) cannot hold. $\square$

Corollary 28. *If every tight cut of $G - e'$ is laminar to C, then $r(G - e') > r(G - e)$.*

Proof. If $G - e'$ is a brick then this assertion holds immediately. We may thus assume that $G - e'$ has nontrivial tight cuts. Cut C is not tight in $G - e'$, and by Lemma 27, every nontrivial tight cut of $G - e'$ has its bipartite shore as a proper subset of X. We conclude that in any tight cut decomposition of $G - e'$, the brick G'_0 thereby obtained contains all the vertices of $\overline{X}$ and at least three other vertices. Thus $r(G - e') \geqslant |\overline{X}| + 3$.

On the other hand, consider a tight cut decomposition of $G - e$ that uses cut C. Let G_0 be the brick thereby obtained. The bipartite shore of C is X, whence $V(G_0)$ has at most $|\overline{X}|+1$ vertices. We conclude that $r(G-e') > |\overline{X}|+1 \geqslant r(G-e)$. As asserted, $r(G - e') > r(G - e)$. $\square$

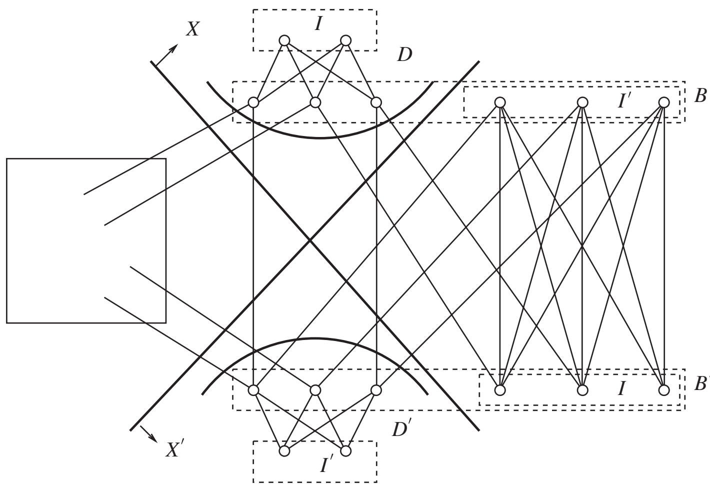


Fig. 8. Cuts $\nabla(X)$ and $\nabla(X')$ in $G - e - e'$.

Corollary 29. *If some nontrivial tight cut C_0 of $G - e'$ crosses C, then there exists an extremal tight cut of $G - e'$ that crosses C.*

Proof. Let C' be a tight cut of $G - e'$ such that $X_0 \subseteq X'$. Since C_0 crosses C, X_0 is not a subset of X, therefore X' is not a subset of X. By Lemma 27, C' crosses C. □

Lemma 30. *Let C' be a tight cut of $G - e'$ that crosses cut C. Then, $X \cap X'$ and $\overline{X} \cap \overline{X'}$ are both even and the following properties hold*:

(1) *cut $D := \nabla_{G-e-e'}(X - X')$ is a nontrivial tight cut of $G - e - e'$, $X - X'$ is its bipartite shore, $B - X'$ the external part and $I - X'$ the internal part of $X - X'$,*
(2) *cut $D' := \nabla_{G-e-e'}(X' - X)$ is a tight cut of $G - e$, $X' - X$ is its bipartite shore, $B' - X$ the external part and $I' - X$ the internal part of $X' - X$. (See Fig. 8.)*
(3) *edge e' has one end in $I - X'$, the other in $B \cap I'$.*

Proof. Let us first show that $X \cap X'$ is even. Suppose that $X \cap X'$ is odd. As C and $C' - e$ are tight cuts of $G - e - e'$, then, by Proposition 2, cuts $\nabla(X \cap X')$ and $\nabla(\overline{X} \cap \overline{X'})$ are also tight cuts in $G - e - e'$. As X and X' are bipartite, so too is $G[X \cap X'] - e - e'$. Moreover, $G[X \cap X'] - e - e'$ is connected, whence its bipartition is unique. Thus, $\{B \cap X', I \cap X'\} = \{B' \cap X, I' \cap X\}$. Let Y' denote one of B' and I', let Z' denote the other, so that $B \cap X' = Y' \cap X$ and $I \cap X' = Z' \cap X$. No edge of $G - e - e'$ joins a vertex of $X - X'$ to a vertex of $X' - X$. It follows that $\{B \cup Y', I \cup Z'\}$ is a bipartition of $G[X \cup X'] - e - e'$. Cut $C'' := \nabla_{G-e-e'}(X \cup X')$ is tight in $G - e - e'$. Therefore, $X \cup X'$ is its bipartite shore in $G - e - e'$. Moreover, edge e' has one end in B, the other in I, whence $G[X \cup X'] - e$ is bipartite. In sum, cut C'' is tight in $G - e$, $X \cup X'$ its bipartite shore in $G - e$. On the other hand, cut C is extremal in $G - e$, by Lemma 17. This is a contradiction. We conclude that $X \cap X'$ is even. It follows that $\overline{X} \cap \overline{X'}$ is also even.

It follows from Proposition 2 that D and D' are tight cuts of $G - e - e'$.

Every tight cut of $G - e - e'$ that has a subset of $\overline{X}$ as a shore is a tight cut in $G - e$, because cut C is tight in both $G - e$ and $G - e - e'$, and edge e' has both ends in X. In particular, D' is a tight cut in $G - e$.

Graph $G[X] - e$ is bipartite, whence so too is $G[X - X'] - e - e'$. Thus, $X - X'$ is the bipartite shore of D. Set $\overline{X'}$ is a shore of tight cut $C' - e$ of $G - e - e'$, whence $G[\overline{X'}] - e - e'$ is connected. Thus, at least one edge joins a vertex, say v, of $X - X'$, to a vertex of $\overline{X} \cap \overline{X'}$. That vertex lies in the external parts of both X and $X - X'$. Therefore, vertex v lies in B, whence B meets the external part of $X - X'$. But $\{B - X', I - X'\}$ is the only bipartition of $G[X - X'] - e - e'$. Thus, $B - X'$ is the external part of $X - X'$ and $I - X'$ its internal part. A similar argument shows that $X' - X$ is the bipartite shore of D', $B' - X$ its external part, $I' - X$ its internal part.

In order to complete the proof, we must now show that D is nontrivial and also that edge e' has one end in $I - X'$, the other in $B \cap I'$. For this, let v_I denote the end of e' in I.

Assume, to the contrary, that v_I lies in X'. Vertex v_I does not lie in B', otherwise B' would be a nontrivial barrier of G. Thus, v_I lies in I'. Thus, v_I lies in $I \cap I'$. Let f be any edge of $G - e - e'$ that is incident with vertex v_I. The end of f distinct from v_I lies in $B \cap B'$. But B is a maximal barrier of $G - e - e'$ and B' is a barrier of $G - e - e'$. Therefore, $B' \subseteq B$. It follows that $B' - X$, the external part of D', is empty. This is a contradiction. We conclude that v_I lies in $I - X'$. We have seen that $I - X'$ is the internal part of D. Thus, D is nontrivial. Finally, edge e' must have one end in I', otherwise B' would be a nontrivial barrier of G. Thus, the end v_B of e' in B lies in I'. $\square$

In all the subsequent lemmas in this section, we assume that there is an extremal tight cut C' of $G - e'$ such that C and $C' - e$ are crossing tight cuts of $G - e - e'$. We shall adopt the notation in the statement of the above lemma. (Note that the neighbours in $G - e$ of vertices in I are in B and the neighbours in $G - e'$ of vertices in I' are in B'. Since $G - e - e'$ is a subgraph of both $G - e$ and $G - e'$, all neighbours of I in $G - e - e'$ are in B and all neighbours of I' in $G - e - e'$ are in B'.)

Lemma 31. *If e' is of index one, then $r(G - e') > r(G - e)$.*

Proof. Assume that e' is of index one, let C' be the extremal (nontrivial) tight cut of $G - e'$. Then,

$$r(G - e') = |\overline{X'}| + 1 = |V(G)| - (|X'| - 1). \tag{1}$$

By Lemma 26,

$$r(G - e) = r(G - e - e'). \tag{2}$$

If no tight cut of $G - e'$ crosses C then the asserted inequality holds, by Corollary 28. We may thus assume that some tight cut of $G - e'$ crosses C. By Corollary 29, cuts C' and C cross. By Lemma 30, cut D is nontrivial and tight in $G - e - e'$, $X - X'$ is its bipartite shore. Moreover, cut C' is tight in $G - e - e'$, with bipartite shore X'. Therefore,

$$r(G - e - e') \leqslant |V(G)| - (|X'| - 1) - (|X - X'| - 1),$$

whence

$$r(G - e - e') < |V(G)| - (|X'| - 1). \tag{3}$$

From (1)–(3), we conclude that $r(G - e') > r(G - e)$. $\square$

Lemma 32. *Suppose that e is of index two and $r(G - e) = r(G - e')$. Then the maximal nontrivial barrier B_2 of $G\{X \to x\} - e$ is a subset of a barrier of $G - e'$.*

Proof. By hypothesis, $r(G - e) = r(G - e')$. By Corollaries 28 and 29, graph $G - e'$ has an extremal tight cut C' that crosses cut C. By Lemma 31, edge e' has index two.

Let B'_2 denote the maximal nontrivial barrier of $G\{X' \to x'\} - e'$. Note that B_2 and B'_2 are barriers of $G - e - e'$. We prove the required assertion by showing that $B_2 = B' - X$.

We achieve this by first showing that $X'_2 \subseteq X - X'$. By Lemma 30, edge e' has one end v_I in $I - X'$, the other v_B in $B \cap I'$. But e' has an end in I'_2, in turn disjoint with X'. Thus, v_I lies in I'_2 and is therefore adjacent in $G - e'$ only to vertices of B'_2. Hence, this is also true in $G - e - e'$. The nonnull set of neighbours of v_I in $G - e - e'$ lie all in B. Recall that B is a maximal barrier of $G - e - e'$. Therefore, B'_2, which meets B, is a subset of B. Indeed, B'_2 is a subset of $B - X'$.

Let v be a vertex of I'_2. Vertex v is adjacent in $G - e - e'$ to a vertex of $B'_2 \subseteq B$. Thus, v does not lie in B. In $G - e - e' - B'_2$, v is isolated, whence it is isolated in $G - e - e' - B$. Therefore, v lies in I. This conclusion holds for each vertex v in I'_2. Thus, $I'_2 \subseteq I$, whence $X'_2 \subseteq X$. We conclude that $X'_2 \subseteq X - X'$.

We now proceed to show that $X'_2 = X - X'$ and $X_2 = X' - X$. Since $X'_2 \subseteq X - X'$,

$$r(G - e') = 2 + |\overline{X} \cap \overline{X'}| + |X - X' - X'_2| \geqslant 2 + |\overline{X} \cap \overline{X'}|,$$

with equality only if $X_2' = X - X'$. On the other hand, by Lemma 30,

$$r(G-e) \leqslant 2 + |\overline{X} \cap \overline{X'}|,$$

with equality only if $X_2 = X' - X$. By hypothesis, $r(G-e) = r(G-e')$. We conclude that $X_2' = X - X'$ and $X_2 = X' - X$. Finally, B_2 is by definition the external part of X_2 and $B' - X$ the external part of $X' - X$, by Lemma 30. Thus, $B_2 = B' - X$. As asserted, $B_2 \subseteq B'$. □

With the aid of the above lemmas we can now complete the proof of the Main Theorem.

Proof of the Main Theorem. Let G be a brick distinct from K_4, $\overline{C_6}$ and the Petersen graph. Our task is to show that G has a b-invariant edge that is thin. By Theorem 5, graph G has a b-invariant edge. By Lemma 18, every edge that is b-invariant in G is special. By Lemma 20, for every b-invariant edge e of G, graph $G - e$ has index at most two.

Among the b-invariant edges of G, choose one, e, such that the rank of $G - e$ is maximum. If possible, choose e so that its index is two.

We assert that e is thin. For this, assume the contrary. Then, $G - e$ has a (nontrivial special) barrier B_0 that is not thin. Let B denote the maximal nontrivial barrier of $G - e$ that includes B_0. Then B is also special and not thin. Let v denote a vertex of I that has degree greater than two in $G - e$. By Theorem 24, $\nabla(v) - e$ has at least two edges that are b-invariant in $G - e$. Let e' and e'' be any two such edges. By Theorem 24 at least one of e' and e'', say e', is b-invariant in G.

By Lemma 26, $r(G-e') \geqslant r(G-e)$. By the choice of e, $r(G-e') = r(G-e)$. By Corollaries 28 and 29, there is an extremal tight cut C' of $G - e'$ that crosses C. By Lemma 31, edge e' has index two. By the choice of e, edge e also has index two. We deduce now, from Lemma 32, that the maximal nontrivial barrier B_2 of $G\{X \to x\} - e$ is a subset of barrier B' of $G - e'$.

Since e has index two, then, by Theorem 24, e'' is also b-invariant in G. Arguing as above, we may deduce that B_2 is also a subset of a barrier B'' of $G - e''$.

Let x and y be any two vertices in B_2. Since G is a brick, there is a perfect matching M of $G - \{x, y\}$. On the other hand, since B_2 is a subset of B' which is a barrier of $G - e'$, there is no perfect matching in $G - e' - \{x, y\}$. Thus e' must be an edge in M. By an analogous argument, we may conclude that e'' is also in M. This is impossible because e' and e'' are adjacent edges. Hence e must be thin. □

By adapting the above proof and using Theorem 6 one may prove the following theorem.

Theorem 33. *Every brick that is not a Petersen brick and is distinct from K_4 and $\overline{C_6}$ has a b-invariant thin edge e such that the brick of $G - e$ is not a Petersen brick.*

6. Building bricks

In this section we give a precise definition of the four operations alluded to in the Introduction. The first operation simply consists of adding an edge to a given brick H to obtain a new brick G. But the other three operations are more complicated and involve the notion of splitting vertices in graphs. We begin with a definition of this notion.

Let H be a graph and let x be a vertex of H. A graph

$$H' := H\{x \to (b_1, b_2, \ldots, b_k)\}$$

is said to be obtained from H by *splitting x into $b_1, b_2, \ldots, b_k$* if:

(i) $V(H') = V(H - x) \cup \{b_1, b_2, \ldots, b_k\}$,
(ii) $E(H') = E(H)$,
(iii) $H' - \{b_1, b_2, \ldots, b_k\} = H - x$, and
(iv) every edge of H that joins a vertex $v \in V(H - x)$ to x in H joins v to one of the vertices $b_1, b_2, \ldots, b_k$ in H'. (Thus $(\nabla_{H'}(b_1), \nabla_{H'}(b_2), \ldots, \nabla_{H'}(b_k))$ is a partition of $\nabla_H(x)$.)

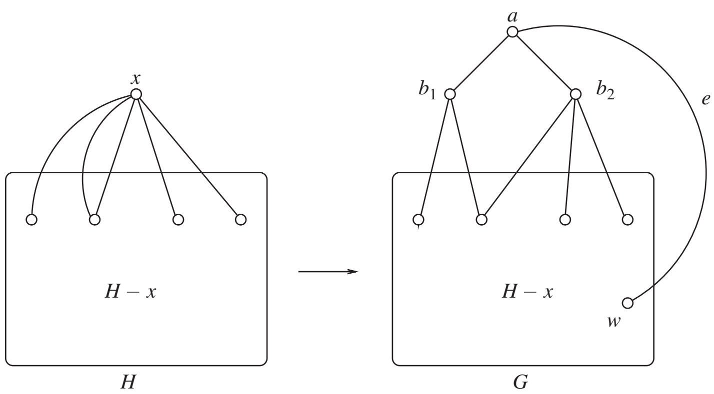


Fig. 9. Expansion of a vertex x by a barrier of size two.

The fourth operation on bricks that we shall define involves splitting two distinct vertices of a brick. Thus, suppose that x and x' are two distinct vertices of a graph H, then $H'' := H\{x \to (b_1, b_2); x' \to (b'_1, b'_2)\}$ is the graph obtained by splitting x into vertices b_1 and b_2 and then splitting x' into vertices b'_1 and b'_2 in the resulting graph.

Let H be a graph and let $H' := H\{x \to (b_1, b_2, \ldots, b_k)\}$ be a graph obtained from H by a splitting a vertex x. Then H can be recovered from H' by identifying the vertices $b_1, b_2, \ldots, b_k$ to form a single vertex x. The following propositions, which will be used in the proof of Theorem 36, are simple consequences of this observation:

Proposition 34. *A graph obtained from a connected graph by splitting a vertex into two vertices has at most two components.*

Proposition 35. *A graph obtained from a* 2*-connected graph by splitting a vertex into* (*any number of*) *vertices of positive degree is connected.*

6.1. The four operations

We are now in a position to define the four operations on bricks that can be used to generate all bricks from the three basic bricks. We shall refer to these four operations as *expansions* of a brick.

Edge addition: Let H be a brick and let x and y be two distinct vertices of H. Obtain G from H by adding a new edge joining x and y.

Expansion of a vertex by a barrier of size two: Let H be a brick and let x be a vertex of H whose degree is at least four. Let $H' := H\{x \to (b_1, b_2)\}$ such that, in the underlying simple graph of H', b_1 and b_2 have degree at least two. Now obtain G from H' by adding a new vertex a and joining it to b_1 and b_2 and to a vertex w of $H - x$ by an edge labelled e (see Fig. 9). We shall refer to the graph G thus constructed as a graph obtained from H by *an expansion of x by a barrier of size two*. We shall see (Theorem 36) that G is a brick for any vertex w of $H - x$.

Expansion of a vertex by a barrier of size three: Let H be a brick and let x be a vertex of H whose degree is at least five. Let $H' := H\{x \to (b_1, b_2, b_3)\}$ such that, in the underlying simple graph of H', the degrees of b_1 and b_3 are at least two and the degree of b_2 is at least one. Now obtain G from H' by adding two new vertices a_1 and a_2, and joining a_1 to b_1 and b_2, joining a_2 to b_2 and b_3, and joining a_1 and a_2 by an edge labelled e (see Fig. 10). We shall refer to the graph G thus constructed as a graph obtained from H by *an expansion of x by a barrier of size three*. We shall see (Theorem 36) that G is a brick.

Expansion of two vertices by barriers of size two: Let H be a brick and let x and x' be two vertices of H whose degrees are at least four. Let $H'' := H\{x \to (b_1, b_2); x' \to (b'_1, b'_2)\}$ such that, in the underlying simple graph of H'', each of b_1, b_2, b'_1 and b'_2 has degree at least two and has at least one neighbour in $V(H - \{x, x'\})$. Now obtain G from H'' by adding two new vertices a and a', joining a to b_1 and b_2, joining a' to b'_1 and b'_2, and joining a and a' by an edge labelled e (see Fig. 11). We shall refer to the graph G thus constructed as a graph obtained from H by *an expansion of x*

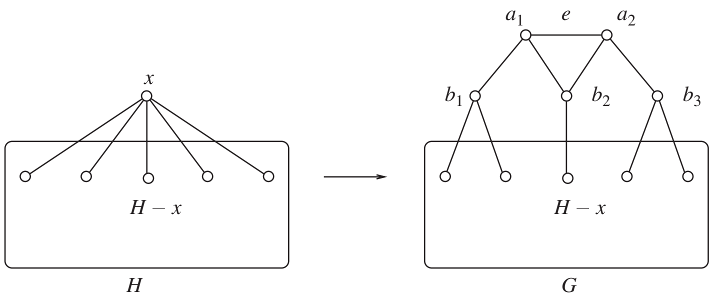


Fig. 10. Expansion of a vertex by a barrier of size three.

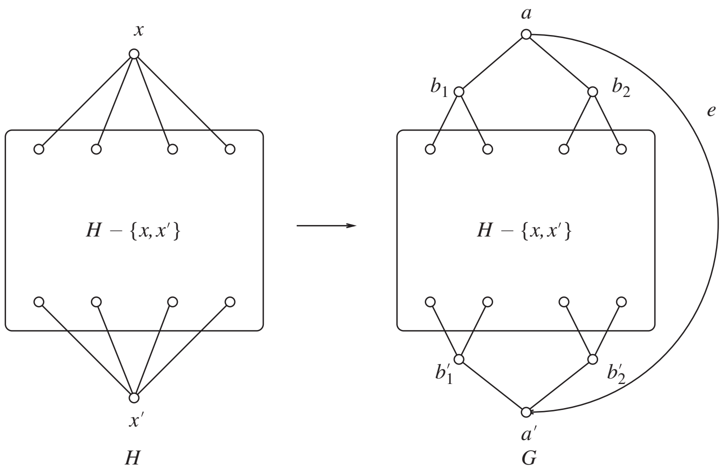


Fig. 11. Expansion of two vertices by barriers of size two.

and x' by barriers of size two. (Note that the order in which the two vertices x and x' are split is immaterial.) We shall see (Theorem 36) that G is a brick.

Theorem 36. *Let H be a brick and let G be a graph obtained from H by one of the four operations of expansion. Then G is a brick.*

Proof. The proof is obvious in the case of the first operation which simply consists of adding an edge to a brick. In each of the other three cases, it is straightforward to verify that $G-e$ is a near-brick. Using the definitions of expansions and the fact that H is a brick, we shall verify that G is bicritical. This, in particular means that G is matching covered and, by Proposition 4, it follows that G is a near-brick. The required assertion now follows because, by Lemma 13, every bicritical near-brick is a brick.

If possible suppose that B is a nontrivial barrier of G. Since $G-e$ is a spanning matching covered subgraph of G, it follows that B is also a barrier of $G-e$. Therefore, by Theorem 7, B cannot contain any pair $\{u, v\}$ of vertices such that $G-e-\{u, v\}$ has a perfect matching. In particular B cannot contain both the ends of any edge of $G-e$.

Throughout the proof we shall implicitly use the fact that, since H is a brick, $H-\{u, v\}$ has a perfect matching for any two vertices u and v of H. We shall also use Theorem 7 without specifically referring to it. Now we proceed to deal with each of the three remaining cases separately.

Expansion of a vertex x by a barrier of size two: See Fig. 9.

For any two vertices u and v of $H-x$ and any perfect matching M of $H-\{u, v\}$, either $M\cup\{b_2a\}$ or $M\cup\{b_1a\}$ is a perfect matching of $G-\{u, v\}$ depending on whether M has an edge incident with b_1 or with b_2. It follows that $|B\cap V(H-x)|\leqslant 1$. We consider two cases depending on the value of $|B\cap V(H-x)|$.

Case 1: $|B \cap V(H-x)| = 0$. In this case, $B \subseteq \{b_1, b_2, a\}$. Since ab_1 and ab_2 are edges of $G-e$, a cannot be in B. Therefore, $B = \{b_1, b_2\}$. However, $G - \{b_1, b_2\}$ is connected. A contradiction.

Case 2: $|B \cap V(H-x)| = 1$. Say $B \cap V(H-x) = \{v\}$. For any perfect matching M of $H - \{x, v\}$, $M \cup \{b_2a\}$ is a perfect matching of $G - \{b_1, v\}$ and $M \cup \{b_1a\}$ is a perfect matching of $G - \{b_2, v\}$. Since v is in B, it follows that $B \cap \{b_1, b_2\} = \emptyset$. So, the only possibility left to be examined is $B = \{a, v\}$. But $G - \{a, v\}$ is connected, a contradiction. We conclude that G is a brick.

Expansion of a vertex x by a barrier of size three: See Fig. 10.

Suppose that u and v are any two vertices of $H-x$ and let M be a perfect matching of $H-\{u, v\}$. Then $M \cup \{b_2a_1, b_3a_2\}$ or $M \cup \{b_1a_1, b_3a_2\}$ or $M \cup \{b_1a_1, b_2a_2\}$ is a perfect matching of $G - \{u, v\}$. It follows that $|B \cap V(H-x)| \leqslant 1$. We consider two cases depending on the value of $|B \cap V(H-x)|$.

Case 1: $|B \cap V(H-x)| = 0$. In this case $B \subseteq \{b_1, b_2, b_3, a_1, a_2\}$. For any perfect matching M_f of H containing an edge f incident with b_3, $M_f \cup \{a_1b_2\}$ is a perfect matching of $G - \{a_2, b_1\}$. Thus, B cannot contain both a_2 and b_1. Similarly, B cannot contain both a_1 and b_3. All other pairs of a_ib_j are edges of $G-e$. Therefore, B is a subset of either $\{a_1, a_2\}$ or $\{b_1, b_2, b_3\}$. $G - \{a_1, a_2\}$ is connected, $G - S$ is connected for any 2-subset S of $\{b_1, b_2, b_3\}$ and $G - \{b_1, b_2, b_3\}$ has two components. Therefore, there are no nontrivial barriers of G contained in $\{b_1, b_2, b_3, a_1, a_2\}$.

Case 2: $|B \cap V(H-x)| = 1$. Say $B \cap V(H-x) = \{v\}$. Let M be a perfect matching of $H - \{x, v\}$. Then, $M \cup \{b_2a_1, b_3a_2\}$ is a perfect matching of $G - \{b_1, v\}$ and $M \cup \{b_1a_1, b_3a_2\}$ is a perfect matching of $G - \{b_2, v\}$ and $M \cup \{b_1a_1, b_2a_2\}$ is a perfect matching of $G - \{b_3, v\}$. It follows that $|B \cap \{b_1, b_2, b_3\}| = 0$. Hence $B \subseteq \{a_1, a_2, v\}$. For any 2-subset S of $\{a_1, a_2, v\}$, $G - S$ is connected. So $B = \{a_1, a_2, v\}$. The graph $G - \{a_1, a_2, v\}$ is the same as the graph $H' - v$. It is obtained from the 2-connected graph $H - v$ by splitting x into three vertices b_1, b_2 and b_3. If all these three vertices have positive degree in $H' - v$ then, by Proposition 35, $H' - v$ is connected. On the other hand, if$v$ is the unique neighbour of b_2 in $V(H-x)$, then only b_1 and b_3 have positive degree in $H' - v$. In either case, $G - \{a_1, a_2, v\}$ has at most two components and hence $\{a_1, a_2, v\}$ is not a barrier of G. We conclude that G is a brick.

Expansion of two vertices x and x′ by barriers of size two: See Fig. 11.

Let u and v be two vertices of $H - \{x, x'\}$ and let M be a perfect matching of $H - \{u, v\}$. Then $M \cup \{b_1a, b_1'a'\}$ or $M \cup \{b_1a, b_2'a'\}$ or $M \cup \{b_2a, b_1'a'\}$ or $M \cup \{b_2a, b_2'a'\}$ is a perfect matching of $G - \{u, v\}$. It follows that $|B \cap V(H - \{x, x'\})| \leqslant 1$. We consider two cases depending on the value of $B \cap V(H - \{x, x'\})$.

Case 1: $|B \cap V(H - \{x, x'\})| = 0$. In this case B is a subset of $\{b_1, b_2, a, b_1', b_2', a'\}$. For $i = 1, 2$ and $j = 1, 2$, any perfect matching M of $H - \{x, x'\}$ can be extended to a perfect matching of $G - \{b_i, b_j'\}$. It now follows that B is a subset of either $\{b_1, b_2, a'\}$ or $\{b_1', b_2', a\}$. Suppose that $B \subseteq \{b_1, b_2, a'\}$ (the other case is similar). For any 2-subset S of $\{b_1, b_2, a'\}$, $G - S$ is connected. So, $B = \{b_1, b_2, a'\}$. In this case, one component of $G - B$ has just a as its vertex. The component $G - B - a$ is obtained from splitting the 2-connected graph $H - x$ into two vertices of positive degree and hence, by Proposition 35, is connected. Thus $G - B$ has two components and B is not a barrier.

Case 2: $|B \cap V(H - \{x, x'\})| = 1$. Say $B \cap V(H - \{x, x'\}) = \{v\}$. For any perfect matching M of $H - \{x, v\}$, $M \cup \{b_2a, b_1'a'\}$ or $M \cup \{b_2a, b_2'a'\}$ is a perfect matching of $G - \{b_1, v\}$. Thus, $b_1 \notin B$. Arguing similarly with other elements of $\{b_1, b_2\}$ and $\{b_1', b_2'\}$ we may now conclude that $B \cap \{b_1, b_2\} = \emptyset$ and $B \cap \{b_1', b_2'\} = \emptyset$. Thus $B \subseteq \{a, a', v\}$. For any 2-subset S of $\{a, a', v\}$, $G - S$ is connected. Therefore, $B = \{a, a', v\}$. The graph $G - \{a, a', v\}$ is obtained from the 2-connected graph $H - v$ by splitting x first into two vertices of positive degree and then, in the resulting graph, splitting x' into two vertices of positive degree. By Propositions 35 and 34, $G - \{a, a', v\}$ has at most two components. Hence $\{a, a', v\}$ is not a barrier of G. We conclude that G is a brick. □

We remark that the above proof also shows that, in every case, the edge e is a thin edge of the brick G and that H is the brick of $G - e$.

As a complement to the above result, we now prove the following theorem that justifies the title of this paper.

Theorem 37. *Every brick different from the three basic bricks can be obtained from one of them by a sequence of applications of the four operations of expansion.*

Proof. By induction on the number of edges. Let G be a brick that is distinct from the three basic bricks. By Theorem 10, there exists an edge e of G that is thin. Let H denote the brick obtained by the tight cut-contractions associated with the thin barriers of $G - e$. Clearly, H has fewer edges than G. By induction, H can be obtained from one of the three basic bricks by a sequence of the four operations of expansion. Thus, to complete the proof, it suffices to show that G

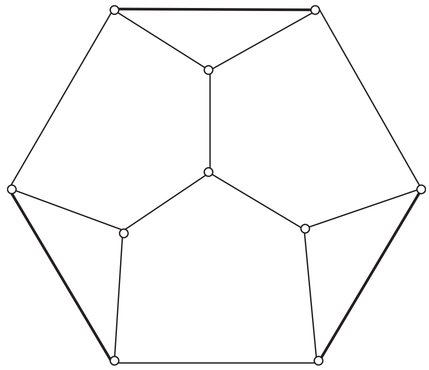


Fig. 12. A brick and its three thin edges.

can be obtained from H by means of an expansion operation. We consider four cases depending on the index and the cardinalities of the nontrivial thin barriers of $G-e$.

- Edge e is of index zero in $G-e$. In this case, $H=G-e$ and G is obtained from H by the addition of an edge.
- Edge e is of index one. Let B be the unique nontrivial thin barrier of $G-e$. Since e is thin, either $|B|=2$ and e joins the vertex of I to a vertex of $V(G)\backslash(B\cup I)$ or $|B|=3$ and e joins the two isolated vertices of $G-e-B$. In the former case, G is obtained from H by an expansion of a vertex by a barrier of size two and, in the latter case, G is obtained from H by an expansion of a vertex by a barrier of size three.
- Edge e is of index two. In this case, $G-e$ has two thin barriers B and B', each of size two, and e joins the vertex of I and the vertex of I' and G is obtained by the expansion of two vertices of H by barriers of size two. $\square$

A natural question that arises at this stage is whether all the four operations are really required for building bricks. We proceed to show that this is indeed the case: (1) First suppose that G is any brick of minimum degree at least four. For any thin edge e of G, the graph $G-e$ is a brick. Thus G can only be obtained from a smaller brick by the addition of an edge. (2) Let G be any odd wheel of order at least five. The only thin edges of G are its spokes, and for any spoke e, the graph $G-e$ has precisely one nontrivial barrier and that barrier is of size two. Thus the only way to obtain G from a smaller brick is by the expansion of a vertex by a barrier of size two. (3) Let G be the graph shown in Fig. 12. It can be shown that the only thin edges of this graph are the three edges indicated by solid lines. If e is any one of these edges, $G-e$ has precisely one maximal nontrivial barrier and that barrier is of size three. It follows that the only way to obtain G from a smaller brick is by the expansion of a vertex by a barrier of size three. (4) Finally, let G be any triangle-free cubic brick distinct from the Petersen graph. Clearly, for any thin edge e of G, the graph $G-e$ must have two maximal nontrivial barriers, both of size two. It follows that the only way to obtain G from a smaller brick is by expansion of two vertices by barriers of size two.

We conclude this section with the observation that, in view of Theorem 33, every brick that is not a Petersen brick may be obtained from K_4 and $\overline{C_6}$ by a sequence of applications of the four operations of expansion.

7. Solid bricks

A matching covered graph G is *solid* if the only separating cuts it has are tight cuts. It can be shown that every bipartite matching covered graph is solid. However, not every nonbipartite matching covered graph is solid. For example, both $\overline{C_6}$ and P have separating cuts that are not tight.

Analogous to a tight cut decomposition one may define a separating cut decomposition of a matching covered graph. Thus one may obtain any matching covered graph by splicing together graphs that are free of nontrivial separating cuts. The bipartite matching covered graphs that have no nontrivial separating cuts are braces. Nonbipartite matching covered graphs that have no nontrivial separating cuts are solid bricks.

Solid bricks have many interesting properties. (For example, every removable edge in a solid brick is also b-invariant, see [1].) There is also an interesting interpretation of solid bricks in terms of their perfect matching polytopes. For any graph G, Edmonds showed that the perfect matching polytope of G is the set of solutions to: (i) $x \geqslant 0$, (ii) $x(\nabla(v)) = 1$, for all $v \in V$, and (iii) $x(\nabla(S)) \geqslant 1$, for all odd subsets S of V. It is well-known that, when G is bipartite, this polytope may be defined just in terms of the first two conditions. We showed that when G is a brick, (i) and (ii) imply (iii) if and only if G is solid [4]. This is related to the result, first proved by Lucchesi [10], which shows that every solid r-graph is r-edge colourable. (Seymour [12] conjectured that every r-graph that does not have the Petersen graph as a minor is r-edge colourable.)

Although separating cut decompositions do not have many of the nice properties that tight cut decompositions have, in view of the results cited above, we believe that an understanding of the structure of solid bricks would be helpful in the resolution of many important problems concerning bricks. Unfortunately, we do not even know if the problem of deciding whether a given brick is solid is in $\mathcal{NP}$.

The following proposition provides a simple necessary condition for a separating cut in a matching covered graph to be not tight.

Proposition 38 (*See de Carvalho et al. [1]*). *Let $C = \nabla(X)$ be a separating cut in a matching covered graph G that is not tight. Then both the shores $G[X]$ and $G[\overline{X}]$ of C are nonbipartite.*

A graph G is *odd-intercyclic* if any two odd circuits of G have at least one vertex in common. It follows from the above proposition that every odd-intercyclic brick is solid. Odd wheels, described below, are an infinite class of odd-intercyclic bricks. (See [1] for a description of odd-intercyclic graphs.)

A *wheel* is a simple graph obtained from a circuit by adding a new vertex and joining that vertex to each vertex of the circuit; the circuit is called the *rim*, the new vertex the *hub* and each edge joining the hub to the rim a *spoke*. The *order* of the wheel is the number of vertices of its rim; a wheel of order n is denoted W_n. A wheel is *even* or *odd*, according to the parity of n. Note that the hub of a wheel is uniquely identified, except for W_3, which is K_4, the complete graph on four vertices; in this case, we may take any of its vertices to be its hub. It is easy to see that every odd wheel is a brick and that it is odd-intercyclic. Thus:

Proposition 39. *Every odd wheel is a solid brick.*

A matching covered graph G is *extremal* if the number of perfect matchings in G is equal to the dimension of the matching lattice of G. In [5] we showed that every simple extremal solid brick is an odd wheel and used that result to give a characterization of all extremal bricks. In this section, using Theorem 37, we shall show that every simple planar solid brick is an odd wheel. The following two theorems play a useful role in establishing this fact.

Theorem 40 (*See de Carvalho et al. [1]*). *A matching covered graph G is solid if and only if each of its bricks is solid. (Conversely, if one of the bricks of G is nonsolid, then G is nonsolid.)*

Theorem 41 (*See de Carvalho et al. [1]*). *Let G be a solid brick and let e be a removable edge of G. Then e is b-invariant and the brick of $G - e$ is also solid.*

We shall also need the following propositions that show that certain graphs that can be obtained from odd wheels are nonsolid.

Proposition 42. *Let W be an odd wheel of order five or more and let u and v be two nonadjacent vertices on its rim. Then $W + uv$ is nonsolid.*

Proof. Since the rim of W is an odd circuit, it consists of two internally disjoint paths connecting u and v, where exactly one of them, say P, has an odd number of vertices. Let $C := \nabla(V(P))$. Then, both C-contractions of $W + uv$ are odd wheels (up to multiple edges). As W is a brick, C is not tight in W. Then C is not tight in $W + uv$ either and so, $W + uv$ is nonsolid. $\square$

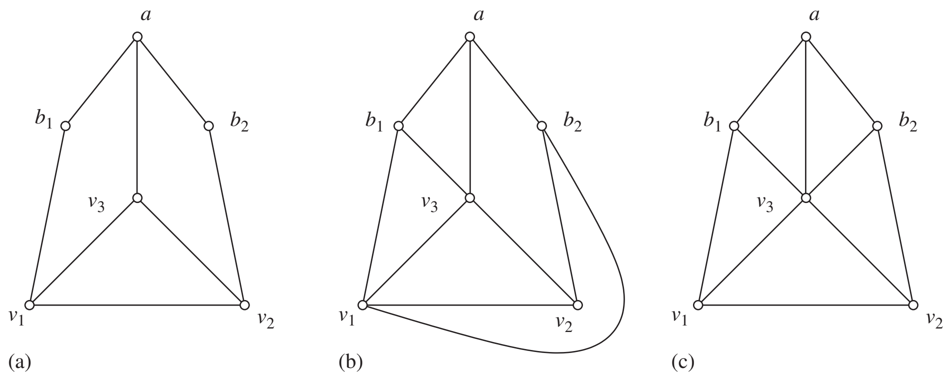


Fig. 13. An unavoidable subgraph of a solid brick on six vertices.

By Theorem 41, every supergraph of a nonsolid brick is nonsolid. Thus, it follows from the above proposition that every proper simple supergraph of an odd wheel is nonsolid. The proof of the following proposition is similar to the above proof. It uses the fact that any spoke of an odd wheel of order five or more is removable.

Proposition 43. *Let W be an odd wheel of order five or more and let u and v be two nonadjacent vertices on its rim. Then $W + uv - uh$, where h is the hub of W, is nonsolid.*

As a warm-up to the characterization of planar solid bricks, we shall first present a characterization of solid bricks on eight or fewer vertices.

7.1. Small solid bricks

The odd wheel $W_3 = K_4$ is the only simple brick on four vertices. Thus, W_3 is the only simple solid brick on four vertices. There are several simple bricks on six vertices, but only one of them is solid, namely W_5. The following theorem is clearly equivalent to the assertion that W_5 is the only simple solid brick on six vertices.

Theorem 44. *Let G be a simple brick on six vertices. Then G is either nonsolid or is the* 5*-wheel W_5.*

Proof. By induction on $|E|$. Since the minimum degree of G is at least three, $|E|$ is at least nine. The only brick on six vertices and nine edges is $\overline{C_6}$. Since $\overline{C_6}$ is nonsolid, the statement is true when $|E| = 9$. So, we may assume that $|E| \geqslant 10$.

Clearly G is not one of the three basic bricks. Therefore, by the Main Theorem, G has thin edges. We shall consider two cases depending on whether or not G has a thin edge of index zero.

Case 1: G has thin edges of index zero. Let e be a thin edge of G of index zero. Then, by definition, $G - e$ is a brick. By induction, $G - e$ is either nonsolid or is W_5. If $G - e$ is nonsolid then, by Theorem 41, G is also nonsolid. On the other hand, if $G - e = W_5$, by Proposition 42, G is nonsolid and the assertion holds.

Case 2: G has no thin edge of index zero. Let e be a thin edge of G of positive index. Since $|V| = 6$, it is easy to see that the index of e must be one and that the nontrivial barrier of $G - e$ has cardinality two. Let b_1, b_2, a, v_1, v_2, and v_3 denote the vertices of G, where $B := \{b_1, b_2\}$ is the unique nontrivial barrier of $G - e$, a is the isolated vertex of $G - e - B$ and $\{v_1, v_2, v_3\}$ is the vertex set of the nontrivial component of $G - e - B$. Clearly the underlying simple graph of the brick of $G - e$ is K_4. Thus, the subgraph of G induced by $\{v_1, v_2, v_3\}$ is K_3. Since G is a brick, all its vertices have degree at least three. Therefore a must have a neighbour distinct from b_1 and b_2. We may assume without loss of generality that a is joined to v_3. A perfect matching of G containing the edge av_3 matches $\{b_1, b_2\}$ with $\{v_1, v_2\}$. So, we may assume without loss of generality that b_1 is joined to v_1 and b_2 is joined to v_2. Thus G must contain the graph shown in Fig. 13(a) as a subgraph.

Since $\{b_1, b_2\}$ is a barrier of $G - e$, b_1b_2 is not an edge of G. Therefore, a perfect matching of G containing the edge v_1v_2 must contain an edge joining v_3 to one of b_1 and b_2. Thus, we may assume without loss of generality

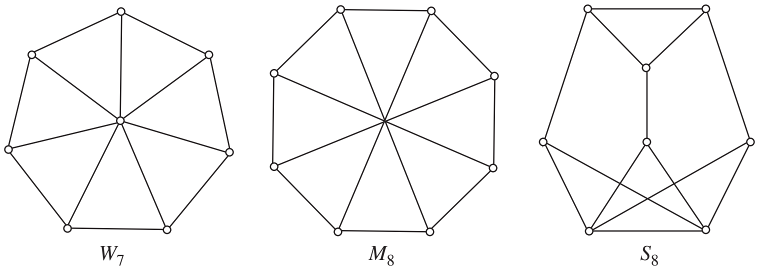


Fig. 14. Minimal solid bricks on eight vertices.

that b_1 is joined to v_3 in G. Since b_2 must have degree at least three in G, it must be joined to either v_3 or v_1 or both.

If b_2 is joined to v_1, then $\nabla(\{a, b_1, v_3\})$ would be a separating cut of G (Fig. 13(b)). This is impossible because G is solid. Therefore, b_2 must be joined to v_3 and G contains W_5 with v_3 as its hub (Fig. 13(c)). If G has any additional edges, they would be thin edges of index zero. By the hypothesis of the case, G has no thin edges of index zero. Therefore, it follows that $G = W_5$. □

Now we proceed to describe solid bricks on eight vertices. Fig. 14 shows three bricks on eight vertices. The first of these is the odd wheel W_7. The second is the Möbius ladder M_8. Bricks W_7 and M_8 are odd-intercyclic and hence are solid. The third graph S_8 is the smallest example of a solid brick that is not odd-intercyclic.

No simple graph on eight vertices that properly contains either W_7 or S_8 is solid, but there are simple solid bricks that can be obtained by adding edges to M_8. However, the following theorem shows that these three graphs are the only minimal solid bricks on eight vertices. (A brick G is *minimal* if it has no edge e such that $G - e$ is a brick. Thus, every thin edge of a minimal brick has index greater than zero.)

Theorem 45. *Every minimal solid brick on eight vertices is isomorphic to one of the three bricks shown in Fig.* 14.

Proof. Let G be a minimal solid brick on eight vertices. Clearly, G is distinct from the three basic bricks. Therefore, by the Main Theorem, G has a thin edge e. By minimality, the index of e is greater than zero. There are three possible cases depending on the cardinalities and the number of nontrivial maximal barriers of $G - e$. We analyse these separately.

Case 1: The index of e is two. In this case, $G - e$ has two barriers of size two and the underlying simple graph of the brick of $G - e$ is K_4. Let $B := \{b_1, b_2\}$ and $B' := \{b'_1, b'_2\}$ be the two nontrivial barriers of $G - e$, a_1 and a_2 be the isolated vertices of $G - e - B$ and $G - e - B'$, respectively, and let v_1 and v_2 be the remaining two vertices of G. As v_1 and v_2 are vertices of the underlying simple brick K_4 of $G - e$, it follows that v_1 and v_2 are adjacent in G. A perfect matching of $G - B'$ must match a_1 with a_2 and the vertices of B with $\{v_1, v_2\}$. Similarly, a perfect matching of $G - B$ must match a_1 with a_2 and the vertices of B' with $\{v_1, v_2\}$. Thus, by renaming the vertices, if necessary, we may assume that the graph H shown in Fig. 15 is a subgraph of G.

As G is a brick, all vertices have degree at least three in G. Thus there must be edges incident with the vertices of B and B' in addition to the edges shown in Fig. 15. Since G is simple, it has no multiple edges and since B and B' are barriers of $G - e$, b_1b_2 and $b'_1b'_2$ cannot be edges. However, since the brick of $G - e$ contains K_4, there must be at least one edge joining B and B'. Now we consider various possibilities.

First let us show that $b_1b'_1$ cannot be an edge of G. For, suppose that $b_1b'_1$ is an edge of G. In this case, regardless of where the additional edges incident with b_2 and b'_2 are, it is easy to see that $\nabla(\{v_1, b_1, b'_1\})$ would be a nontrivial separating cut of G. This is impossible because G is solid. Therefore $b_1b'_1$ cannot be an edge of G. Similarly, $b_2b'_2$ cannot be an edge of G.

As noted above, there must be at least one edge joining B and B'. Thus, either $b_1b'_2$ or b'_1b_2 is an edge of G. Using the symmetries of H, we may assume that $b_1b'_2$ is an edge of G. Suppose that b'_1b_2 is not an edge of G. Then, in order for b'_1 and b_2 to have degree at least three, both b'_1v_2 and b_2v_1 must be edges of G. It can be seen that $H - v_1v_2 + b_1b'_2 + b'_1v_2 + b_2v_1$ is isomorphic to M_8 and hence $G - v_1v_2$ is a brick. This is not possible by the hypothesis. Therefore, b'_1b_2 must be an

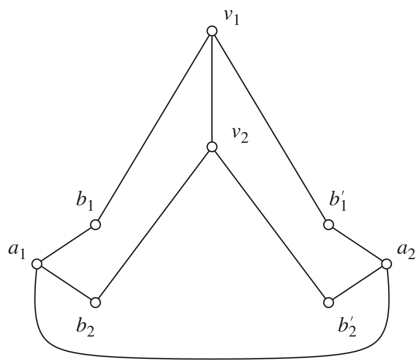


Fig. 15. A subgraph H of G when e is of index two.

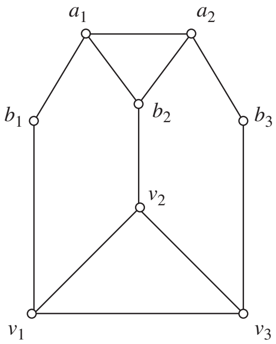


Fig. 16. A subgraph H of G when $G-e$ has one barrier of size three.

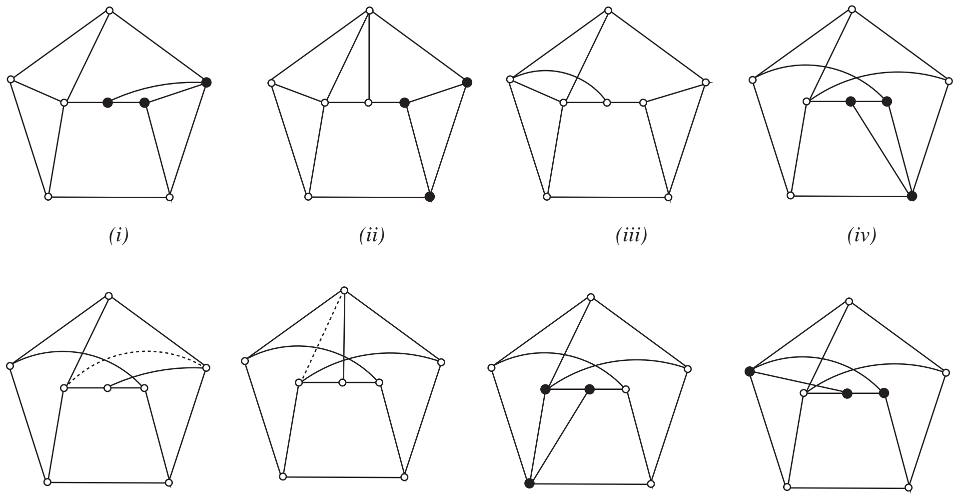
edge of G. The graph $H+b_1b_2'+b_1'b_2$ is isomorphic to M_8. As $H+b_1b_2'+b_1'b_2$ is a spanning subgraph of G, by the minimality of G it follows that G is isomorphic to M_8.

Case 2: The index of e is one and the maximal nontrivial barrier of $G-e$ has cardinality three. Let $B:=\{b_1,b_2,b_3\}$ denote the barrier of $G-e$ with $I:=\{a_1,a_2\}$. Let v_1, v_2 and v_3 denote the three vertices that are not in $B\cup I$. Since the underlying simple graph of the brick of $G-e$ must be K_4, it follows that v_1v_2, v_2v_3 and v_1v_3 are edges of G. Furthermore, any perfect matching of G containing the edge a_1a_2 must match the vertices of B with the vertices in $\{v_1,v_2,v_3\}$. Thus, without loss of generality, we may assume that the graph H shown in Fig. 16 is a subgraph of G.

Note that all vertices of G must have degree at least three and that no edge can join two vertices of B or a vertex of I to a vertex that is not in $B\cup I$. Moreover, as e is thin, both ends of e must have degree two in $G-e$, whence b_1 is not joined to a_2. Therefore, we may assume that b_1 is joined to v_2 or v_3.

We now prove that b_1 is not joined to v_2. For this, assume the contrary. We consider two cases, depending on whether or not b_3 is also joined to v_2. Consider first the case in which b_3 is also adjacent to v_2. Let $C:=\nabla_G(\{a_1,a_2,b_2\})$. One of the C-contractions of G is K_4, up to multiple edges. The other C-contraction of G has W_5 with hub v_2 as a spanning subgraph. But W_5 is a brick. By Corollary 9, both C-contractions of G are bricks, whence C is a nontrivial separating cut of G. This is a contradiction to the hypothesis that G is solid. Consider last the case in which b_3 is not joined to v_2. In that case, b_3 and v_1 are adjacent. Let $D:=\nabla_G(\{b_3,v_1,v_3\})$. One of the D-contractions of G is K_4, up to multiple edges. The other D-contraction of G has $\overline{C_6}$ as a spanning subgraph. But $\overline{C_6}$ is a brick. By Corollary 9, both D-contractions of G are bricks, whence D is a nontrivial separating cut of G. This is a contradiction to the hypothesis that G is solid. In both cases we derived a contradiction. As asserted, b_1 is not joined to v_2.

We conclude that vertices b_1 and v_3 are adjacent in G. Likewise, vertices b_3 and v_1 are adjacent in G. Thus, G has $G':=H+b_1v_3+b_3v_1$ as a spanning subgraph. But G' is S_8. By the minimality of G and Corollary 9, it follows that G itself is S_8.



(i) *(ii)* *(iii)* *(iv)*

(v) *(vi)* *(vii)* *(viii)*

Fig. 17. The case in which x is the hub of G'.

Case 3: The index of e is one and the nontrivial barrier of $G-e$ has cardinality two. Let $B:=\{b_1,b_2\}$ denote the nontrivial barrier of $G-e$ and let a denote the isolated vertex of $G-e-B$. Since e is of index one, $G':=(G-e)\{\{b_1,b_2,a\}\to x\}$ must be a brick. Since G is solid, G' is solid. By Theorem 44, the underlying simple graph of G' is the odd wheel W_5. Let h denote the hub of G' and let $C:=(v_0,v_1,v_2,v_3,v_4,v_0)$ denote its rim. We consider two subcases depending on whether x is the hub of G' or a vertex on the rim of G'.

Subcase: *Vertex x is the hub of G'*. In G', x is adjacent to each v_i. Therefore in G, each v_i must be adjacent to at least one of b_1 and b_2. Furthermore, since the degrees of b_1 and b_2 must be at least three, each of them must be adjacent to at least two distinct v_i. Therefore, we may assume that there exists a partition of $V(C)$ into two parts S_1 and S_2 with $|S_1|=3$, $|S_2|=2$, such that b_1 is adjacent to all vertices in S_1 and b_2 is adjacent to all vertices in S_2. We consider two cases depending on whether or not S_1 consists of three consecutive vertices of C. In the first case, we may assume that b_1 is adjacent to v_1, v_2 and v_3 and b_2 is adjacent to v_4 and v_5. Edge e joins a to one of the vertices of C. Thus, in this case, graph G must be isomorphic to one of the three graphs (i)–(iii) in Fig. 17. In the alternative case, where the vertices of S_1 are not consecutive vertices of C, G must be isomorphic to one of the five graphs (iv)–(viii) shown in Fig. 17.

The graph labelled (iii) in Fig. 17 is isomorphic to S_8. By deleting the edges indicated in dotted lines in the graphs labelled (v) and (vi) in Fig. 17, we obtain graphs isomorphic to M_8. In each other case, the graph has a separating cut consisting of the coboundary of the set of the three vertices indicated by solid dots.

Subcase: *Vertex x is on the rim of G'*. Suppose that $x=v_0$. Note that the only neighbours of x in G' are v_1, v_4 and h. Therefore, a perfect matching of $G-\{h,a\}$ must match the vertices in $\{b_1,b_2\}$ with those in $\{v_1,v_4\}$. Thus, we may assume without loss of generality that b_1v_1 and b_2v_4 are edges of G. Furthermore, since x is adjacent to h in G', in G, at least one of b_1, b_2 must be adjacent to h. Assume without loss of generality that b_1 is adjacent to h. Also, since b_2 has degree at least three in G, either it is also adjacent to h or it is adjacent to v_1. The edge e joins a to h or to one of the vertices v_1, v_2, v_3 and v_4 on the rim of C. Thus, it can be seen that, up to isomorphism, G must contain one of the eight graphs shown in Fig. 18 as a subgraph. The first graph in the list is the odd wheel W_7. The graph obtained by deleting the edges indicated by dotted lines from the sixth graph is isomorphic to M_8. In all other cases, the coboundary of the set of vertices indicated by solid dots is a separating cut. $\square$

7.2. Planar solid bricks

Here we shall prove that odd wheels are the only simple planar solid bricks. The following simple facts can be gathered from the proofs of the theorems concerning solid bricks of order six and eight.

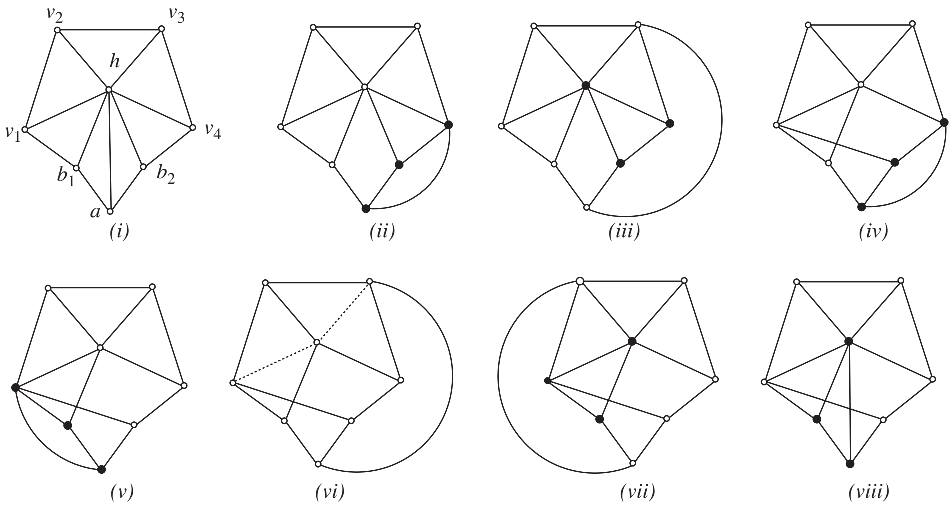


Fig. 18. The case in which x is on the rim of G'.

Lemma 46. *No planar brick that is obtained by an expansion of the hub of* W_5 (*up to multiple edges*) *by a barrier of size two is solid.*

Lemma 47. *No planar brick that is obtained by an expansion of a vertex of* W_3 (*up to multiple edges*) *by a barrier of size three is solid.*

Lemma 48. *No planar brick that is obtained by an expansion of two vertices of* W_3 (*up to multiple edges*) *by barriers of size two is solid.*

More generally, it turns out that a simple planar solid brick G with $|V(G)| \geqslant 6$ can only be built from an odd wheel W on $|V(G)|-2$ vertices by means of an expansion of a vertex on the rim of W by a barrier of size two. In the remainder of this section, we shall let W denote an odd wheel (up to multiple edges) with hub h and rim $R := (v_0, v_1, v_2, \ldots, v_{2n}, v_0)$.

Lemma 49. *Let G be a planar brick obtained from an odd wheel W*, *of order five or more*, *by the expansion of the hub h by a barrier of size two*. *Then G is nonsolid.*

Proof. By induction on $|E|$. If G has any multiple edges, we may delete a multiple edge and apply induction. Therefore, we may suppose that G is simple. Also, by Lemma 46, the statement is true when W has order five. Hence we may assume that the order of W is at least seven.

Since every vertex on the rim is adjacent to the hub h in W, every vertex on the rim is adjacent to at least one of b_1, b_2 in G. Thus, as the rim has at least seven vertices, we may assume without loss of generality that b_1 has at least four neighbours on the rim. Also, by the definition of the expansion under consideration, both b_1 and b_2 have at least two distinct neighbours apart from a. Suppose first that there is a vertex v_i on the rim that is adjacent to both b_1 and b_2 in G. This means that the edges $e_1 := b_1 v_i$ and $e_2 := b_2 v_i$ of G correspond to multiple edges of W incident with h. After the deletion of e_1, both b_1 and b_2 still have at least two distinct neighbours apart from a. Thus $G - e_1$ is obtained from $W - e_1$ by the expansion of h by a barrier of size two. By the induction hypothesis, $G - e_1$ is nonsolid and hence, by Theorem 41, G is also nonsolid. Thus we may suppose that every vertex on the rim is adjacent to precisely one of b_1 and b_2.

Let M be a perfect matching of G containing e. Then M matches the vertices in $\{b_1, b_2, a\}$ with three vertices on the rim of W. Thus there is a spoke f of W incident with b_1 that does not belong to M. The cut $C := \nabla(\{b_1, b_2, a\})$ is a tight cut of $G - e$. One of the C-contractions of $G - e$ is W and f is removable in W. The other C-contraction of $G - e$ is a brace on four vertices and f is a multiple edge in it. Thus, f is removable in both the C-contractions of $G - e$ and, therefore, is removable in $G-e$. But since f does not belong to the perfect matching M, it follows that f is removable in G itself.

Suppose that v_i is the end of f on the rim of W. It is adjacent to the vertices v_{i-1} and v_{i+1} on the rim and to b_1 and no other vertices. Thus, in $G-f$ the vertex v_i has degree two. Let $Y := \{v_{i-1}, v_i, v_{i+1}\}$. The graph $W' := (W-f)\{Y \to y\}$ is an odd wheel (up to multiple edges) of order two less than that of W and $G' := (G-f)\{Y \to y\}$ is obtained from W' by the expansion of h by a barrier of size two. (Since b_1 has at least four distinct neighbours in G, it can be seen that in G' both b_1 and b_2 have at least two distinct neighbours on the rim of W'.) It follows from Theorem 36 that G' is a brick. Furthermore, since G is planar, G' is planar. By the induction hypothesis, G' is nonsolid. As G' is clearly the only brick of $G-f$, it follows from Theorem 40 that $G-f$ is nonsolid. Finally, it follows from Theorem 41 that G itself is nonsolid. □

In Lemma 50 we shall consider a planar brick G that is obtained by the expansion of a vertex of an odd wheel W by a barrier of size three and in Lemma 51 a planar brick that is obtained by an expansion of two vertices of W by barriers of size two. In the proofs of these lemmas, under certain conditions, it is possible to directly find a separating cut in G and deduce that it is nonsolid. In other cases, we find a suitable spoke f of W that is removable in G and apply induction. The proofs in these cases are similar to the proof of the above lemma.

Lemma 50. *Let G be a planar brick obtained from an odd wheel W by the expansion of a vertex x by a barrier of size three. Then G is nonsolid.*

Proof. By induction on the order of $|E|$. If G has any multiple edges, we may delete a multiple edge and apply induction. Therefore, we may suppose that G is simple. Also, by Lemma 47, the statement is true if W has order three. So, we may assume that the order of W is at least five. We shall divide the proof into two cases depending on whether the vertex x is a vertex on the rim of W or is the hub h. (In the former case, we shall take x to be v_0.)

Case 1: $x=v_0$. Let M be a perfect matching of G containing the edge e. Then, M matches $\{b_1, b_2, b_3\}$ with $\{v_1, h, v_{2n}\}$. The rest of the edges of M are from the rim of W. Let f be a spoke of W joining h to v_2. As in the proof of Lemma 49, it follows that f is removable in G. As G is simple, f is not a multiple edge. Thus, v_2 has degree two in $G-f$ and $\nabla(Y)$, where $Y=\{v_1, v_2, v_3\}$, is a tight cut of $G-f$. Since all neighbours of b_1, b_2 and b_3 in G are in the set $\{v_1, h, v_{2n}\}$, the numbers of neighbours of these vertices in $G' := (G-f)\{Y \to y\}$ are the same as those in G. Hence G' may be regarded as an expansion of the vertex v_0 on the rim of $(W-f)\{Y \to y\}$ by a barrier of size three. By the induction hypothesis, G' is nonsolid. As in the proof of Lemma 49, it follows that G is nonsolid.

Case 2: $x=h$. In this case, it is straightforward to verify that $\nabla(\{a_1, a_2, b_2\})$ is a separating cut of G. Hence G is nonsolid. □

Lemma 51. *Let G be a planar brick obtained from an odd wheel W by the expansion of two vertices x and x′ by barriers of size two. Then G is nonsolid.*

Proof. By induction on $|E|$. As in the proof of the previous lemma, we may suppose that G is simple and that the order of W is at least five. We shall consider three different cases depending on the locations of the vertices x and x' in W.

Case 1: One of the two vertices x and x' is the hub $x := h$ of W. We may assume, without loss of generality, that G is obtained from W by the expansion of the hub h by $B := \{b_1, b_2\}$ and the vertex $x' := v_0$ on the rim by $B' := \{b_1', b_2'\}$. Using the notation in Section 6, G has an edge e joining a and a' where a is an isolated vertex of $G-B$ and a' is an isolated vertex of $G-B'$. A perfect matching of $G-B$ matches the vertices of B' with the vertices in $\{v_1, v_{2n}\}$. Thus, we may assume without loss of generality that $v_1 b_1'$ and $b_2' v_{2n}$ are edges of G. Furthermore, there is an edge in W joining h and v_0. Thus, there must be at least one edge of G joining a vertex in B to a vertex in B'. Thus, we may assume without loss of generality that $b_1 b_1'$ is an edge of G.

We may extend the rim R of W to the circuit $C := (a', b_1', v_1, v_2, \ldots, v_{2n}, b_2', a')$. By planarity, the neighbours of b_1 and b_2 on C must consist of nonoverlapping segments of that circuit. Each of these segments must have at least two vertices. Let S_1 and S_2 denote the segments of C, consisting of the neighbours of b_1 and b_2, respectively. Vertices b_1' and v_1 are certainly in S_1 and v_{2n} is certainly in S_2. Vertex b_2' may or may not belong to S_2. If b_2' belongs to S_2, then $b_2 b_2'$ is an edge of G and $M := \{b_1 b_1', aa', b_2 b_2', v_1 v_2, \ldots, v_{2n-1} v_{2n}\}$ is a perfect matching of

G containing $e = aa'$. On the other hand, if b_2' does not belong to S_2, it must be adjacent to v_1 in G. In this case, $M := \{b_1b_1', aa', b_2'v_1, b_2v_{2n}, v_2v_3, \ldots, v_{2n-2}v_{2n-1}\}$ is a perfect matching of G containing $e = aa'$.

If the vertex v_2 is in both S_1 and S_2, then there are two spokes of W incident with v_2. Let $f := b_1v_2$. In $G - f$, each of the four vertices in $B \cup B'$ has at least two neighbours not in $\{a, a'\}$ and at least one neighbour in $W - \{h, v_0\}$. Thus $G - f$ may be regarded as an expansion of $W - f$ at h and v_0 by B and B', respectively, and induction can be applied. So, suppose that there is only one spoke, say f, of W incident with v_2. Then consider the graph $G' : (G - f)\{Y \to y\}$ and the odd wheel $W' := (W - f)\{Y \to y\}$, where $Y = \{v_1, v_2, v_3\}$. It can be verified that G' may be regarded as an expansion of W' at h and v_0 by B and B'. By the induction hypothesis G' is nonsolid. As in the proof of Lemma 49, it follows that G is nonsolid.

Case 2: $x = v_i$ and $x' = v_j$ are nonadjacent vertices on the rim of W. A perfect matching of $G - \{h, a\}$ matches the vertices of B with the vertices in $\{v_{i-1}, v_{i+1}\}$ and a perfect matching of $G - \{h, a'\}$ matches the vertices of B' with the vertices in $\{v_{j-1}, v_{j+1}\}$. Therefore, we may assume without loss of generality that

$$C := (b_1, a, b_2, v_{i+1}, \ldots, v_{j-1}, b_1', a', b_2', v_{j+1}, \ldots, v_{i-1}, b_1)$$

is a circuit of G. The edge $e = aa'$ and the edges incident with h belong to two overlapping bridges of C. Hence they must belong to different residual domains of the Jordan curve C. We may assume that h is in the interior and e is the exterior of C. Using planarity and the facts that $N_W(x) = \{v_{i-1}, h, v_{i+1}\}$ and $N_W(x') = \{v_{j-1}, h, v_{j+1}\}$, it is now easy to deduce that $N_G(b_1) = \{v_{i-1}, a, h\}$, $N_G(b_2) = \{v_{i+1}, a, h\}$, $N_G(b_1') = \{v_{j-1}, a', h\}$ and $N_G(b_2') = \{v_{j+1}, a', h\}$.

Since C is an odd circuit, one of the two components, say P, of $C - \{a, a'\}$ is even. It can be verified that both $\nabla(P \cup \{h\})$-contractions of G are odd wheels (up to multiple edges). It follows that G is nonsolid.

Case 3: x and x' are adjacent vertices on the rim of W. We may adjust notation so that $x = v_0$ and $x' = v_1$. A perfect matching of $G - B$ matches the vertices of B' with the two vertices in $\{h, v_2\}$ and a perfect matching of $G - B'$ matches the vertices of B with the two vertices in $\{h, v_{2n}\}$. So, we may assume without loss of generality that $v_{2n}b_1$, b_2h, $b_1'h$ and $b_2'v_2$ are edges of G. Thus

$$C := (b_1, a, a', b_2', v_2, v_3, \ldots, v_{2n}, b_1)$$

is a circuit of G. The circuit C together with the spokes $hv_2, hv_3, \ldots,$ and hv_{2n} and the two paths (h, b_2, a) and (h, b_1', a') is a subdivision of an odd wheel and so has a unique embedding in the plane. Now, using planarity, we may deduce that b_1b_1' and b_2b_2' cannot be edges of G. Since v_0 and v_1 are adjacent in W, there must be at least one edge in G between B and B'. Thus, we may conclude that the set of edges between B and B' is a nonempty subset of $\{b_1b_2', b_2b_1'\}$. If both b_1b_2' and b_2b_1' are edges of G, then

$$\{e, b_1b_2', b_2b_1', v_2h, v_3v_4, \ldots, v_{2n-1}v_{2n}\}$$

is a perfect matching of G containing the edge e. If b_1b_2' is an edge and b_2b_1', is not an edge, then b_2v_{2n} and $b_1'v_2$ will have to be edges and

$$\{e, b_1b_2', b_1'v_2, b_2h, v_3v_4, \ldots, v_{2n-1}v_{2n}\}$$

is a perfect matching of G. If b_2b_1' is an edge and b_1b_2', is not an edge, then b_1h and $b_2'h$ will have to be edges and $\{e, b_2b_1', b_1h, b_2'v_2, v_3v_4, \ldots, v_{2n-1}v_{2n}\}$ is a perfect matching of G. In every case, one can easily verify that the spoke $f = hv_3$ can be deleted and induction applied to deduce that G is nonsolid. □

Theorem 52. *Every simple solid planar brick G is an odd wheel.*

Proof. By induction on $|E|$. If G has four vertices, the theorem is clearly true. Therefore, we may assume that $|V| \geqslant 6$. Thus, clearly, G is distinct from the three basic bricks. Therefore, by Theorem 36, G is obtained by applying one of the four operations of expansion to a planar solid brick G' with $|E(G')| < |E(G)|$. By the induction hypothesis, the underlying simple graph of G' is an odd wheel. To conform to the notation we have been using, let us set $W := G'$.

From Proposition 42 and Lemmas 49–51, it follows that G is obtained by expanding a vertex on the rim of W by a barrier of size two. Let us take that vertex x to be v_0. A perfect matching of $G - \{a, h\}$ matches the vertices of B with $\{v_1, v_{2n}\}$. Without loss of generality, we may assume that v_1b_1 and b_2v_{2n} are edges of G. Since x and h are adjacent in W, at least one of b_1 and b_2 must be adjacent to h in G. Assume without loss of generality that b_1h is an edge of G. We shall consider two separate cases depending on whether or not b_2h is also an edge of G.

Case 1: $b_2h \notin E(G)$. In this case b_2v_1 must be an edge of G. Then, by planarity, edge e must join a to either h or to v_{2n}. If ah is an edge, then G is nonsolid by Proposition 43. If av_{2n} is an edge, then $\nabla(\{a, b_2, v_{2n}\})$ is a separating cut of G and hence G is nonsolid.

Case 2: $b_2h \in E(G)$. In this case, if e joins a to a vertex other than h, then G has a separating cut by Proposition 43. Thus, we may assume that ah is an edge. In this case G is an odd wheel. □

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