ELSEVIER

Available online at www.sciencedirect.com

ScienceDirect

J. Math. Anal. Appl. 328 (2007) 370–379

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

On meromorphically harmonic starlike functions with respect to symmetric conjugate points

Hakan Bostanci, Sibel Yalçin, Metin Öztürk *

Department of Mathematics, Faculty of Science and Art, Uludag University, 16059 Bursa, Turkey

Received 16 December 2005

Available online 19 June 2006

Submitted by H.M. Srivastava

Abstract

In a previous paper M.P. Chen, Z.-R. Wu and Z.-Z. Zou [M.P. Chen, Z.-R. Wu, Z.-Z. Zou, On functions α-starlike with respect to symmetric conjugate points, J. Math. Anal. Appl. 201 (1996) 25–34] developed a method, using some operators, to deal with functions analytic and starlike with respect to symmetric conjugate points in the unit disc. Then, the same method is employed to functions meromorphic by Z.Z. Zou and Z.-R. Wu [Zhong Zhu Zou, Zhuo-Ren Wu, On meromorphically starlike functions and functions meromorphically starlike with respect to symmetric conjugate points, J. Math. Anal. Appl. 261 (2001) 17–27]. Now, the method can be employed to functions meromorphic harmonic in the punctured disc $0 < |z| < 1$. Especially, a sharp coefficient estimate and a structural representation of such functions are obtained.

Keywords: Harmonic, meromorphic, starlike, and convex functions

1. Introduction

A continuous function $f = u + iv$ is a complex valued harmonic function in a complex domain D if both u and v are real harmonic in D. In any simply connected domain $D \subset \mathbb{C}$ we can write $f = h + \bar{g}$, where h and g are analytic in D. A necessary and sufficient condition for f to be locally univalent and sense preserving in D is that $|h'(z)| > |g'(z)|$ in D (see [2]). In [6], there is a more comprehensive study on harmonic univalent functions. In [3], Hengartner and

* Corresponding author.
E-mail address: ometin@uludag.edu.tr (M. Öztürk).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2006.05.044

Schober investigated functions harmonic in the exterior of the unit disc $\tilde{U} = \{z\colon |z| > 1\}$. They showed that complex valued, harmonic, sense preserving, univalent mapping f must admit the representation

$$f(z) = h(z) + \overline{g(z)} + A\log|z|,$$

where

$$h(z) = \alpha z + \sum_{n=1}^{\infty} a_n z^{-n} \quad \text{and} \quad g(z) = \beta\bar{z} + \sum_{n=1}^{\infty} b_n z^{-n},$$

$0 \leqslant |\beta| < |\alpha|$, $A \in \mathbb{C}$. Also, Jahangiri and Silverman [8], Jahangiri [5] and Murugusundaramoorthy [10,11] have studied the classes of meromorphic harmonic functions.

Let MH denote the class of functions

$$f(z) = h(z) + \overline{g(z)} = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n} \tag{1}$$

which are univalent harmonic in the punctured unit disc $U \setminus \{0\}$. $h(z)$ and $g(z)$ are analytic in $U \setminus \{0\}$ and U, respectively and $h(z)$ has a simple pole at the origin with residue 1 here. Denote by the class PH of all the functions of the form $p = s + \bar{t}$ that are harmonic in U and such that for $z \in U$, $\operatorname{Re} p(z) > 0$, where

$$s(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \quad \text{and} \quad t(z) = \sum_{n=1}^{\infty} q_n z^n$$

are analytic in U (see [9]).

A function $f(z) \in MH$ is said to be in the subclass MHS^* of meromorphically harmonic starlike in $U \setminus \{0\}$ if it satisfies the condition

$$\operatorname{Re}\left\{-\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}}\right\} > 0, \quad z \in U \setminus \{0\}. \tag{2}$$

Also, a function $f(z) \in MH$ is said to be in the subclass MHK of meromorphically harmonic convex in $U \setminus \{0\}$ if it satisfies the condition

$$\operatorname{Re}\left\{-\frac{z^2h''(z) + zh'(z) + \overline{z^2g''(z) + zg'(z)}}{zh'(z) - \overline{zg'(z)}}\right\} > 0, \quad z \in U \setminus \{0\}. \tag{3}$$

This classification (3) but for harmonic univalent functions was first used by Jahangiri [4].

We now define two operators D and T as follows [1,12]:

(1) The operator T: For

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n} \in MH,$$

$$Tf(z) = \frac{1}{2}\left[f(z) - \overline{f(-\bar{z})}\right]$$

$$= \frac{1}{z} + \frac{1}{2}\sum_{n=1}^{\infty}\left[a_n - (-1)^n\bar{a}_n\right]z^n + \frac{1}{2}\sum_{n=1}^{\infty}\left[\bar{b}_n - (-1)^n b_n\right]\bar{z}^n.$$

(2) The operator D:

$$D^m f(z) = D^m h(z) + (-1)^m \overline{D^m g(z)}, \quad m = 0, 1, 2, \ldots,$$

where

$$D^m h(z) = (-1)^m \frac{1}{z} + \sum_{n=1}^{\infty} n^m a_n z^n \quad \text{and} \quad D^m g(z) = \sum_{n=1}^{\infty} n^m b_n z^n.$$

The operator D was first applied to Salagean-type harmonic univalent functions by Jahangiri et al. [7].

It is easily verifiable that D and T are well-defined in MH and have the following properties:

(I) Let $\lambda \in \mathbb{C}$ and $f, F \in MH$. Then

$$D\big[\lambda f + (1-\lambda)F\big] = \lambda Df + (1-\lambda)DF$$

and

$$T\big[\lambda f + (1-\lambda)F\big] = \lambda Tf + (1-\lambda)TF.$$

(II) $DT = TD$.
(III) $TT = T$.

Note that the condition (2) is

$$\operatorname{Re}\left\{-\frac{Df(z)}{f(z)}\right\} > 0, \quad z \in U \setminus \{0\}, \tag{4}$$

and the condition (3) is

$$\operatorname{Re}\left\{-\frac{D^2 f(z)}{Df(z)}\right\} > 0, \quad z \in U \setminus \{0\}. \tag{5}$$

Now we define a new class $MHS^*_{SC}(\alpha)$ (see [1,13]).

Definition 1.1. A function $f \in MH$ with $\frac{h(z)+\overline{g(z)}}{zh'(z)-\overline{zg'(z)}} \neq 0$ in $U \setminus \{0\}$ is said to be meromorphically harmonic α-starlike with respect to symmetric conjugate points, if it satisfies

$$\operatorname{Re}\left\{-\frac{D(\alpha D + (1+\alpha)D^0)f(z)}{(\alpha D + (1+\alpha)D^0)Tf(z)}\right\} > 0, \quad z \in U \setminus \{0\}, \tag{6}$$

for some $\alpha \geqslant 0$. This class is denoted $MHS^*_{SC}(\alpha)$.

In this paper, the classes MHS^*, MHK and $MHS^*_{SC}(\alpha)$ are discussed and some properties of these classes such as coefficient estimates and a structural formula are obtained.

Let us adopt the symbol $D_\alpha = \alpha D + (1+\alpha)D^0$, and $f^*(z) = D_\alpha f(z) = (1+\alpha) \times (h(z) + \overline{g(z)}) + \alpha(zh'(z) - \overline{zg'(z)})$. We see that $f \in MHS^*_{SC}$ is equivalent to $f^*(z) \in MHS^*_{SC}(0) = MHS^*_{SC}$.

2. Some results of the class *MHS**

Theorem 2.1. *If* $f(z)=h(z)+\overline{g(z)}$ *of the form* (1) *and the condition*

$$\sum_{n=1}^{\infty} n\big(|a_n|+|b_n|\big)\leqslant 1 \tag{7}$$

is satisfied, then $f\in MHS^*$.

Proof. Set

$$p(z)=\begin{cases}-\frac{Df(z)}{Tf(z)}, & z\in U\setminus\{0\},\\ 1, & z=0.\end{cases} \tag{8}$$

It is sufficient to prove that

$$\big|p(z)-1\big|<\big|p(z)+1\big|,\quad z\in U. \tag{9}$$

From (7) we obtain, for $z\in U$,

$$\left|\sum_{n=1}^{\infty}(n+1)a_n z^{n+1}+|z|^2\sum_{n=1}^{\infty}(n-1)\bar{b}_n\bar{z}^{n-1}\right|<1+\sum_{n=1}^{\infty}|a_n|-\sum_{n=1}^{\infty}|b_n| \tag{10}$$

and

$$\begin{aligned}&\left|2-\sum_{n=1}^{\infty}(n-1)a_n z^{n+1}+|z|^2\sum_{n=1}^{\infty}(n+1)\bar{b}_n\bar{z}^{n-1}\right|\\ &\quad>2-\sum_{n=1}^{\infty}(n-1)a_n z^{n+1}-|z|^2\sum_{n=1}^{\infty}(n+1)\bar{b}_n\bar{z}^{n-1}\\ &\quad\geqslant 1+\sum_{n=1}^{\infty}|a_n|-\sum_{n=1}^{\infty}|b_n|,\quad z\in U.\end{aligned} \tag{11}$$

Since

$$\big|p(z)-1\big|=\frac{|\sum_{n=1}^{\infty}(n+1)a_n z^{n+1}+|z|^2\sum_{n=1}^{\infty}(n-1)\bar{b}_n\bar{z}^{n-1}|}{|1+\sum_{n=1}^{\infty}a_n z^{n+1}+|z|^2\sum_{n=1}^{\infty}\bar{b}_n\bar{z}^{n-1}|},\quad z\in U\setminus\{0\},$$

and

$$\big|p(z)+1\big|=\frac{|2-\sum_{n=1}^{\infty}(n-1)a_n z^{n+1}+|z|^2\sum_{n=1}^{\infty}(n+1)\bar{b}_n\bar{z}^{n-1}|}{|1+\sum_{n=1}^{\infty}a_n z^{n+1}+|z|^2\sum_{n=1}^{\infty}\bar{b}_n\bar{z}^{n-1}|},\quad z\in U\setminus\{0\},$$

by (10) and (11), we see that inequality (9) holds in $U\setminus\{0\}$. The case $z=0$ is trivial. Hence $p(z)\in PH$. So the proof of Theorem 2.1 is complete. □

Corollary 2.2. *If* $f(z)=h(z)+\overline{g(z)}\in MH$, *then* $f\in MHK$ *when*

$$\sum_{n=1}^{\infty} n^2\big(|a_n|+|b_n|\big)\leqslant 1.$$

Theorem 2.3. *Let $f(z)=h(z)+\overline{g(z)}\in MH$. If $f\in MHS^*$, then*

$$\sum_{n=1}^{\infty} n\left(|a_n|^2-|b_n|^2\right)\leqslant 1. \tag{12}$$

The estimate (12) *is sharp and the equality is attained for the function* $f_0(z)=\frac{1}{z}+\frac{1}{\sqrt{n}}z^n$.

Proof. Since $f\in MHS^*$, hence the inequality (9) holds. This implies that

$$\begin{aligned}&\left|\sum_{n=1}^{\infty}(n+1)a_n z^{n+1}-\sum_{n=1}^{\infty}(n-1)\overline{b_n z^n}z\right|\\ &\quad<\left|2-\sum_{n=1}^{\infty}(n-1)a_n z^{n+1}+\sum_{n=1}^{\infty}(n+1)\overline{b_n z^n}z\right|,\quad z\in U.\end{aligned}$$

Therefore for every $r\in(0,1)$ and letting $z=re^{i\theta}$, $0\leqslant\theta\leqslant 2\pi$, we have

$$\begin{aligned}&\int_0^{2\pi}\left|\sum_{n=1}^{\infty}(n+1)a_n z^{n+1}-\sum_{n=1}^{\infty}(n-1)\overline{b_n z^n}z\right|^2 d\theta\\ &\quad\leqslant\int_0^{2\pi}\left|2-\sum_{n=1}^{\infty}(n-1)a_n z^{n+1}+\sum_{n=1}^{\infty}(n+1)\overline{b_n z^n}z\right|^2 d\theta,\end{aligned}$$

and hence

$$\sum_{n=1}^{\infty} n\left(|a_n|^2-|b_n|^2\right)r^{2(n+1)}\leqslant 1. \tag{13}$$

Let $r\to 1^-$; (13) yields (12).

It is easily seen that $f_0(z)\in MHS^*$. Thus we complete the of Theorem 2.3. □

Corollary 2.4. *Let $f(z)=h(z)+\overline{g(z)}\in MH$. If $f\in MHK$, then*

$$\sum_{n=1}^{\infty} n^3\left(|a_n|^2-|b_n|^2\right)\leqslant 1.$$

3. The class $MHS^*_{SC}(\alpha)$

Theorem 3.1. *If*

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_n z^n+\overline{\sum_{n=1}^{\infty}b_n z^n}\in MHS^*_{SC}(\alpha),$$

then $Tf\in MHS^*$.

Proof. Setting

$$-\frac{Df(z)}{Tf(z)}=-\frac{2(zh'(z)-\overline{zg'(z)})}{f(z)-\overline{f(-\bar z)}}=p(z) \tag{14}$$

we have $\operatorname{Re}\{p(z)\} > 0$. We also have

$$\overline{p(-\bar{z})} = -\frac{2(\overline{-\bar{z}h'(-\bar{z})} - (-\bar{z})g'(-\bar{z}))}{f(z) - \overline{f(-\bar{z})}} = -\frac{D\overline{(-f(-\bar{z}))}}{Tf(z)}. \tag{15}$$

From (14) and (15), we obtain

$$-\frac{DTf(z)}{Tf(z)} = -\frac{1}{2}\left\{\frac{Df(z) + D\overline{(-f(-\bar{z}))}}{Tf(z)}\right\} = \frac{1}{2}\big(p(z) + \overline{p(-\bar{z})}\big), \quad z \in U \setminus \{0\}. \tag{16}$$

Since $\operatorname{Re}\{p(z)\} > 0$ for all $z \in U \setminus \{0\}$, and $\operatorname{Re}\{\overline{p(-\bar{z})}\} = \operatorname{Re}\{p(-\bar{z})\} > 0$ for all $z \in U \setminus \{0\}$, therefore from (16) we have

$$\operatorname{Re}\left\{-\frac{DTf(z)}{Tf(z)}\right\} > 0, \quad z \in U \setminus \{0\}.$$

This means that $Tf \in MHS^*$. So the proof of Theorem 3.1 is complete. □

Since

$$Tf(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{1}{2}\big[a_n - (-1)^n \bar{a}_n\big] z^n + \sum_{n=1}^{\infty} \frac{1}{2}\big[\bar{b}_n - (-1)^n b_n\big] \bar{z}^n,$$

hence from Theorems 2.3 and 3.1, we obtain

Corollary 3.2. *If*

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n} \in MHS^*_{SC},$$

then

$$\sum_{n=1}^{\infty} \frac{n}{4}\big[\big|a_n - (-1)^n \bar{a}_n\big|^2 - \big|\bar{b}_n - (-1)^n b_n\big|^2\big] \leqslant 1,$$

i.e.,

$$\sum_{n=1}^{\infty} n\big[|a_n|^2 - |b_n|^2 - (-1)^n\big(\operatorname{Re}(a_n)^2 - \operatorname{Re}(b_n)^2\big)\big] \leqslant 2.$$

Theorem 3.3. *Let $\alpha \geqslant 0$. If $f \in MHS^*(\alpha)$, then $D_\alpha Tf \in MHS^*$ and $Tf \in MHS^*_{SC}(\alpha)$.*

Proof. Since $f \in MHS^*(\alpha)$ if and only if $f^* = D_\alpha f = (\alpha D + (1+\alpha)D^0)f \in MHS^*_{SC}$, then we can see from Theorem 3.1 that $Tf^* = TD_\alpha f \in MHS^*$. Further, by using $TD = DT$ we get $TD_\alpha = D_\alpha T$. Hence, we have $D_\alpha Tf(z) = TD_\alpha f(z) \in MHS^*$.

Moreover, $TT = T$ yields

$$\operatorname{Re}\left\{-\frac{D(\alpha D + (1+\alpha)D^0)Tf(z)}{(\alpha D + (1+\alpha)D^0)TTf(z)}\right\} = \operatorname{Re}\left\{-\frac{D(D_\alpha Tf(z))}{D_\alpha Tf(z)}\right\} > 0, \quad z \in U \setminus \{0\}.$$

This means that $Tf \in MHS^*(\alpha)$. The proof of Theorem 3.3 is complete. □

Theorem 3.4. *If*

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n} \in MH$$

and the condition

$$\sum_{n=1}^{\infty} \sqrt{n^2+1}\big(|a_n| + |b_n|\big) \leqslant 1 \tag{17}$$

is satisfied, then $f \in MHS^*_{SC}$.

Proof. Let $p(z) = -\frac{Df(z)}{Tf(z)}$, $z \in U \setminus \{0\}$, and $p(0) = 1$. We need only to prove that

$$\big|p(z) - 1\big| < \big|p(z) + 1\big|, \quad z \in U \setminus \{0\}.$$

After some computations, we see that we need only

$$\begin{aligned}
&\left|\sum_{n=1}^{\infty}\left\{\left(n+\frac{1}{2}\right)a_n - (-1)^n\frac{1}{2}\bar{a}_n\right\}z^{n+1} - |z|^2\sum_{n=1}^{\infty}\left\{\left(n-\frac{1}{2}\right)\bar{b}_n + \frac{1}{2}(-1)^n b_n\right\}\bar{z}^{n-1}\right| \\
&\quad < \left|2 - \sum_{n=1}^{\infty}\left\{\left(n-\frac{1}{2}\right)a_n + (-1)^n\frac{1}{2}\bar{a}_n\right\}z^{n+1}\right. \\
&\quad\quad \left. + |z|^2\sum_{n=1}^{\infty}\left\{\left(n+\frac{1}{2}\right)\bar{b}_n - \frac{1}{2}(-1)^n b_n\right\}\bar{z}^{n-1}\right|.
\end{aligned} \tag{18}$$

It is sufficient to prove that

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left|\left(n-\frac{1}{2}\right)\bar{b}_n + \frac{1}{2}(-1)^n b_n\right| + \left|\left(n+\frac{1}{2}\right)a_n - (-1)^n\frac{1}{2}\bar{a}_n\right| \\
&\quad + \left|\left(n-\frac{1}{2}\right)a_n + (-1)^n\frac{1}{2}\bar{a}_n\right| + \left|\left(n+\frac{1}{2}\right)\bar{b}_n - \frac{1}{2}(-1)^n b_n\right| \leqslant 2.
\end{aligned} \tag{19}$$

There after some easy computations for n odd and even, we have

$$\begin{aligned}
&\left|\left(n-\frac{1}{2}\right)\bar{b}_n + \frac{1}{2}(-1)^n b_n\right| + \left|\left(n+\frac{1}{2}\right)a_n - (-1)^n\frac{1}{2}\bar{a}_n\right| \\
&\quad\quad + \left|\left(n-\frac{1}{2}\right)a_n + (-1)^n\frac{1}{2}\bar{a}_n\right| + \left|\left(n+\frac{1}{2}\right)\bar{b}_n - \frac{1}{2}(-1)^n b_n\right| \\
&\quad \leqslant 2\sqrt{n^2+1}\big(|a_n| + |b_n|\big).
\end{aligned} \tag{20}$$

From (17) and (20), we see that inequality (19) holds. This complete proof of Theorem 3.4. □

Remark 3.1. If the a_n $(n = 1, 2, \ldots)$ are real and satisfy the condition

$$\sum_{n=1}^{\infty} n\big(|a_n| + |b_n|\big) \leqslant 1, \tag{21}$$

then

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_nz^n+\overline{\sum_{n=1}^{\infty}b_nz^n}\in MHS^*_{SC}.$$

If f and F are

$$f(z)=h(z)+\overline{g(z)}=\frac{1}{z}+\sum_{n=1}^{\infty}a_nz^n+\overline{\sum_{n=1}^{\infty}b_nz^n}$$

and

$$F(z)=H(z)+\overline{G(z)}=\frac{1}{z}+\sum_{n=1}^{\infty}A_nz^n+\overline{\sum_{n=1}^{\infty}B_nz^n},$$

then the convolution (or Hadamard product) of f and F if defined to be the function

$$(f*F)(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_nA_nz^n+\overline{\sum_{n=1}^{\infty}b_nB_nz^n}.$$

Theorem 3.5. *A function*

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_nz^n+\overline{\sum_{n=1}^{\infty}b_nz^n}$$

belongs to MHS^*_{SC}, there exist a function $p\in PH$ and a function $K\in MHS^*$ with real coefficients such that K is satisfies

$$-\frac{DK(z)}{K(z)}=\frac{1}{2}\big(p(iz)+\overline{p(i\bar{z})}\big),\quad z\in U\setminus\{0\},\tag{22}$$

and

$$Df(z)=zh'(z)-\overline{zg'(z)}=ip(z)K(-iz),\quad z\in U\setminus\{0\}.\tag{23}$$

Proof. Suppose $f\in MHS^*_{SC}$. Then there exist a function $p\in PH$ such that

$$Df(z)=zh'(z)-\overline{zg'(z)}=-p(z)Tf(z),\quad z\in U\setminus\{0\},\tag{24}$$

where $p(z)=-\frac{Df(z)}{Tf(z)}$, and

$$Tf(z)=\frac{1}{z}+\frac{1}{2}\sum_{n=1}^{\infty}\big[a_n-(-1)^n\bar{a}_n\big]z^n+\frac{1}{2}\sum_{n=1}^{\infty}\big[\bar{b}_n-(-1)^nb_n\big]\bar{z}^n.$$

By Theorem 3.3, $Tf(z)\in MHS^*$, and hence, for $z\in U\setminus\{0\}$,

$$\mathrm{Re}\left\{-\frac{Df(z)}{f(z)}\right\}=\mathrm{Re}\left\{\frac{\frac{1}{z}-\sum_{n=1}^{\infty}\frac{n}{2}[a_n-(-1)^n\bar{a}_n]z^n+\sum_{n=1}^{\infty}\frac{n}{2}[\bar{b}_n-(-1)^nb_n]\bar{z}^n}{\frac{1}{z}+\sum_{n=1}^{\infty}\frac{1}{2}[a_n-(-1)^n\bar{a}_n]z^n+\sum_{n=1}^{\infty}\frac{1}{2}[\bar{b}_n-(-1)^nb_n]\bar{z}^n}\right\}>0.$$

Let

$$s(z)=\frac{\frac{1}{z}-\sum_{n=1}^{\infty}\frac{n}{2}[a_n-(-1)^n\bar{a}_n]z^n+\sum_{n=1}^{\infty}\frac{n}{2}[\bar{b}_n-(-1)^nb_n]\bar{z}^n}{\frac{1}{z}+\sum_{n=1}^{\infty}\frac{1}{2}[a_n-(-1)^n\bar{a}_n]z^n+\sum_{n=1}^{\infty}\frac{1}{2}[\bar{b}_n-(-1)^nb_n]\bar{z}^n}.$$

Since $s(z) \in PH$,

$$s(z) \prec \frac{1}{(1-z)^2} - \frac{\bar{z}^2}{(1-\bar{z})^2}, \quad z \in U,$$

where $p(z) = (1-z)^{-2} - \bar{z}^2(1-\bar{z})^{-2} \in PH$. Moreover, we see that

$$\begin{aligned} s(iz) &= \frac{1 - \sum_{n=1}^{\infty} \frac{n}{2}[a_n - (-1)^n \bar{a}_n] i^{n+1} z^{n+1} + \sum_{n=1}^{\infty} \frac{n}{2}[\bar{b}_n - (-1)^n b_n](-1)^n i^{n+1} z\bar{z}^n}{1 + \sum_{n=1}^{\infty} \frac{1}{2}[a_n - (-1)^n \bar{a}_n] i^{n+1} z^{n+1} + \sum_{n=1}^{\infty} \frac{1}{2}[\bar{b}_n - (-1)^n b_n](-1)^n i^{n+1} z\bar{z}^n} \\ &= s(z) * \left(\frac{1}{1-iz} + \frac{1}{1+i\bar{z}}\right) \prec \frac{1}{(1-z)^2} - \frac{\bar{z}^2}{(1-\bar{z})^2}, \quad z \in U \setminus \{0\}. \end{aligned} \tag{25}$$

From (25) we see that

$$\begin{aligned} K(z) &= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{1}{2}\left[a_n - (-1)^n \bar{a}_n\right] i^{n+1} z^n + \sum_{n=1}^{\infty} \frac{1}{2}\left[\bar{b}_n - (-1)^n b_n\right](-1)^n i^{n+1} \bar{z}^n \\ &= Tf(z) * \left(\frac{1}{z} + \frac{1}{1-iz} + \frac{1}{1+i\bar{z}}\right) \end{aligned} \tag{26}$$

is in the class MHS^* with real coefficients. Therefore

$$Tf(z) = K(z) * \left(\frac{1}{z} - \frac{z}{1+iz} + \frac{\bar{z}}{1+i\bar{z}}\right) = -iK(-iz). \tag{27}$$

From (24) and (27), we obtain (23).

We now show that the function K satisfies the condition (22).

Since $p(z) = -\frac{Df(z)}{Tf(z)}$, $z \in U \setminus \{0\}$, we obtain

$$\overline{p(-\bar{z})} = -\frac{D\overline{(-f(-\bar{z}))}}{Tf(z)}$$

and

$$-\frac{DTf(z)}{Tf(z)} = -\frac{1}{2}\left\{\frac{Df(z) + D\overline{(-f(-\bar{z}))}}{Tf(z)}\right\} = \frac{1}{2}\big(p(z) + \overline{p(-\bar{z})}\big), \quad z \in U \setminus \{0\}.$$

Moreover, we have

$$\begin{aligned} -\frac{DK(z)}{K(z)} &= \left(-\frac{DTf(z)}{Tf(z)}\right) * \left(\frac{1}{1-iz} + \frac{1}{1+i\bar{z}}\right) \\ &= \frac{1}{2}\big(p(z) + \overline{p(-\bar{z})}\big) * \left(\frac{1}{1-iz} + \frac{1}{1+i\bar{z}}\right) \\ &= \frac{1}{2}\big(p(iz) + \overline{p(i\bar{z})}\big), \quad z \in U \setminus \{0\}. \end{aligned}$$

Since $p \in PH$, hence the condition (22) holds.

Conversely, for $f(z) \in MH$, if there exist a function $p \in PH$ and a function $K \in MHS^*$ with real coefficients such that conditions (22) and (23) hold, then we can show that $f \in MHS^*_{SC}$.

To do this, we need only show that $Tf(z) = -iK(-iz)$. From (23), we get

$$p(z) = \frac{Df(z)}{iK(-iz)}$$

and

$$\overline{p(-\bar{z})} = -\frac{D\overline{(f(-\bar{z}))}}{i\overline{K(-i\bar{z})}}.$$

Hence, from (22)

$$\begin{aligned} -\frac{DK(-iz)}{K(-iz)} &= \frac{1}{2}\big(p(z) + \overline{p(-\bar{z})}\big) \\ &= \frac{1}{2}\left(\frac{Df(z)}{iK(-iz)} - \frac{D\overline{(f(-\bar{z}))}}{i\overline{K(-i\bar{z})}}\right) \\ &= \frac{DTf(z)}{iK(-iz)}, \quad z \in U \setminus \{0\}, \end{aligned}$$

where $K(-iz) = \overline{K(i\bar{z})}$, since $K(-iz)$ has real coefficients, and $DT = TD$, and

$$D\big(-iK(-iz)\big) = DTf(z). \tag{28}$$

Since the functions $(-iK(-iz))$ and $Tf(z)$ belong to MH, from (28) we have

$$Tf(z) = -iK(-iz).$$

This completes the proof of Theorem 3.5. □

References

[1] M.P. Chen, Z.-R. Wu, Z.-Z. Zou, On functions α-starlike with respect to symmetric conjugate points, J. Math. Anal. Appl. 201 (1996) 25–34.
[2] J. Cluine, T. Sheil-Small, Harmonic functions, Ann. Acad. Sci. Fenn. Ser. Al Math. 9 (1984) 3–25.
[3] W. Hengartner, G. Schober, Univalent harmonic functions, Trans. Amer. Math. Soc. 299 (1987) 1–31.
[4] J.M. Jahangiri, Coefficient bounds and univalence criteria for harmonic functions with negative coefficients, Ann. Univ. Mariae Curie-Sklodowska Sect. A 52 (1998) 57–66.
[5] J.M. Jahangiri, Harmonic meromorphic starlike functions, Bull. Korean Math. Soc. 37 (2) (2000) 291–301.
[6] J.M. Jahangiri, Harmonic univalent functions with varying arguments, Int. J. Appl. Math. 8 (3) (2002) 267–275.
[7] J.M. Jahangiri, G. Murugusundaramoorthy, K. Vijaya, Salagean-type harmonic univalent functions, Southwest J. Pure Appl. Math. 2 (2002) 77–82.
[8] J.M. Jahangiri, H. Silverman, Meromorphic univalent harmonic functions with negative coefficients, Bull. Korean Math. Soc. 36 (1999) 763–770.
[9] Z.J. Jakubowski, W. Majchrzak, K. Skalska, Harmonic mappings with a positive real part, Materialy XIV Konferencji Szkoleniowejz Teorii Zagadnien Ekstremalnych, Lodz, 1993, pp. 17–24.
[10] G. Murugusundaramoorthy, Starlikeness of multivalent meromorphic harmonic functions, Bull. Korean Math. Soc. 40 (4) (2003) 553–564.
[11] G. Murugusundaramoorthy, Harmonic meromorphic convex functions with missing coefficients, J. Indones. Math. Soc. 10 (1) (2004) 15–22.
[12] G.S. Salagean, Subclasses of univalent harmonic functions, in: Fifth Romanian Finish Seminar, Bucharest, Complex Analysis 1 (1981) 362–372.
[13] Zhong Zhu Zou, Zhuo-Ren Wu, On meromorphically starlike functions and functions meromorphically starlike with respect to symmetric conjugate points, J. Math. Anal. Appl. 261 (2001) 17–27.