Asymptotics of Solutions to the Boundary-Value Problem for the Korteweg–de Vries–Burgers Equation on a Half-Line

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We study the following initial–boundary value problem for the Korteweg–de Vries–Burgers equation,

\[
\begin{align*}
 u_t + (-1)^\alpha u u_x - u_{xx} + (-1)^\alpha u_{xxx} &= 0, \quad (x, t) \in \mathbb{R}^+ \times \mathbb{R}^+, \\
 u(x, 0) &= u_0(x), \quad x \in \mathbb{R}^+, \\
 \partial_x u(0, t) &= 0, \quad n = 0, \alpha, t \in \mathbb{R}^+,
\end{align*}
\]

(0.1)

where \( \alpha = 0, 1 \). We prove that if the initial data \( u_0 \in H^0_\omega \cap H^1_0 \), where \( H^s = \{ f \in L^2; \| f \|_{H^s} = \| (x)^s \langle i\partial_x \rangle f \|_{L^2} < \infty \} \), \( \omega \in \left( \frac{1}{4}, \frac{1}{2} \right) \), and the norm \( \| u_0 \|_{H^0_\omega} + \| u_0 \|_{H^1_0} \) is sufficiently small, then there exists a unique solution \( u \in C([0, \infty), H^0_\omega) \cap C((0, \infty), H^1_\omega) \) of the initial–boundary value problem (0.1),
where $\kappa \in (0, 1/2)$. Moreover, if the initial data are such that $x^{1+\kappa} u_0(x) \in L^1$, $\mu = \omega - 1/2$, then there exists a constant $A$ such that the solution has the asymptotics

$$u(x, t) = \frac{A}{t} \Phi_a \left( \frac{x}{2\sqrt{t}}, t \right) + O \left( \min \left( \frac{x}{\sqrt{t}} \right)^{1-\frac{\omega}{2}} \right)$$

for $t \to \infty$ uniformly with respect to $x > 0$, where $\alpha = 0, 1$, $\Phi_a(q, t) = (q/\sqrt{\pi}) e^{-q^2}$, $\Phi_0(q, t) = (1/2\sqrt{\pi}) (e^{-q^2} (2q^2 - 1) + e^{-2q^2})$.

Key Words: dissipative nonlinear evolution equation; large time asymptotics; Korteweg–de Vries–Burgers equation; half-line.

1. INTRODUCTION

We consider the initial–boundary value problem on a half-line for the Korteweg–de Vries–Burgers (KdV-B) equation. The amount of the boundary data for this problem depends on the direction of the half-line, $x \geq 0$ or $x \leq 0$, so we are interested in the two initial–boundary value problems

$$u_t + uu_x - au_{xx} + bu_{xxx} = 0, \quad t > 0, x > 0,$$

$$u(x, 0) = u_0(x), \quad x > 0; \quad u(0, t) = 0, \quad t > 0,$$

and

$$u_t + uu_x - au_{xx} + bu_{xxx} = 0, \quad t > 0, x < 0,$$

$$u(x, 0) = u_0(x), \quad x < 0; \quad u(0, t) = u_x(0, t) = 0, \quad t > 0,$$

where $a, b > 0$. By a change of variables and introduction of a parameter $\alpha = 0, 1$, we can combine these two problems as one problem on the half-line $x > 0$,

$$u_t + (-1)^\alpha uu_x - u_{xx} + (-1)^\alpha u_{xxx} = 0, \quad t > 0, x > 0,$$

$$u(x, 0) = u_0(x), \quad x > 0; \quad u(0, t) = 0, \quad t > 0, j = 0, \alpha.$$

(1.1)

The Korteweg–de Vries–Burgers equation (1.1) is a simple universal model equation which appears as the first approximation in the description of dispersive dissipative nonlinear waves.

The main goal of this paper is to find the large-time asymptotics of solutions to problem (1.1). Some results on the decay estimates of the solutions in different norms to the Cauchy problems for some nonlinear evolution equations with strongly dissipative operators were obtained in papers [1–3]. In papers [4, 7–9] the large–time asymptotic behavior of solutions to the Korteweg–de Vries–Burgers-type equations, on the line was found. In paper [5] the problem

$$u_t + u_x^2 - u_{xx} + u_{xxx} = 0, \quad t > 0, x > 0,$$

$$u(x, 0) = u_0(x), \quad x > 0; \quad u(0, t) = 0, \quad t > 0,$$

and
was studied, and asymptotic behavior of solutions was obtained under weaker conditions on the data since the $L^2$ norm of the nonlinear term decays faster than the case of this paper. This problem corresponds to the case $\alpha = 0$ in this paper. As far as we know there are a few results on the asymptotics of solutions to the initial–boundary value problem (1.1) when $\alpha = 1$, which is different from the case $\alpha = 0$. For the nonlinear nonlocal Schrödinger equation on a half-line the initial–boundary value problem was studied in [6], where the pseudodifferential operator $\mathcal{K}$ with homogeneous symbol $K(p) = Cp^\beta$ was introduced through the Laplace transformation, and it was shown that the number of boundary data which is necessary to show the well-posedness of the problem is equal to $[\frac{\beta}{2}]$, except for the case where $\beta$ is an odd integer. In the case of Eq. (1.1) we have the operator $\mathcal{K}u = -u_{xx} + (-1)^\alpha u_{xxx}$; i.e., the symbol $K_\alpha(p) = -p^2 + (-1)^\alpha p^3$ is not homogeneous and the order of $p$ in the second term is an odd integer. Thus we consider the critical case that the number of the boundary data also depends on the sign of the highest derivative in the equation. Note that in the case of the Cauchy problem for the Korteweg–de Vries–Burgers equation the nonlinearity $uu_x$ of the shallow water type is critical from the point of view of the large time asymptotic behavior of solutions since the nonlinearity in the equation has the same decay rate as the linear terms. In the case of the initial–boundary value problem due to the homogeneous boundary data the solution obtains an additional decay, and as a result the nonlinear term in the boundary-value problem (1.1) appears to be supercritical, in contrast to the corresponding Cauchy problem. The main difficulty in the boundary value problem (1.1) is in the evaluation of the contribution of the boundary data to the large time asymptotic formulas of the solutions. Our approach here is based on the $L^p$ estimates of the Green function.

Let us denote by $H^{s, \lambda} = \{ f \in L^2; \| f \|_{H^{s, \lambda}} = \| \langle x \rangle^k (i\partial_x)^s f \|_{L^2} < \infty \}$ the weighted Sobolev space; here and below $\langle x \rangle = \sqrt{1 + x^2}$. The symbol of the operator $\mathcal{K}$ has the form $K_\alpha(p) = -p^2 + (-1)^\alpha p^3$, so that it satisfies the dissipation condition $\Re K_\alpha(p) > 0$ for all $p$ on the imaginary axis $\Re p = 0, p \neq 0$. By the same letter $C$ we denote different positive constants. First of all we formulate the local existence of the solutions of the initial–boundary value problem (1.1).

Now we state our results.

**Theorem 1.1.** Suppose that the initial data $u_0(x) \in H^{0, \omega} \cap H^{1, 0}$, where $\omega \in (\frac{1}{2}, \frac{3}{2})$. Then for some $T > 0$ there exists a unique solution

$u(x, t) \in C([0, T], H^{0, x}) \cap C((0, T], H^{1, \omega})$

of the initial–boundary value problem (1.1), where $x \in (0, \frac{1}{2})$.

Now we give sufficient conditions for the global existence of solutions.
Theorem 1.2. Suppose that the initial data $u_0 \in H^{0, \omega} \cap H^{1,0}$, where $\omega \in (1, \frac{3}{2})$ and the norm $\|u_0\|_{H^{0, \omega}} + \|u_0\|_{H^{1,0}} \leq \epsilon$, where $\epsilon > 0$ is small enough. Then there exists a unique solution

$$u \in C([0, \infty), H^{0,x}) \cap C((0, \infty), H^{1,\omega})$$

of the initial–boundary value problem (1.1), where $x \in (0, \frac{1}{2})$. Moreover, if the initial data are such that

$$x^{1+\mu}u_0(x) \in L^1, \quad \mu = \omega - \frac{1}{2},$$

then there exists a constant $A$ such that the solution has the asymptotics

$$u(x, t) = \frac{A}{t} \Phi_\alpha \left( \frac{x}{2\sqrt{t}}, t \right) + O \left( \min \left( \frac{x}{\sqrt{t}}, 1 \right) t^{-1-\frac{q}{2}} \right)$$

for $t \to \infty$ uniformly with respect to $x > 0$, where $\alpha = 0, 1$, and

$$\Phi_0(q, t) = \frac{q}{\sqrt{\pi}} e^{-q^2}, \quad \Phi_1(q, t) = \frac{1}{2\sqrt{\pi} t} \left( e^{-q^2} (2q\sqrt{t} - 1) + e^{-2q\sqrt{t}} \right).$$

Remark 1.1. Estimates obtained in Theorems 1.1 and 1.2 are sufficient to obtain the large-time asymptotic formula for the solution, which is the main aim of this paper. By virtue of the explicit formulas for the Green function (see (2.9)) we can see that there is a smoothing effect in the problem (1.1), so the solution can be proved to be classical; namely, we have $u \in C^\infty((0, \infty) \times (0, \infty))$.

Remark 1.2. Note that the asymptotics of solutions to the problem (1.1) obtained in Theorem 1.2 have the decay rate $t^{-1}$, which is more rapid in comparison with the case of the Cauchy problem $t^{-\frac{1}{2}}$. This happens due to the homogeneous boundary data, similarly to the problems for the heat equation on the line and the half-line.

We organize our paper as follows. In Section 2 we consider the linear initial–boundary value problem corresponding to (1.1) and discuss the question of the number of boundary data which are necessary to prove the existence and uniqueness of solutions. We construct the Green function of the solution of the linear problem and formulate Theorem 2.1 on the existence and uniqueness of the solution. In Section 3 we prove some preliminary estimates in Lemma 3.1. Section 4 is devoted to the proof of Theorem 2.1 for the linear problem. In Section 5 we prove Theorem 1.1 on the local existence of solutions to the nonlinear problem (1.1). Theorem 1.2 is proved in Section 6.
2. LINEAR PROBLEM

We consider the linear initial-boundary value problem corresponding to (1.1),

\[ u_t - u_{xx} + (-1)\alpha u_{xxx} = f(x, t), \quad t > 0, x > 0, \]

\[ u(x, 0) = u_0(x), \quad x > 0, \]

\[ u_j'(0, t) = 0, \quad t > 0, \quad \text{for} \quad j = 0, \alpha, \]

where \( \alpha = 0, 1. \)

In this section we obtain the explicit formula for the solution of the linear problem (2.1) under the condition

\[ u_0 \in L^1(\mathbb{R}^+), \quad f \in L^q(0, T; L^1(\mathbb{R}^+)), \]

with \( q > 2. \) Taking the Laplace transformation of the problem (2.1) and integrating the result with respect to \( t, \) we obtain a representation for the Laplace transform of the solution,

\[ \hat{u}(p, t) = e^{-K_\alpha(p)t} \hat{u}_0(p) + \int_0^t e^{-K_\alpha(p)(t-\tau)} f_1(p, \tau) d\tau, \]

where \( K_\alpha(p) = -p^2 + (-1)^\alpha p^3 \) and

\[ f_1(p, \tau) = -p^2 \sum_{j=1}^2 \partial_x^{j-1} u(0, \tau) \frac{p}{p_j} \]

\[ + (-1)^\alpha p^3 \sum_{j=1}^3 \partial_x^{j-1} u(0, \tau) \frac{p}{p_j} + \hat{f}(p, \tau). \]

The condition

\[ |\hat{u}(p, t)| \leq M(1 + |p|)^\beta, \quad \text{for all} \quad \Re p \geq 0, \]

with some \( M, \beta > 0 \) is necessary and sufficient for the existence of the inverse Laplace transformation. It is easy to see that condition (2.4) is fulfilled in domains \( \Re K_\alpha(p) > 0 \) of the right half-complex plane \( \Re p \geq 0. \)

In domains \( \Re K_\alpha(p) < 0 \) of the right half-complex plane \( \Re p \geq 0, \) we rewrite formula (2.2) as

\[ \hat{u}(p, t) = e^{-K_\alpha(p)t} \left( \hat{u}_0(p) + \int_0^{+\infty} e^{K_\alpha(p)\tau} f_1(p, \tau) d\tau \right) \]

\[ - \int_t^{+\infty} e^{-K_\alpha(p)(t-\tau)} f_1(p, \tau) d\tau. \]
Since the last integral
\[
\int_0^{+\infty} e^{-K_\alpha(p)(t-\tau)} f_1(p, \tau) d\tau
\]
satisfies the condition (2.4) for all \( \Re p \geq 0, \Re K_\alpha(p) < 0 \), we have to assume the condition

\[
(2.5) \quad \hat{u}_0(p) + \int_0^{+\infty} e^{K_\alpha(p)\tau} f_1(p, \tau) d\tau = \hat{u}_0(p) - \sum_{j=1}^{2} \int_0^{+\infty} e^{K_\alpha(p)\tau} \partial^j_x u(0, \tau) d\tau
\]

\[
+ \int_0^{+\infty} e^{K_\alpha(p)\tau} \hat{f}(p, \tau) d\tau = 0
\]

for all \( \Re p \geq 0, \Re K_\alpha(p) < 0 \). We use Eq. (2.5) to find some of the boundary functions \( u_{\xi}^{-1}(0, t) \) involved in (2.3). Making the change of the variable \(-K_\alpha(p) = \xi\), we transform domains \( \Re K_\alpha(p) < 0 \) of the right half-complex plane \( \Re p \geq 0 \) to the half-complex plane \( \Re \xi > 0 \). Since \( K_\alpha(p) \) is a cubic polynomial, there exist three functions \( \phi_l(\xi) = K_\alpha^{-1}(\xi) = p, \ l = 1, 2, 3 \), which transform the half-complex plane \( \Re \xi > 0 \) to domains, where \( \Re K_\alpha(p) < 0 \). Therefore condition (2.5) can be written as an equation or a system of equations in the half-complex plane \( \Re \xi > 0 \),

\[
(2.6) \quad \hat{u}_0(\phi_l(\xi)) - \sum_{j=1}^{2} \frac{\hat{v}_{j-1}(\xi)}{\phi_l(\xi)^{j-2}} + (-1)^{3} \sum_{j=1}^{3} \frac{\hat{v}_{j-1}(\xi)}{\phi_l(\xi)^{j-2}} + \hat{f}(\phi_l(\xi), \xi) = 0,
\]

for all \( l \) such that \( \Re \phi_l(\xi) > 0 \) for \( \Re \xi > 0 \), where

\[
\hat{f}(\phi_l(\xi), \xi) = \int_0^{+\infty} \int_0^{+\infty} e^{-(\phi_l(\xi)x+\xi \tau)} f(x, \tau) dx d\tau,
\]

\[
\hat{u}_0(\phi_l(\xi)) = \int_0^{+\infty} e^{-\phi_l(\xi)x} u_0(x) dx, \quad \hat{v}_j(\xi) = \int_0^{+\infty} e^{-\xi \tau} u^{(j)}(0, \tau) d\tau
\]

are the Laplace transforms with respect to space and time for the initial data \( u_0(x) \), the boundary values \( \partial^j_x u(0, t) \), and the source \( f(x, t) \), respectively. The number of equations in system (2.6) is equal to the number of functions \( \phi_l(\xi) = K_\alpha^{-1}(\xi) \) such that \( \Re \phi_l(\xi) > 0 \) for all \( \Re \xi > 0 \). For example, in the case \( \alpha = 0 \) the equation

\[
-K_0(p) = p^2 - p^3 = \xi
\]
has two roots \( \phi_1(\xi) \) and \( \phi_2(\xi) \) such that \( \Re \phi_l(\xi) > 0, \ l = 1, 2 \), for all

\[
\xi \in D_0 \equiv \{ \xi \in \mathbb{C} : \Re \xi > 0, \xi \notin \left[ 0, \frac{4}{27} \right] \},
\]

and in the case \( \alpha = 1 \) the equation

\[
-K_1(p) = p^2 + p^3 = \xi
\]

has only one root \( \tilde{\phi}_3(\xi) \) such that \( \Re \tilde{\phi}_3(\xi) > 0 \) for all

\[
\xi \in D_1 \equiv \{ \xi \in \mathbb{C} : \Re \xi > 0 \}.
\]

Therefore in the case \( \alpha = 0 \) condition (2.6) is a system of two equations, and so we assume in problem (2.1) one boundary datum, \( u(0,t) = 0 \), and in the case \( \alpha = 1 \) condition (2.6) is one equation, and so we assume in problem (2.1) two boundary data, \( u(0,t) = u_x(0,t) = 0 \). Solving (2.6) we define the Laplace transforms of the remainder boundary functions \( v_j(t) = \partial_x \tilde{u}(0,t) \), \( j = 1, 2 \), for the case \( \alpha = 0 \) as follows:

\[
(2.7) \quad \tilde{v}_1(\xi) = \frac{\tilde{u}_0(\phi_2) - \tilde{u}_0(\phi_1) + \tilde{f}(\phi_2, \xi) - \tilde{f}(\phi_1, \xi)}{\phi_1 - \phi_2},
\]

\[
(2.8) \quad \tilde{v}_2(\xi) = \frac{(\tilde{u}_0(\phi_2) + \tilde{f}(\phi_2, \xi))(1 - \phi_1) + (\tilde{u}_0(\phi_1) + \tilde{f}(\phi_1, \xi))(\phi_2 - 1)}{\phi_1 - \phi_2}.
\]

And for the case \( \alpha = 1 \) from condition (2.6) we obtain for the Laplace transform of the remainder boundary function \( v_2(t) = \partial_x^2 u(0,t) \)

\[
\tilde{v}_2(\xi) = \tilde{u}_0(\tilde{\phi}_3(\xi)) + \tilde{f}(\tilde{\phi}_3(\xi), \xi).
\]

To find the remaining boundary functions \( \tilde{v}_j(t) = u_x^{(j)}(0,t) \), we take the inverse Laplace transforms

\[
u_x^{(j)}(0,t) = \frac{1}{2\pi i} \int_{\Gamma} e^{\xi t} \tilde{v}_j(\xi) \, d\xi,
\]

where \( \Gamma = \partial D_\alpha \); i.e.,

\[
\Gamma_0 = (-i\infty, -i0) \cup \left[ -i0, \frac{4}{27} - i0 \right] \cup \left[ \frac{4}{27} + i0, i0 \right] \cup (i0, i\infty),
\]

\[
\Gamma_1 = (-i\infty, i\infty),
\]

since condition (2.4) is fulfilled for the functions \( \tilde{v}_j(\xi) \) which are analytic in the domain \( D_\alpha \).
Now we prove an integral representation of the solution \( u(x, t) \) to the problem (2.1),

\[
(2.9) \quad u(x, t) = \int_0^\infty u_0(y) F_0(x, y, t) \, dy \\
+ \int_0^t d\tau \int_0^\infty f(y, \tau) F_0(x, y, t - \tau) \, dy,
\]

where

\[
F_0(x, y, t) = -\frac{1}{4\pi^2} \int^{-\infty}_{-\infty} dpe^{-py} \left( \int_{-\infty}^{\infty} e^{\xi t} \frac{\phi_3(\xi) e^{\phi_3(\xi)\xi} + e^{\phi_3(\xi)}}{K_0(p) + \xi} \, d\xi \right).
\]

\( \phi_3(\xi) \) is the root of the equation \( K_0(p) = -\xi \), such that \( \Re \phi_3(\xi) < 0 \) for \( \Re \xi > 0 \), and for the case \( \alpha = 1 \),

\[
F_1(x, y, t) = \frac{1}{4\pi^2} \int^{-\infty}_{-\infty} dpe^{px} \left( \int_{-\infty}^{\infty} e^{\xi t} \frac{-\phi_3(\xi) y - e^{-\phi_3(\xi)y}}{K_1(p) + \xi} \, d\xi \right).
\]

\( \phi_3(\xi) \) is the root of the equation \( K_1(p) = -\xi \), such that \( \Re \phi_3(\xi) > 0 \) for \( \Re \xi > 0 \). Note that the integrals in the definition of the functions \( F_0, F_1 \) are convergent absolutely for all \( t > 0 \). Indeed, since

\[
\frac{1}{|a + ib|} \leq \frac{1}{|a|^\nu |b|^{1-\nu}}
\]

for any \( a, b \in \mathbb{R} \), where \( \nu \in [0, 1] \), we have

\[
(2.10) \quad \frac{1}{|K_0(p) + \xi|} \leq |p|^{-2\nu} |\xi + (-1)^\nu p^3|^{1+\nu}
\]

for all \( \Re p = 0, \Re \xi = 0, p, \xi \neq 0 \), where \( \nu \in [0, 1] \). Therefore, integrating by parts with respect to \( \xi \) in the domain \( |\xi| > 1 \), we get

\[
|F_\alpha| \leq C \int_{-\infty}^{\infty} |d| \int_{-\infty}^{\infty} \left| \frac{d|}{|K_\alpha(p) + \xi|^2} \right| + \frac{C}{t} \int_{-\infty}^{\infty} |d| \int_{|\xi| = 0, |\xi| > 1} \frac{d|}{|K_\alpha(p) + \xi|^2} \int_{|\xi| = 0, |\xi| > 1} \frac{d|}{|K_\alpha(p) + \xi|^2}
\]

\[
+ \frac{C}{t} \int_{-\infty}^{\infty} |d| \int_{|\xi| = 0, |\xi| > 1} \frac{d|}{|K_\alpha(p) + \xi|^2} \int_{|\xi| = 0, |\xi| > 1} \frac{d|}{|K_\alpha(p) + \xi|^2} < C \max(t^{-1}, 1).
\]

We prove formula (2.9) for the more difficult case \( \alpha = 0 \) only; for the case \( \alpha = 1 \) the proof is analogous. Substituting representations (2.7), (2.8) into (2.2) and taking the inverse Laplace transformation with respect to \( p \), we obtain

\[
(2.11) \quad u(x, t) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} e^{-K_0(p)x + px} \left( \hat{u}_0(p) + \int_0^t e^{K_0(p)p} \hat{f}(p, \tau) \, d\tau \right)
\]

\[
+ \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} e^{-K_0(p)(x-\tau)} \left( \int_0^\infty e^{K_0(p)(x-\tau)} \, d\tau \int_{|\xi| > 1} e^{\xi \tau \eta} H(p, \xi) \, d\xi \right)
\]

\[
\equiv I_1 + I_2,
\]
where

\[ H(p, \xi) = \frac{(p - \phi_2)(\hat{u}_0(\phi_1) + \hat{\phi}_1(\phi_1, \xi)) + (\phi_1 - p)(\hat{u}_0(\phi_2) + \hat{\phi}_2(\phi_2, \xi))}{\phi_1 - \phi_2}. \]

Substitution of \( \hat{u}_0(p) = \int_0^\infty e^{-px} u_0(y) \, dy \) and \( \hat{\phi}(p, \tau) = \int_0^\infty e^{-px} f(y, \tau) \, dy \) into the first integral \( I_1 \) yields

\[
(2.12) \quad I_1 = \frac{1}{2\pi i} \left( \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-K_0(p)\tau + px} \int_0^\infty e^{-py} u_0(y, t) \, dy \, dp \right.
\]
\[
+ \int_{-\infty}^{+\infty} \int_0^t \int_0^\infty e^{-K_0(p)(t-\tau)} e^{-px} f(y; \tau) \, dy \, dp \, d\tau \bigg) \bigg)
\]
\[
= \int_0^\infty u_0(y) G(x - y, t) \, dy + \int_0^t \int_0^\infty f(y, \tau) G(x - y, t - \tau) \, d\tau \, dy,
\]

where

\[ G(x, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{px - K_0(p)\tau} \, dp. \]

The change of the order of integration in formula (2.12) is justified under the condition

\[ u_0 \in L^1, \quad f \in L^q(0, t; L^1(R^+)) \]

with \( q > 2 \), by the Fubini theorem. Now we consider the integral \( I_2 \). We prove the formula

\[
(2.13) \quad I_2 = \int_0^\infty u_0(y) G_1(x, y, t) \, dy + \int_0^t \int_0^\infty f(y, \tau) G_1(x, y, t - \tau) \, dy \, d\tau,
\]

where

\[ G_1(x, y, t) = -\frac{1}{4\pi^2} \int_{-\infty}^{+\infty} e^{-py} \int_{-\infty}^{+\infty} e^{\xi t} \frac{\phi_1 - \phi_2}{\phi_1 - \phi_2} \, d\xi. \]

For simplicity we consider only the part of the function \( H(p, \xi) \) with the function \( u_0(x) \); the part of the function \( H(p, \xi) \) containing the function \( f(x, t) \) is considered in the same manner. Substituting the Laplace transform of \( u_0 \) into the definition of the function \( H(p, \xi) \), changing the order of integration, and integrating with respect to \( t > 0 \), we get

\[
I_2 = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} e^{px} \int_0^t e^{-K_0(p)(t-\tau)} d\tau \int_{-\infty}^{+\infty} e^{\xi \tau} \frac{\phi_1 - \phi_2}{\phi_1 - \phi_2} \, d\xi \times
\]
\[
\left. \left( \int_0^{+\infty} e^{-\phi_1 y} u_0(y) \, dy + (\phi_1 - p) \int_0^{+\infty} e^{-\phi_2 y} u_0(y) \, dy \right) \right) \int_{\Gamma_0} \tilde{H}(p, \phi_1, \phi_2, y) \frac{e^{(K_0(p) + \xi)\tau} - 1}{K_0(p) + \xi} \, d\xi,
\]

where

\[ \tilde{H}(p, \phi_1, \phi_2, y) = \int_{\Gamma_0} H(p, \phi_1, \phi_2, y) \, d\xi. \]
where
\[\tilde{H}(p, \phi_1, \phi_2, y) = \frac{(p - \phi_2)e^{-\phi_1 y} + (\phi_1 - p)e^{-\phi_2 y}}{\phi_1 - \phi_2} .\]

Since \(\Re K_0(p) > 0\) for all \(\Re p = 0, p \neq 0\) and due to \(\phi_1(\xi) \neq \phi_2(\xi)\) for all \(\xi \in D_0\), the function \(\tilde{H}(p, \phi_1, \phi_2, y)\) is analytic in the domain \(D_0\); therefore by the Cauchy theorem we have
\[\int_{\Gamma_0} \frac{\tilde{H}(p, \phi_1, \phi_2, y)}{K_0(p) + \xi} d\xi = 0.\]

Using the relations
\[\overline{\phi_1(\xi)} = \phi_2(\overline{\xi})\]
for all
\[\xi \in D_0 \equiv \{\xi \in \mathbb{C}: \Re \xi > 0, \xi \notin \left[0, \frac{4}{\pi}\right]\}\]
and \(\tilde{H}(p, \phi_1, \phi_2, y) = \tilde{H}(p, \phi_2, \phi_1, y)\), we can change the contour of integration \(\Gamma_0\) to the imaginary axis \((-i\infty, i\infty)\) to get
\[\int_{-i\infty}^{i\infty} \tilde{H}(p, \phi_1, \phi_2, y) e^{\xi} K_0(p) + \xi d\xi = \int_{-i\infty}^{i\infty} \tilde{H}(p, \phi_1, \phi_2, y) e^{\xi} K_0(p) + \xi d\xi;\]
therefore we can write the integral \(I_2\) in the form
\[I_2 = \frac{1}{4\pi^2} \int_0^{\infty} u_0(y) dy \int_{-i\infty}^{i\infty} \tilde{H}(p, \phi_1, \phi_2, y) e^{\xi} K_0(p) + \xi d\xi;\]
(since \(\Re \phi_1, 2(\xi) > 0, |\phi_1 - \phi_2| \geq 1\) for all \(\Re \xi = 0\), and from inequality (2.10), we see that the last integral in the above formula converges absolutely). Hence applying the identities
\[K_0(p) + \xi = (p - \phi_1)(p - \phi_2)(p - \phi_3),\]
\[\frac{1}{(\phi_1 - \phi_2)(\phi_1 - \phi_3)} = \phi_1'(\xi), \quad \frac{1}{(\phi_2 - \phi_1)(\phi_2 - \phi_3)} = \phi_2'(\xi),\]
and using the theory of residues, we obtain
\[\int_{-i\infty}^{i\infty} \frac{\tilde{H}(p, \phi_1, \phi_2, y)}{K_0(p) + \xi} e^{\xi} dp = 2\pi i \int_{-i\infty}^{i\infty} \tilde{H}(p, \phi_1, \phi_2, y) e^{\phi_3(\xi)} dp\]
\[= 2\pi i \int_{-i\infty}^{i\infty} \tilde{H}(p, \phi_1, \phi_2, y) e^{\phi_3(\xi)} dp\]
\[= 2\pi i e^{\phi_3(\xi)} (e^{-\phi_1(\xi)} \phi_1'(\xi) + e^{-\phi_2(\xi)} \phi_2'(\xi))\]
\[= -e^{\phi_3(\xi)} \int_{-i\infty}^{i\infty} \frac{1}{K_0(p) + \xi} e^{-\xi} dp.\]
Thus the formula (2.9) is proved in the case where

\begin{equation}
G_1(x, y, t) = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} e^{-py} dp \int_{-\infty}^{+\infty} e^{\xi y + \phi_3 x} \frac{1}{K_0(p) + \xi} d\xi.
\end{equation}

Also since \(e^{-K_0(p)t} = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} e^{\xi t} \frac{1}{K_0(p) + \xi} d\xi\), we can rewrite the function \(G(x, t)\) in the form

\begin{equation}
G(x, t) = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} e^{-py} dp \int_{-\infty}^{+\infty} e^{\xi t} \frac{1}{K_0(p) + \xi} d\xi.
\end{equation}

Therefore, using (2.11)–(2.13) for solution \(u(x, t)\), we obtain

\begin{equation}
u(x, t) = \int_0^t u_0(y) F_0(x, y, t) dy + \int_0^t d\tau \int_0^t f(y, \tau) F_0(x, y, t - \tau) dy,
\end{equation}

where for the function \(F_0(x, y, t) = G(x, y, t) + G_1(x, y, t)\) we have

\begin{equation}
F_0(x, y, t) = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} e^{-py} dp \int_{-\infty}^{+\infty} e^{\xi t} \frac{1}{K_0(p) + \xi} d\xi.
\end{equation}

Thus the formula (2.9) is proved in the case \(\alpha = 0\).

Now we formulate the following result, which will be proved in Section 4.

**Theorem 2.1.** Let

\(u_0 \in H^{0, \omega}(\mathbb{R}^+) \cap H^{1,0}(\mathbb{R}^+)\) \(t^\nu f \in L^\infty(0, T; L^1(\mathbb{R}^+))\)

with \(\omega \in (\frac{1}{2}, \frac{3}{2}), \nu \in (\frac{1}{5}, \frac{1}{2})\). Then for some \(T > 0\) there exists a unique solution

\(u \in C((0, T]; H^{1,\omega}) \cap C([0, T]; H^{0,x})\)

of the initial–boundary value problem (2.1) such that

\[\sup_{t \in [0, T]} (\|u\|_{H^{0,\omega}} + t^\nu \|u_t\|_{L^2} + \|x^\alpha u_x\|_{L^2}) \leq C\lambda,\]

where

\[\lambda = \|u_0\|_{H^{0,\omega}} + \|u_0\|_{H^{1,0}} + T^{\frac{1}{2} - \nu} \sup_{t \in [0, T]} t^\nu \|f\|_{L^1}\]

and

\[\nu \in (\frac{1}{5}, \frac{1}{2}), \quad \alpha \in (0, \frac{1}{2}).\]

To prove this result we prove some preliminary estimates in the next section.
3. PRELIMINARIES

As in Section 2 we denote by \( \phi_l(\xi), l = 1, 2, 3 \), the roots of the equation

\[-K_0(\xi) = p^2 - p^3 = \xi,\]

such that

\[\Re \phi_l(\xi) > 0, \quad l = 1, 2, \quad \Re \phi_3(\xi) < 0,\]

for all \( \xi \in D_0 = \{ \xi \in \mathbb{C} : \Re \xi \geq 0, \xi \notin [0, \frac{4}{27}] \} \).

Note that the functions \( \phi_l(\xi) \) are analytic in the domain

\( \{ \xi \in \mathbb{C} : \xi \notin (-\infty, \frac{4}{27}] \} \).

We represent

\[p^2 = \frac{\xi}{1 - p}\] for \( |p| < 1 \) and \[p^3 = \frac{-\xi}{1 - \frac{1}{p}}\] for \( |p| > 1 \).

Hence we get

(3.1) \[\phi_1(\xi) = \begin{cases} \sqrt{\xi} + O(|\xi|), & \xi \to 0, \Re \xi > 0, 1 + O(|\xi|), & \xi \to 0, \Re \xi < 0, \\ e^{\frac{i}{3}} \sqrt{\xi} + O(1), & |\xi| \to \infty, \end{cases}\]

(3.2) \[\phi_2(\xi) = \begin{cases} 1 + O(|\xi|), & \xi \to 0, \Re \xi > 0, \sqrt{\xi} + O(|\xi|), & \xi \to 0, \Re \xi < 0, \\ e^{-i \frac{2}{3}} \sqrt{\xi} + O(1), & |\xi| \to \infty, \end{cases}\]

and

(3.3) \[\phi_3(\xi) = \begin{cases} -\sqrt{\xi} + O(|\xi|), & |\xi| \to 0, \\ -\sqrt{\xi} + O(1), & |\xi| \to \infty, \end{cases}\]

for all \( \xi \in \{ \mathbb{C} : \xi \notin (-\infty, \frac{4}{27}] \} \)

(by \( \sqrt{\xi} \) and \( \sqrt[3]{\xi} \) we denote the main value of the analytic function; i.e., \( \sqrt{1} = \sqrt[3]{1} = 1 \)). In the case \( \alpha = 1 \) the equation

\[-K_1(p) = p^2 + p^3 = \xi\]

has roots

\( \tilde{\phi}_l(\xi) = -\phi_l(\xi), \quad l = 1, 2, 3, \)
such that
\[ \Re \tilde{\phi}_1(\xi) < 0, \quad \Re \tilde{\phi}_2(\xi) < 0, \quad \Re \tilde{\phi}_3(\xi) > 0 \]
for all
\[ \xi \in D_1 = \{ \xi \in \mathbb{C} : \Re \xi \geq 0, \quad \xi \neq 0 \}. \]

Note that the functions \( \tilde{\phi}_l(\xi) \) are analytic in the domain
\[ \{ \xi \in \mathbb{C} : \xi \neq (-\infty, 0] \} \]
and have the asymptotic representations
\[
\begin{align*}
\tilde{\phi}_1(\xi) &= 
\begin{cases}
-\sqrt{\xi} + O(\xi), & \xi \to 0, \Im \xi > 0, \\
-1 + O(\xi), & \xi \to 0, \Im \xi < 0, \\
-e^{-i\pi/3} \sqrt{\xi} + O(1), & |\xi| \to \infty,
\end{cases} \\
\tilde{\phi}_2(\xi) &= 
\begin{cases}
-1 + O(\xi), & \xi \to 0, \Im \xi > 0, \\
-\sqrt{\xi} + O(\xi), & \xi \to 0, \Im \xi < 0, \\
-e^{i\pi/3} \sqrt{\xi} + O(1), & |\xi| \to \infty,
\end{cases} \\
\tilde{\phi}_3(\xi) &= 
\begin{cases}
\sqrt{\xi} + O(\xi), & |\xi| \to 0, \\
\sqrt{\xi} + O(1), & |\xi| \to \infty,
\end{cases}
\end{align*}
\]
for all
\[ \xi \in \{ \mathbb{C} : \xi \neq (-\infty, 0] \}. \]

As in the previous section we denote
\[
\begin{align*}
F_0(x, y, t) &= -\frac{1}{4\pi^2} \int_{-\infty}^{\infty} dp e^{-p^2 y} \int_{-\infty}^{\infty} d\xi e^{\xi t} \frac{e^{\phi_1(\xi) x + e^{p x}}}{K_0(p) + \xi}, \\
F_1(x, y, t) &= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dp e^{p x} \int_{-\infty}^{\infty} d\xi e^{\xi t} \frac{e^{-(\phi_2(\xi))(y - e^{-p y})}}{K_1(p) + \xi},
\end{align*}
\]
for all \( x, y, t \geq 0 \), where \( K_0(p) = -p^2 + p^3 \) and \( K_1(p) = -p^2 - p^3 \).

In the next lemma we prove the asymptotics of the functions \( F_\alpha \) for large \( t \).

**Lemma 3.1.** We have
\[
\| \partial_x^n \partial_y^n F_\alpha(\cdot, y, t) \|_{L^2} \leq Cy t^{-\alpha_n + \frac{3}{2} + \gamma},
\]
where
\[ \alpha_n = \frac{3 + 2n}{4}, \quad \gamma > 0, \]
for the function changing the order of integration and using the theory of residues we get

\[ y^{1+\mu}O\left(t^{-1-\xi} \min \left(\frac{x}{\sqrt{t}}, 1\right)\right) \]

for any \( x, y, t > 0 \), where

\[ \mu \in (0, 1), \quad \alpha = 0, 1, \quad \Lambda_0(y) = e^{-y} - 1 + y, \quad \Lambda_1(y) = y, \]

\[ \Phi_0(q, t) = \frac{q}{\sqrt{\pi}} e^{-q^2}, \quad \Phi_1(q, t) = \frac{1}{\sqrt{\pi t}} \left( e^{-q^2} (2q\sqrt{t} - 1) + e^{-2q\sqrt{t}} \right). \]

**Proof.** First we show the estimate (3.9) for the function \( F_1(x, y, t) \). Since \( K'_1(p) = 0 \) for \( p = 0 \) and \( p = -\frac{1}{2} \), we have \( \tilde{\phi}_1(\xi) \neq \tilde{\phi}_2(\xi) \) for all \( \Re \xi = 0 \). Also we easily get \( \sum_{j=1}^3 \tilde{\phi}_j(\xi) = -1 \) and \( \tilde{\phi}_1 \tilde{\phi}_2 \tilde{\phi}_3 = \xi \). Since \( K'_1(\tilde{\phi}_j(\xi)) = -1/\tilde{\phi}_j'(\xi) \), by the theory of residues,

\[ \int_{-\infty}^{\infty} \frac{e^{pt}}{K_1(p) + \xi} d\xi = 2\pi i e^{-K_1(p)t} \]

and

\[ \int_{-\infty}^{\infty} \frac{e^{px}}{\xi + K_1(p)} dp = 2\pi i \sum_{j=1}^2 \text{res}_{p=\tilde{\phi}_j(\xi)} \frac{e^{px}}{\xi + K_1(p)} \]

\[ = -2\pi i \sum_{j=1}^2 e^{\tilde{\phi}_j(\xi)x} \tilde{\phi}_j'(\xi), \]

changing the order of integration and using the theory of residues we get for the function \( F_1 \),

\[ F_1(x, y, t) = \frac{1}{2\pi i} \left( \int_{-\infty}^{\infty} d\xi e^{\xi(-\tilde{\phi}_1(\xi)y)} \sum_{j=1}^2 e^{\tilde{\phi}_j(\xi)x} \tilde{\phi}_j'(\xi) \right. \]

\[ + \left. \int_{-\infty}^{\infty} dp e^{p(x-y) - K_1(p)t} \right) \]

\[ = M(x, y, t) + R(x, y, t), \]

where

\[ M(x, y, t) = \frac{1}{2\pi i} \left( \int_{-\infty}^{\infty} d\xi e^{\xi(1 - \tilde{\phi}_1(\xi)y)} \sum_{j=1}^2 e^{\tilde{\phi}_j(\xi)x} \tilde{\phi}_j'(\xi) \right. \]

\[ + \left. \int_{-\infty}^{\infty} dp (1 - py) e^{p(x-K_1(p)t)} \right) \]
and

\[ R(x, y, t) = \frac{1}{2\pi i} \left( \int_{-\infty}^{\infty} d\xi e^{\xi t} (e^{-\phi_3(\xi)y} - 1 + \tilde{\phi}_3(\xi)y) \sum_{j=1}^{2} e^{\phi_j(\xi)x} \tilde{\phi}_j(\xi) 
\]
\[ + \int_{-\infty}^{\infty} dp(e^{-py} - 1 + py) e^{px - K_1(p)y} \right). \]

Using the formulas (3.10) and (3.12) and changing the order of integration, we have

\[ \int_{-\infty}^{\infty} e^{px - K_1(p)y} (1 - py) dp = \int_{-\infty}^{\infty} e^{\xi t} \int_{-\infty}^{\infty} e^{px} (1 - py) dp \]
\[ = -\int_{-\infty}^{\infty} e^{\xi t} \sum_{i=1}^{2} e^{\phi_i(\xi)x} (1 - \tilde{\phi}_i(\xi)y) \tilde{\phi}_i(\xi) d\xi. \]

Then via the identity

\[ (\tilde{\phi}_3(\xi) - \tilde{\phi}_j(\xi)) \tilde{\phi}_j(\xi) = \frac{(-1)^j}{\tilde{\phi}_1 - \tilde{\phi}_2}, \quad j = 1, 2, \]

we write the function \( M(x, y, t) \) as

\[ M(x, y, t) = \frac{y}{2\pi i} \int_{-\infty}^{\infty} e^{\xi t} \frac{e^{\phi_3(\xi)x} - e^{\phi_2(\xi)x}}{\tilde{\phi}_1 - \tilde{\phi}_2} d\xi. \]

Similarly for the function \( R(x, y, t) \) we obtain

\[ \int_{-\infty}^{\infty} dpe^{-K_1(p)y} (e^{-py} - 1 + py) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{\xi t} \int_{-\infty}^{\infty} e^{-py} - 1 + py dp \]
\[ = -\int_{-\infty}^{\infty} e^{\xi t} (e^{-\phi_3(\xi)y} - 1 + \phi_3(\xi)y) \sum_{j=1}^{3} \tilde{\phi}_j(\xi) d\xi, \]

since

\[ \sum_{j=1}^{3} \tilde{\phi}_j(\xi) = 0, \quad \Re \tilde{\phi}_3 > 0, \]

and \( K_1(p) = O(|p|^3) \) as \( |p| \to \infty \). Therefore we can rewrite the function \( R(x, y, t) \) as

\[ (3.13) \quad R(x, y, t) = \frac{1}{2\pi i} \left( \int_{-\infty}^{\infty} d\xi e^{\xi t} (e^{-\phi_3(\xi)y} - 1 + \tilde{\phi}_3(\xi)y) \right. \]
\[ \times \sum_{j=1}^{2} (e^{\phi_j(\xi)x} - 1) \tilde{\phi}_j(\xi) + \int_{-\infty}^{\infty} dpe^{-K_1(p)y} \]
\[ \times \left. (e^{px} - 1)(e^{-py} - 1 + py) \right). \]
First we estimate the function $R$. We have $\Re \tilde{\phi}_l(\xi) < 0$, $l = 1, 2$, $\Re \tilde{\phi}_3(\xi) > 0$ on the imaginary axis $\Re \xi = 0$, $\xi \neq 0$. There is a contour in the complex left half-plane $\Re \xi < 0$ such that $\Re \tilde{\phi}_l(\xi) = 0$; it is defined by equation $K_l(iy) = -\xi$, i.e., $-iy^2 - y^2 = \xi$ with $y = \Delta \tilde{\phi}_l(\xi) \in \mathbb{R}$. Therefore there exists a contour

\[ \mathcal{C} = \{ \xi \in \mathbb{C}, \Re \xi < 0, \Re \xi = O(|\xi|), |\xi| \leq 1, \Re \xi = O(|\xi|^{\frac{3}{2}}), |\xi| > 1 \} \]

such that

$\Re \tilde{\phi}_l(\xi) \leq 0$, $l = 1, 2$, $\Re \tilde{\phi}_3(\xi) \geq 0$, for $\xi \in \mathcal{C}$.

Note that the asymptotic formulas (3.4)–(3.6) are valid on the contour $\mathcal{C}$. Then using (3.6) we have for $\xi \in \mathcal{C}$, $|\xi| < 1$,

\[ e^{-\phi_j(\xi)y} - 1 + \tilde{\phi}_3(\xi)y = y^{1+\mu}O(|\xi|^{\frac{1}{2}}), \]

where $\mu \in [0, 1]$. Since by (3.4) and (3.5) we have

\[ \tilde{\phi}_j(\xi) = O(|\xi|^{-\frac{1}{2}}) \quad \text{for} \ \xi \to -(1)^\dagger i0 \]

and

\[ \tilde{\phi}_j(\xi) = O(1) \quad \text{for} \ \xi \to (-1)^\dagger i0, \xi \in \mathcal{C}, \]

where $j = 1, 2$, we get in the case $|\xi| \leq 1$, $\xi \in \mathcal{C}$,

\[ \left| \sum_{j=1}^{2} (e^{\phi_j(\xi)x} - 1) \tilde{\phi}_j(\xi) \right| \leq \frac{C}{\sqrt{|\xi|}} (e^{-C\sqrt{|\xi|}x} - 1) + C(e^{-Cx} - 1) \]

\[ \leq C \min(x, |\xi|^{-\frac{1}{2}}). \]

In the case $|\xi| > 1$, $\xi \in \mathcal{C}$, we have

\[ \left| \sum_{j=1}^{2} (e^{\phi_j(\xi)x} - 1) \tilde{\phi}_j(\xi) \right| \leq C|\xi|^{-\frac{1}{2}} (e^{-C\sqrt{|\xi|}x} - 1) \leq C \min \left(x|\xi|^{-\frac{1}{2}}, |\xi|^{-\frac{1}{2}} \right) \]

\[ \leq C \min(x, |\xi|^{-\frac{1}{2}}). \]

Thus

\[ \sum_{j=1}^{2} (e^{\phi_j(\xi)x} - 1) \tilde{\phi}_j(\xi) = O \left( \min \left(x\sqrt{|\xi|}, 1 \right)|\xi|^{-\frac{1}{2}} \right) \]

for all $\xi \in \mathcal{C}$. Therefore we have

\[
\left( e^{-\phi_3(\xi)y} - 1 + \tilde{\phi}_3(\xi)y \right) \sum_{j=1}^{2} (e^{\phi_j(\xi)x} - 1) \tilde{\phi}_j(\xi) \\
= y^{1+\mu}O \left( \min \left(x\sqrt{|\xi|}, 1 \right)|\xi|^{-\frac{1}{2}} \right)
\]
for all \(x, y > 0, \xi \in \mathbb{C}\). In the same manner we obtain
\[
(e^{px} - 1)(e^{-py} - 1 + py) = y^{1+\mu}O\left( \min(x|p|, 1)|p|^{1+\mu} \right)
\]
for all \(x, y > 0, \Re p = 0\), where \(\mu \in [0, 1]\). Therefore, making the changes in the variables of integration \(p^2 t = z^2\) and \(\xi t = q\) in the formula (3.13), we get
\[
R(x, y, t) = y^{1+\mu}O\left( \min\left(\frac{x}{\sqrt{t}}, 1\right) t^{-1-\frac{\mu}{2}} \right).
\]

Now we estimate the main term of asymptotics \(M(x, y, t)\). Via (3.4) and (3.5) we can write in the case \(x\sqrt{\xi} \leq 1, |\xi| < 1, \xi \in \mathbb{C}\)
\[
\frac{e^{\phi_1(\xi)x} - e^{\phi_2(\xi)x}}{\phi_1(\xi) - \phi_2(\xi)} = \frac{e^{-\sqrt{\xi}x + O(|\xi|x)} - e^{-1 + O(|\xi|x)}}{1 - \sqrt{\xi} + O(|\xi|)}
\]
\[
= \left( e^{-\sqrt{\xi}x}(1 + O(|\xi|x)) - e^{-x}(1 + O(|\xi|x)) \right) 
\times (1 + \sqrt{\xi} + O(|\xi|))
\]
\[
= (e^{-\sqrt{\xi}x} - e^{-x})(1 + \sqrt{\xi}) + O(|\xi|x),
\]
and in the case \(x\sqrt{\xi} > 1, |\xi| < 1, \xi \in \mathbb{C}, \Re \xi > 0\) (the case \(\Re \xi < 0\) is considered in the same manner), we have
\[
|e^{\phi_1(\xi)x} - e^{-\sqrt{\xi}x}| \leq e^{-C\sqrt{|\xi|}x}O(|\xi|x) = O\left( \sqrt{|\xi|} \right)
\]
and
\[
|e^{\phi_2(\xi)x}| \leq \frac{C}{x} \leq C\sqrt{|\xi|}, \quad e^{-x} \leq \frac{C}{x} \leq C\sqrt{|\xi|};
\]
therefore we obtain
\[
\frac{e^{\phi_1(\xi)x} - e^{\phi_2(\xi)x}}{\phi_1(\xi) - \phi_2(\xi)} = (e^{-\sqrt{\xi}x} - e^{-x})(1 + \sqrt{\xi})
\]
\[
+ O\left( \min\left(\sqrt{|\xi|x}, 1\right) \sqrt{|\xi|} \right)
\]
for all \(|\xi| < 1, \xi \in \mathbb{C}\). Again by virtue of asymptotics (3.4) and (3.5) in the case \(x\sqrt{\xi} \leq 1\) we get
\[
|e^{\phi_1(\xi)x} - e^{\phi_2(\xi)x}| \leq C|\phi_1(\xi) - \phi_2(\xi)||x| \leq Cx\sqrt{|\xi|},
\]
and if
\[
x\sqrt{|\xi|} \geq 1
\]
we have
\[ |e^{\phi_1(x)} - e^{\phi_2(x)}| \leq C, \]
and hence
\[ \left| \frac{e^{\phi_1(x)} - e^{\phi_2(x)}}{\phi_1(x) - \phi_2(x)} \right| x \leq C \min(\sqrt{|x|}, 1) \frac{1}{\sqrt{|x|}} \]
\[ \leq C \sqrt{|x|} \min\left(\sqrt{|x|}, 1\right) \]
for \(|\xi| \geq 1, \xi \in \mathbb{C}\). Hence we see that representation (3.15) is valid for all \(\xi \in \mathbb{C}\). By virtue of the estimate
\[ \int_{\mathbb{C}} e^{\xi t} O(\min(\sqrt{|x|}, 1)\sqrt{|\xi|}) d\xi = O\left(\min\left(\frac{x}{\sqrt{t}}, 1\right) t^{-\frac{3}{2}}\right) \]
we write the representation for the function \(M(x, y, t)\),
\[ M(x, y, t) = \frac{y}{2\pi i} \int_{\mathbb{C}} e^{\xi t} (e^{-\sqrt{\xi}x} - e^{-x})(1 + \sqrt{\xi}) d\xi \]
\[ + y O\left(\min\left(\frac{x}{\sqrt{t}}, 1\right) t^{-\frac{3}{2}}\right). \]
Since
\[ \int_{-\infty}^{\infty} e^{-q^2} dq = \sqrt{\pi} \]
and
\[ \int_{-\infty}^{\infty} e^{-q^2} q^2 dq = \frac{\sqrt{\pi}}{2}, \]
making the change of variables of integration \(\xi t - \sqrt{\xi} x = -\eta^2\) and \(\xi t = -q^2\) we obtain
\[ \int_{\mathbb{C}} e^{\xi t} (e^{-\sqrt{\xi}x} - e^{-x})(1 + \sqrt{\xi}) d\xi = \sqrt{\pi} it^{-\frac{3}{2}}\left(e^{-\frac{\xi}{4}}(x - 1) + e^{-x}\right). \]
Therefore we get
\[ (3.16) \quad M(x, y, t) = \frac{y}{2\sqrt{\pi}} t^{-\frac{3}{2}}\left(e^{-\frac{\xi}{4}}(x - 1) + e^{-x}\right) \]
\[ + y O\left(\min\left(\frac{x}{\sqrt{t}}, 1\right) t^{-\frac{3}{2}}\right). \]
From (3.12)–(3.16) we have the asymptotics (3.9) with $\alpha = 1$. Now we show the asymptotics (3.9) for the case $\alpha = 0$. Similarly to the case $\alpha = 1$, we rewrite the function $F_0(x, y, t)$ as

\begin{equation}
F_0(x, y, t) = \frac{1}{2\pi i} \left( \int_{-i\infty}^{i\infty} d\xi e^{\xi t} \sum_{j=1}^{3} e^{-\phi_j(\xi) y} \phi_j'(\xi) \sum_{j=1}^{3} e^{\phi_j(\xi) y} \phi_j'(\xi) \right)
\end{equation}

Since

$$\sum_{j=1}^{3} \phi_j'(\xi) = 0, \quad \sum_{j=1}^{3} \phi_j \phi_j'(\xi) = 0,$$

and $K_0(p) = O(|p|^3)$ for $|p| \geq 1$, we have

$$\int_{-i\infty}^{i\infty} d\xi e^{\xi t} \sum_{j=1}^{2} (1 - \phi_j(\xi)y)\phi_j'(\xi) = -\int_{-i\infty}^{i\infty} dpe^{p(x-y)-K_0(p)t}(1 - py)$$

and

$$\int_{-i\infty}^{i\infty} d\xi e^{\xi t} \sum_{j=1}^{2} (e^{-\phi_j(\xi)y} - 1 + \phi_j(\xi)y)\phi_j'(\xi)
= -\int_{-i\infty}^{i\infty} dpe^{-K_0(p)t}(e^{-py} - 1 + py).$$

Also we change the contour of integration with respect to $\xi$ into $\mathcal{C}$. Denote

$$\mathcal{C}^+ = \{ \xi \in \mathcal{C}, \; \exists \xi > 0 \}$$

and

$$\mathcal{C}^- = \{ \xi \in \mathcal{C}, \; \exists \xi < 0 \}.$$  

Therefore we can write the function $F_0(x, y, t) = M(x, y, t) + R(x, y, t)$, where

$$M(x, y, t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} d\xi e^{\xi t} (e^{\phi_1(\xi)x} - 1)(e^{-\phi_1(\xi)y} - 1 + \phi_1(\xi)y)\phi_1'(\xi)
+ \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} d\xi e^{\xi t} (e^{\phi_2(\xi)x} - 1)(e^{-\phi_2(\xi)y} - 1 + \phi_2(\xi)y)\phi_2'(\xi),$$

and the remainder

$$R(x, y, t) = \frac{1}{2\pi i} \left( \int_{-i\infty}^{i\infty} d\xi e^{\xi t} (e^{\phi_3(\xi)x} - 1)(e^{-\phi_3(\xi)y} - 1 + \phi_3(\xi)y)\phi_3'(\xi)
+ \int_{-i\infty}^{i\infty} d\xi e^{\xi t} (e^{\phi_1(\xi)x} - 1)(e^{-\phi_1(\xi)y} - 1 + \phi_1(\xi)y)\phi_1'(\xi)
+ \int_{-i\infty}^{i\infty} dpe^{-K_0(p)t}(e^{px} - 1)(e^{-py} - 1 + py) \right).$$
By virtue of (3.1)–(3.3) we have
\[
(e^{\phi_{\beta}(\xi)x} - 1)(e^{-\phi_{\beta}(\xi)y} - 1 + \phi_{j}(\xi)y)\phi'_{j}(\xi)
= y^{1+\mu}O\left(\min\left(\sqrt{|\xi|, 1}, |\xi|^{\frac{\mu}{2}}\right)\right)
\]
for $\xi \in \mathbb{C}^+$ if $j = 1$ and for $\xi \in \mathbb{C}^-$ if $j = 2$, where $\mu \in [0, 1]$. Also we have
\[
(e^{px} - 1)(e^{-py} - 1 + py) = y^{1+\mu}O(\min(|p|x, 1), |p|^{1+\mu})
\]
for $\Re p = 0$. Therefore by analogy to (3.14) we obtain
\[
R(x, y, t) = y^{1+\mu}O\left(\min\left(\frac{x}{\sqrt{t}}, 1\right)t^{-\frac{\mu}{2}}\right).
\]
We use (3.1)–(3.3) to find that
\[
(e^{\phi_{\beta}(\xi)x} - 1)(e^{-\phi_{\beta}(\xi)y} - 1 + \phi_{j}(\xi)y)\phi'_{j}(\xi)
= (e^{-\sqrt{t}x} - 1)(e^{-y} - 1 + y) + yO\left(\min(1, x\sqrt{|\xi|}, \sqrt{|\xi|})\right)
\]
for $\xi \in \mathbb{C}^+$ if $j = 1$ and for $\xi \in \mathbb{C}^-$ if $j = 2$. Therefore similarly to (3.16) we get
\[
M(x, y, t) = \frac{e^{-y} - 1 + y}{2\pi i}
\times \int_{\gamma} d\xi e^{\xi t}(e^{-\sqrt{t}x} - 1) + yO\left(\min\left(\frac{x}{\sqrt{t}}, 1\right)t^{-\frac{\mu}{2}}\right)
= \frac{e^{-y} - 1 + y}{t\sqrt{\pi}} e^{-\frac{x}{2\sqrt{t}}} x + yO\left(\min\left(\frac{x}{\sqrt{t}}, 1\right)t^{-\frac{\mu}{2}}\right).
\]
By virtue of formulas (3.18) and (3.19) we obtain (3.9) with $\alpha = 0$. Now we prove estimate (3.8) in the case $\alpha = 0$, $n = 0$ (the case $\alpha = 0$, $n = 1$ and cases $\alpha = 1$, $n = 0, 1$ are considered by analogy). Since
\[
\int_{-\infty}^{\infty} d\xi e^{\xi t} e^{\phi_{\beta}(\xi)x} \sum_{j=1}^{2}(1 - \phi_{j}(\xi)y)\phi'_{j}(\xi) = -\int_{-\infty}^{\infty} dpe^{px-K_{0}(p)t}(1 - py)
\]
via (3.17) we have
\[
F_{0}(x, y, t) = \frac{1}{2\pi i}\left(\int_{-\infty}^{\infty} d\xi e^{\xi t} e^{\phi_{\beta}(\xi)x} \sum_{j=1}^{2}(e^{-\phi_{j}(\xi)y} - 1 + \phi_{j}(\xi)y)\phi'_{j}(\xi)
+ \int_{-\infty}^{\infty} dpe^{px-K_{0}(p)t}(e^{-py} - 1 + py)\right).
\]
By virtue of the inequalities
\[
\left| e^{\phi_1(\xi)x} \sum_{j=1}^{2} (e^{-\phi_j(\xi)y} - 1 + \phi_j(\xi)y) \phi_j'(\xi) \right| \leq \begin{cases} 
C |\xi|^{-\frac{7}{2}} yx^{-\mu}, & \text{if } |\xi| \leq 1, \\
C |\xi|^{-\frac{11}{2}} yx^{-\mu}, & \text{if } |\xi| > 1,
\end{cases}
\]
and
\[
\left\| x^\delta \int_{-\infty}^{\infty} pe^{x-K_\delta(p)y} (e^{-py} - 1 + py) \right\| \leq Cy \int_{-\infty}^{\infty} e^{-Re K_\delta(p)y} |p|^{1-\delta} |dp| \\
\leq Cy^{-\frac{1}{2}+\frac{\delta}{2}},
\]
we easily get
\[
\| x^\delta F(x, y, t) \|_{L^2} \leq Cy \int_{\xi \in \mathcal{C}} |\xi|^{-\frac{7}{2}-\gamma} e^{Re \xi t} d\xi + Cy^{-\frac{1}{2}+\gamma} \\
\leq Cy^{-\frac{1}{2}+\gamma},
\]
Lemma 3.1 is proved. \( \blacksquare \)

4. PROOF OF THEOREM 2.1

From Section 2 we see that the solution of problem (2.1) can be represented as
\[
(4.1) \quad u(x, t) = \int_0^\infty u_0(y) F_a(x, y, t) dy \\
+ \int_0^t d\tau \int_0^\infty dy f(y, \tau) F_a(x, y, t - \tau),
\]
where the functions \( F_a(x, y, t) \) are defined in (3.7). Let us prove the estimate
\[
(4.2) \quad \| u \|_{X_{T, \nu}^{\omega}} \leq C\lambda,
\]
where
\[
\| u \|_{X_{T, \nu}^{\omega}} = \sup_{t \in (0, T]} (\| (\cdot)^x u(\cdot, t) \|_{L^2} + \| (\cdot)^{\nu} u_x(\cdot, t) \|_{L^2} + t^{\nu} \| u_x(\cdot, t) \|_{L^2}),
\]
\[
\lambda = \| u_0 \|_{H^{\frac{1}{2}}} + \| u_0 \|_{H^{\frac{1}{2}} + \| T^{1-\nu} \sup_{t \in (0, T]} t^{\nu} \| (\cdot)^\omega f(\cdot, t) \|_{L^1}},
\]
\[
T > 0, \quad \nu \in \left( \frac{1}{2}, \frac{3}{2} \right), \quad \kappa \in \left( 0, \frac{1}{2} \right), \quad \omega \in \left( \frac{1}{2}, \frac{3}{2} \right),
\]
and the norm is taken with respect to the space variable $x$, which is denoted by the dot. From (4.1) we have

$$
(4.3) \|\cdot \|^{\alpha} F_{\alpha}(\cdot, y, t) \|_{L^2} \leq C \left\| \int_0^\infty u_0(y)(\cdot)^{\alpha} dF_{\alpha}(\cdot, y, t) dy \right\|_{L^2} + C \int_0^t \int_0^\infty |f(y, \tau)| \left\| (\cdot)^{\alpha} F_{\alpha}(\cdot, y, t) \right\|_{L^2} dy,
$$

for $\alpha = 0, 1, n = 0, 1, \delta \geq 0$. Consider the case $\alpha = 1$; the case $\alpha = 0$ is considered in the same manner. We have

$$
\partial_x^n F_t(x, y, t) = H^{(n)}_x(x, y, t) + G^{(n)}_x(x - y, t),
$$

where

$$
H^{(n)}_x(x, y, t) = -\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{\xi t} e^{-\phi(\xi)y} \sum_{j=1}^2 \bar{\phi}_j(\xi) \bar{\phi}_j(\xi) d\xi,
$$

and

$$
G^{(n)}_x(x, t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dp e^{pt-K(p)t} p^d dp.
$$

Now we prove some estimates for the function $H^{(n)}_x(x, y, t)$. Using (3.4)–(3.6), we obtain

$$
e^{-\phi(\xi)y} \sum_{j=1}^2 \bar{\phi}_j(\xi) \bar{\phi}_j(\xi) = O\left(x^{-\mu} y^{-\theta_1} |\xi|^{-\frac{\mu+\theta_1-n}{2}} + x^{-\mu} y^{-\theta_2} |\xi|^{-\frac{\mu+\theta_2-n}{2}}\right)
$$

for $|\xi| < 1$ and

$$
e^{-\phi(\xi)y} \sum_{j=1}^2 \bar{\phi}_j(\xi) \bar{\phi}_j(\xi) = O\left(x^{-\mu} y^{-\theta_1} |\xi|^{-\frac{2\mu+\theta_1-n}{2}}\right)
$$

for $|\xi| \geq 1$, where $n = 0, 1; \mu, \theta_1, \theta_2 \geq 0$. Therefore changing the contour of integration to the contour

$$
\mathcal{C} = \left\{ \xi \in \mathbb{C}, \Re \xi < 0, \Re \xi = O(|\xi|), \, |\xi| \leq 1, \, \Re \xi = O\left(|\xi|^{\frac{1}{2}}\right), \, |\xi| > 1 \right\}
$$

such that $\Re \phi_j(\xi) \leq 0$, $l = 1, 2$, for $\xi \in \mathcal{C}$, we get

$$
\left| H^{(n)}_x(x, y, t) \right| \leq C x^{-\mu} \int_{\mathcal{C}} \left| \xi \right|^{-\frac{\mu+\theta_1-n}{2}} |d\xi| + C x^{-\mu} \int_{|\xi| < 1} \frac{|d\xi|}{\sqrt{|\xi|}}
$$

$$
+ C x^{-\mu} y^{-\gamma-1} \int_{|\xi| > 1, \xi \in \mathcal{C}} e^{\Re(\xi)} \left| \xi \right|^{-\frac{\mu+\theta_1-n}{2}} |d\xi| = x^{-\mu} \left( O(1) + O\left(y^{-\gamma-1} t^{-\frac{n+\mu}{2}}\right) \right),
$$
where $\mu \in [0, 1 + n)$. So choosing

$$\mu = \frac{1}{2} + \delta \pm \gamma, \quad \delta \in [0, \frac{1}{2} + n)$$

($\gamma > 0$ is small enough), we obtain

$$(4.4) \quad \| (\cdot)^{\delta} H_x^{(n)} (\cdot, y, t) \|_{L^2} \leq C (1 + y^{1 - 1/2}) t^{-\frac{n + 2}{2}}$$

for all $y > 0, t \in (0, T]$. In the same way as in the proof of the above estimate we have the estimate

$$\| G(\cdot, t) \|_{L^1} + \| (\cdot)^{\delta} G(\cdot, t) \|_{L^1} \leq C$$

and

$$(4.5) \quad \| (\cdot)^{\delta} G^{(n)} (\cdot, t) \|_{L^2} \leq C \| (\cdot)^{\delta} G^{(n)} (\cdot, t) \|_{L^2}$$

for all $t \in (0, T]$, where $\delta \in [0, \frac{1}{2} + n)$. Applying the Young inequality

$$\left\| \int_{\theta}^{\infty} g(x - y) v(x) \, dx \right\|_{L^2} \leq C \| g \|_{L^1} \| v \|_{L^1},$$

we have

$$\| (\cdot)^{\delta} \partial^\mu u(\cdot, t) \|_{L^2} \leq C \left( \int_0^\infty | u_0(y) (\cdot)^{\delta} F_x^{(n)} (\cdot, y, t) \, dy \right)_{L^1}$$

$$+ C \int_0^\infty | f(y, \tau) \| (\cdot)^{\delta} F_x^{(n)} (\cdot, y, t - \tau) \|_{L^2} \, dy$$

$$\leq C \left( \int_0^\infty \| u_0(y) \| (\cdot)^{\delta} H_x^{(n)} (\cdot, y, t) \|_{L^2} \, dy \right)$$

$$+ \left( \int_0^\infty | u_0(y) G_x^{(n)} (\cdot, y, t) \|_{L^2} \, dy \right)$$

$$+ \sup_{\tau \in [0, T]} \tau^\nu \| (\cdot)^{\delta} F (\cdot, t) \|_{L^1} \left( \int_0^T \frac{d\tau}{\tau^\nu (1 - \tau)^{\frac{3+2}{2}}} \right)$$

$$\leq C \left( \int_0^\infty | u_0(y) (1 + y^{1 - 1/2}) \, dy \right)$$

$$+ \| u_0 \|_{L^1} \left( \int_0^\infty \| (\cdot)^{\delta} G_x^{(n)} (\cdot, t) \|_{L^1} \right)$$

$$\leq C \left( \| u_0 \|_{H^{1.5}} + T^{1 - \nu} \| (\cdot)^{\delta} F (\cdot, t) \|_{L^1} \right)$$

$$\leq C \left( \| u_0 \|_{H^{1.5}} + T^{1 - \nu} \sup_{\tau \in [0, T]} \tau^\nu \left( \int_0^T \left( \| (\cdot)^{\delta} F (\cdot, t) \|_{L^1} \right) \right) \right) \leq C \lambda,
where
\[ \mu = \frac{1}{4} - \nu, \quad \nu < \frac{1}{4}, \quad \delta \in [0, \frac{1}{2} + n), \quad n = 0, 1, \]
whence we get (4.2). We prove the uniqueness of the solution. Multiplying Eq. (2.1) with \( f = \frac{u}{x + \nu} \), integrating the result with respect to \( x > 0 \), and using the identities
\[ \int_0^\infty u u_x dx = -\int_0^\infty u_x^2 dx \]
and
\[ \int_0^\infty u u_{xxx} dx = \frac{(-1)^\alpha}{2} u_x^2(0, t), \]
we have
\[ \frac{1}{2} \frac{d}{dt} \|u(t)\|_{L^2}^2 + \int_0^\infty u_x^2 dx + \frac{(-1)^\alpha}{2} u_x^2(0, t) = 0. \]
Integrating the above equation with respect to \( t \) and using the homogeneous initial and boundary conditions, we obtain \( \|u(t)\|_{L^2} = 0 \). Thus the solution of the linear problem (2.1) is unique. Theorem 2.1 is proved.

5. PROOF OF THEOREM 1.1

We prove the local existence of solutions by the contraction mapping principle in the space
\[ X_{k, \lambda, \omega}^\nu = \{ \phi \in L^2 : \|\phi\|_{X_{k, \lambda, \omega}^\nu} < \rho \}, \]
where
\[ \|u\|_{X_{k, \lambda, \omega}^\nu} = \sup_{t \in (0, T]} (\|\partial_x^k u(\cdot, t)\|_{L^2} + \|\partial_x^\nu u(\cdot, t)\|_{L^2} + \|\partial_x^\nu u_x(\cdot, t)\|_{L^2}), \]
where
\[ T > 0, \quad \nu \in \left( \frac{1}{6}, \frac{1}{4} \right), \quad \kappa \in (0, \frac{1}{2}), \quad \omega \in \left( \frac{1}{4}, \frac{3}{2} \right). \]
Let \( u(x, t) \) be a solution of the linear problem
\[ u_t + \mathbb{N}(w) - u_{xx} + (-1)^\nu u_{xxx} = 0, \quad t > 0, \quad x > 0, \]
\[ u(x, 0) = u_0(x), \quad x > 0, \]
\[ \partial_x^j u(0, t) = 0, \quad j = 0, \alpha, t > 0, \]
where \( \mathbb{N}(w) = ww_x \) is well defined since \( w \in X_{k, \lambda, \omega}^\nu \).
Note that the initial–boundary value problem (5.1) defines a mapping $\mathbb{M}$ by $u = \mathbb{M}(w)$, and we will show that $\mathbb{M}$ is the contraction mapping from $X_{T, \rho}^{\kappa, \omega}$ into itself for a sufficiently small $T > 0$. Since $w \in X_{T, \rho}^{\kappa, \omega}$ we have

$\sup_{t \in [0, T]} \| t^{\omega} \langle \cdot \rangle^{\omega}(w)(\cdot, t) \|_{L^1} \leq C \rho^2$.

Via Theorem 2.1 problem (5.1) has a unique solution $u(x, t) \in X_{T, \rho}^{\kappa, \omega}$ with the norm

$\| u \|_{X_{T, \rho}^{\kappa, \omega}} \leq C \lambda$,

where

$\lambda = \| u_0 \|_{H^{\omega}} + \| u_0 \|_{H^{1, \omega}} + T^{1-\nu} \sup_{t \in [0, T]} t^{\nu} \| \langle \cdot \rangle^{\omega}(w)(\cdot, t) \|_{L^1}$

and $\mu = \frac{1}{2} - \nu$. Therefore we obtain

(5.2) $\| u \|_{X_{T, \rho}^{\kappa, \omega}} \leq C \| u_0 \|_{H^{\omega}} + C \| u_0 \|_{H^{1, \omega}} + C T^{1-\rho^2},$

whence we get $\| u \|_{X_{T, \rho}^{\kappa, \omega}} \leq \rho$ if $T > 0$ and $\| u_0 \|_{H^{1, \omega}}$ are sufficiently small. Thus the mapping $\mathbb{M}$ transforms the closed ball $X_{T, \rho}^{\kappa, \omega}$ with a center at the origin and a radius $\rho$ into itself. Analogously we can prove the estimate $\sup_{t \in [0, T]} \| u - \tilde{u} \|_{X_{T, \rho}^{\kappa, \omega}} < \sup_{t \in [0, T]} \| w - \tilde{w} \|_{X_{T, \rho}^{\kappa, \omega}}$ for small $T$. Therefore the mapping $\mathbb{M}$ is a contraction mapping in $X_{T, \rho}^{\kappa, \omega}$ and there exists a unique solution $u(x, t) \in X_{T, \rho}^{\kappa, \omega}$ of the initial-value problem (1.1). Theorem 1.1 is proved.

**Remark 5.1.** By (5.2) we see that if the norm of the initial data $u_0$ is sufficiently small, namely, $\| u_0 \|_{H^{\omega}} + \| u_0 \|_{H^{1, \omega}} < \epsilon_1$, then for some time $T > 1$ there exists a unique solution $u$ such that $\| u \|_{X_{T, \rho}^{\kappa, \omega}} < C \epsilon$.

### 6. LARGE TIME ASYMPTOTICS

We consider the initial–boundary value problem (1.1) with small initial data,

(6.1) $\| u_0 \|_{H^{\omega}} + \| u_0 \|_{H^{1, \omega}} < \epsilon_1$,

where

$\omega > \frac{1}{2} + \mu, \quad \mu \in (\frac{1}{2}, 1)$

and $\epsilon_1 > 0$ is sufficiently small. Let us prove the estimate

(6.2) $\sup_{t > 0} \sum_{n=0}^{1} t^{\alpha_n} \left( \| u^{(n)}(\cdot, t) \|_{L^2} + t^{-\frac{\omega}{2}} \| \langle \cdot \rangle^{\omega}(u^{(n)}(\cdot, t)) \|_{L^2} \right) < \epsilon$, 
where
\[ \alpha_n = \frac{3}{4} + \frac{n}{2} - \gamma, \quad \delta_n = \frac{1}{2} + \mu n + \gamma \leq \omega, \]
and \( \gamma, \varepsilon > 0 \) are small enough. We prove (6.2) by the contradiction. We assume that there exists some \( T > 0 \) such that
\[ (6.3) \quad \sup_{t \in [0, T]} \sum_{n=0}^{1} t^{\alpha_n} \left( \left\| u^{(n)}_x (\cdot, t) \right\|_{L^2} + t^{-\frac{\delta_n}{2}} \left\| \cdot \beta u^{(n)}_x (\cdot, t) \right\|_{L^2} \right) = \varepsilon. \]

From Lemma 3.1 we have
\[ (6.4) \quad \left\| \cdot \delta \partial_x^\beta F_a (\cdot, y, t) \right\|_{L^2} \leq Cy^{1-\alpha + \frac{1}{2}} \]
for all \( y, t > 0 \), where \( \delta \geq 0 \). Also by (4.4) and (4.5) we have
\[ \sup_{y > 0} \left\| \partial_t^\beta F_a (\cdot, y, t) \right\|_{L^2} \leq C \]
for all \( t > 0 \). Therefore from (6.4) with \( \delta = 0 \) we obtain for \( n = 0, 1 \)
\[ (6.5) \quad \left\| u^{(n)}_x (\cdot, t) \right\|_{L^2} \leq C t^{-\alpha_n} \left\| yu_0 \right\|_{L^2} + \int_0^t d\tau \int_0^\infty |u u_x (\cdot, \tau)| \left\| F^{(n)}_x (\cdot, y, \tau) \right\|_{L^2} dy \]
\[ \leq \epsilon t^{-\alpha_n} + \int_0^\infty \left\| \cdot \frac{1}{2} \gamma u (\cdot, \tau) \right\|_{L^2} \left\| \cdot \frac{1}{2} + \gamma \right\|_{L^2} (t-\tau)^{-\alpha_n} d\tau \]
\[ + \int \left\| u (\cdot, \tau) \right\|_{L^2} \left\| u_x (\cdot, \tau) \right\|_{L^2} (t-\tau)^{-\alpha_n} d\tau. \]

Using (6.4) with
\[ \delta_n = \frac{1 + 2\mu n}{2} - \gamma, \]
we get
\[ (6.6) \quad \left\| \cdot \beta u^{(n)}_x (\cdot, t) \right\|_{L^2} \leq C t^{-\alpha_n + \frac{\beta}{2}} \left\| yu_0 \right\|_{L^2} \]
\[ + \int_0^t d\tau \int_0^\infty |u u_x (\cdot, \tau)| \left\| \cdot \beta F^{(n)}_x (\cdot, y, \tau) \right\|_{L^2} dy \]
\[ \leq \epsilon t^{-\alpha_n + \frac{\beta}{2}} + \int_0^t \left\| \cdot \frac{1}{2} - \gamma u (\cdot, \tau) \right\|_{L^2} \]
\[ \times \left\| \cdot \frac{1}{2} + \gamma u_x (\cdot, \tau) \right\|_{L^2} (t-\tau)^{-\alpha_n + \frac{\beta}{2}} d\tau. \]

So substituting (6.3) into (6.5) and (6.6), we have
\[ \sup_{t \in [0, T]} \sum_{n=0}^{1} t^{\alpha_n} \left( \left\| u^{(n)}_x (\cdot, t) \right\|_{L^2} + t^{-\frac{\delta_n}{2}} \left\| \cdot \beta u^{(n)}_x (\cdot, t) \right\|_{L^2} \right) < \varepsilon. \]
The contradiction obtained proves (6.2).

Now using estimate (6.2) and Lemma 3.1, we prove that if \( \| (\cdot)^{1+\mu}u_0 \|_{L^1} \leq C \), then the solution has the asymptotics for \( t \to \infty \) uniformly with respect to \( x > 0 \)

\[
(6.7) \quad u(x,t) = \frac{1}{t} A_a \Phi_a \left( \frac{x}{2 \sqrt{t}}, t \right) + O \left( \min \left( \frac{x}{\sqrt{t}}, 1 \right) t^{-\frac{1}{2}} \right),
\]
where \( \mu \in (0,1) \) and

\[
A_a = \int_0^\infty \Lambda_a(y) u_0(y) \, dy + \int_0^t d\tau \int_0^\infty \Lambda_a(y) u(y, \tau) u_y(y, \tau) \, dy < \infty.
\]

Indeed, via (6.2), Lemma 3.1, and (4.1) we have

\[
(6.8) \quad u(x,t) = \frac{1}{t} A_a \Phi_a \left( \frac{x}{2 \sqrt{t}}, t \right) + R_a(x,t),
\]
where

\[
|R_a(x,t)| \leq C \min \left( \frac{x}{\sqrt{t}}, 1 \right) t^{-\frac{1}{2}}
\]
\[
\times \left( \| (\cdot)^{1+\mu}u_0 \|_{L^1} + \int_0^t d\tau \| (\cdot)^{1+\mu}u(\cdot, \tau) u_y(\cdot, \tau) \|_{L^1} \right)
\]
\[
+ \int_0^t d\tau \int_0^\infty dy |uu_y| \| F_a(x, y, t - \tau) - F_a(x, y, t) \|_{L^\infty}.
\]

Using estimate (6.2) and Theorem 1.1, we see that

\[
(6.9) \quad \int_0^t \tau \| (\cdot)^{\frac{1}{2}}u(\cdot, \tau) \|_{L^2} \| (\cdot)^{\frac{3}{2}+\gamma}u_y(\cdot, \tau) \|_{L^2} \, d\tau \leq C + \int_1^t \tau^{-\frac{1}{2}} \, d\tau \leq Ct^{\frac{1}{2}}
\]
for \( t > 1 \). As in the proof of Lemma 3.1 we get

\[
\| \partial_y F_a(\cdot, y, t) \|_{L^\infty} \leq Cy \min \left( \frac{x}{\sqrt{t}}, 1 \right) t^{-2+\gamma},
\]
whence

\[
\| F_a(\cdot, y, t - \tau) - F_a(\cdot, y, t) \|_{L^\infty} \leq Cy \tau \min \left( \frac{x}{\sqrt{t}}, 1 \right) t^{-2+\gamma}.
\]

Therefore using (6.9) we have

\[
(6.10) \quad \int_0^t d\tau \int_0^\infty \left| u(y, \tau) u_y(y, \tau) \right| \left| F_a(x, y, t - \tau) - F_a(x, y, t) \right| dy
\]
\[
\leq Ct^{\frac{1}{2}} \int_0^t \tau \| (\cdot)^{\frac{1}{2}+\gamma}u(\cdot, \tau) \|_{L^2} \| (\cdot)^{\frac{3}{2}+\gamma}u_y(\cdot, \tau) \|_{L^2} \, d\tau
\]
\[
\leq C \min \left( \frac{x}{\sqrt{t}}, 1 \right) t^{-\frac{1}{2}}
\]
with some $\mu \in (0, 1)$. Since $\|e^{i\mu t} u_0\|_{L^1} \leq C$, using (6.2) and (6.10), we get

$$
|R_n(x, t)| \leq C \min \left( \frac{x}{\sqrt{t}}, 1 \right) t^{-1 - \mu},
$$

where $\mu \in (0, 1)$. From (6.8)–(6.11) we obtain the asymptotics (6.7) for the solution. Theorem 1.2 is proved.

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