Europ. J. Combinatorics (1983) **4**, 107–126

On a Theorem of Cooperstein

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A theorem by Cooperstein that partially characterizes the natural geometry $A_{n,d}(F)$ of subspaces of rank $d-1$ in a projective space of finite rank n over a finite field F, is somewhat strengthened and generalized to the case of an arbitrary division ring F.

Moreover, this theorem is used to provide characterizations of $A_{n,2}(F)$ and $A_{5,3}(F)$ which will be of use in the characterization of other (exceptional) Lie group geometries.

1. INTRODUCTION

Theorem A by Cooperstein in [2] provides a partial characterization of the geometry $A_{a,d}(F)$ on all subspaces of rank (= projective dimension) $d-1$ of a projective space of rank a over a finite field F. Though there are more (partial) characterizations (cf. [5], [6]) this one has the advantage of being ready-made for characterizations of geometries corresponding to groups of Lie type, see for instance Theorem B of [2]. This paper deals with a generalization of Theorem A to the case of a projective space of finite rank over an arbitrary division ring F. The present version is stronger than the original theorem in that it describes more specifically what happens in "case (iii)". In fact, it shows that case (iii) does not occur at all if the geometry is finite.

Many steps in the proof are taken from or inspired by Cooperstein's proof of Theorem A. The infinite case (i.e., where the geometry and hence F is infinite) depends on the classification of polar spaces of rank 3 (used in 4.2) as given in [7].

Two applications of the theorem are given: a characterization of the space of lines in a projective space of finite rank, and a characterization of the space of planes in a projective space of rank 5. Precise formulation of the results will be given in Section 2 after some notation and terminology has been introduced.

2. TERMINOLOGY, NOTATION AND MAIN RESULT

An *incidence system* $(P, \mathscr{L})$ is a set P of *points* together with a collection $\mathscr{L}$ of subsets of cardinality >1, called *lines*. If $(P, \mathscr{L})$ is an incidence system then the *collinearity graph* of $(P, \mathscr{L})$ is the graph whose vertex set is P and whose edges consist of the pairs of collinear points. The incidence system is called *connected* whenever its collinearity graph is connected. Likewise terms such as *(co)cliques*, *paths* will be applied freely to $(P, \mathscr{L})$ when in fact they are meant for its collinearity graph. For $x, y \in P$, let $d(x, y)$ denote the ordinary distance in the collinearity graph, and let $x^\perp$ stand for the set of points collinear with x. Instead of $x \in y^\perp$ we shall often write $x \perp y$. For a subset X of P and $y \in P$ we put $d(y, X) = \min_{x\in X} d(y, x)$ and $X^\perp = \bigcap_{x\in X} x^\perp$, $(P, \mathscr{L})$ is called *nondegenerate* if $P^\perp = \varnothing$. A subset X of P is called a *subspace* of $(P, \mathscr{L})$ whenever each point of P on a line bearing two distinct points of X is itself in X. A subspace is called *singular* whenever it induces a clique in $(P, \mathscr{L})$. The length i of a longest chain $X_0 \subset X_1 \subset \cdots \subset X_i = X$ of nonempty singular subspaces X_j of X is called the *rank* of X and denoted by $\mathrm{rk}(X)$.

For a subset X of P, the subspace generated by X is denoted $\langle X\rangle$. Instead of $\langle X\rangle$ we also write $\langle x, Y\rangle$ if $X = \{x\} \cup Y$, and so on.

If $\mathscr{F}$ is a family of subsets of P and X is a subset of P, then $\mathscr{F}(X)$ denotes the family of members of $\mathscr{F}$ contained in X, while $\mathscr{F}_X$ denotes the family of members of $\mathscr{F}$ containing

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X. If $X=\{x\}$ for some $x \in P$, we often write $\mathscr{F}_x$ instead of $\mathscr{F}_{\{x\}}$. Furthermore, if $\mathscr{H}$ is another family of subsets of P, then $\mathscr{F}(\mathscr{H})$ denotes $\{\mathscr{F}(H) \mid H \in \mathscr{H}\}$.

If G is a group of automorphisms of $(P, \mathscr{L})$ such that $L \not\subseteq x^G$ for any $x \in P$ and $L \in \mathscr{L}$, then $(P, \mathscr{L})/G$ denotes *the quotient of* $(P, \mathscr{L})$ *by* G, i.e. the incidence system whose points are the orbits in P of G and whose lines are of the form $\{x^G \mid x \in L\}$ for $L \in \mathscr{L}$. The incidence system $(P, \mathscr{L})$ is called *linear* if any two distinct points are on at most one line. If x, y are collinear distinct points of a linear incidence system, then xy denotes the unique line through them; thus $xy = \langle x, y\rangle$.

A line is called *thick* if there are at least three points on it, otherwise it is called *thin*. Recall (from [2]) that $(P, \mathscr{L})$ is a *polar space* if $|x^\perp \cap L| \neq 1$ implies $L \subseteq x^\perp$ for any $x \in P$ and $L \in \mathscr{L}$, that the *rank* of a polar space is the maximal number $k+1$ such that there exists a singular subspace of rank k in $(P, \mathscr{L})$ and that a *generalized quadrangle* is a polar space of rank 2. The objects under study here are incidence systems $(P, \mathscr{L})$ in which the following four axioms hold:

(P1) For any $x \in P$ and $L \in \mathscr{L}$ with $|x^\perp \cap L| > 1$ the line L is entirely contained in $x^\perp$ (this means $(P, \mathscr{L})$ is a Gamma space in D. G. Higman's terminology).

(P2) The connected components of $(P, \mathscr{L})$ are not complete.

(P3) For any two $x, y \in P$ with $d(x, y) = 2$, the subset $x^\perp \cap y^\perp$ forms a subspace isomorphic to a nondegenerate generalized quadrangle.

(P4) For $x \in P, L \in \mathscr{L}$ such that $x^\perp \cap L = \varnothing$ but $x^\perp \cap L^\perp \neq \varnothing$ the subset $x^\perp \cap L^\perp$ is a line.

For ease of reference and with the result below in mind, an incidence system with thick lines satisfying (P1), (P2), (P3), (P4) (but not necessarily connected) will be called a *Grassmann space*. The incidence structure whose points are the subspaces of rank d of a projective space over a division ring F of rank n and whose lines are the subspaces incident to an incident pair x, y of a subspace x of rank $d-1$ and a subspace y of rank $d+1$, is denoted by $A_{n,d+1}(F)$.

Main Theorem. *$(P, \mathscr{L})$ is a connected Grassmann space whose singular subspaces have finite rank iff one of the following holds*

(a) *$(P, \mathscr{L})$ is a nondegenerate polar space of rank 3 with thick lines.*

(b) *There are $a \geqslant 4$, $d \leqslant (a+1)/2$ and a division ring F such that $(P, \mathscr{L}) \cong A_{a,d}(F)$.*

(c) *There are $d \geqslant 5$, an infinite division ring F and an involutory automorphism σ of $A_{2d-1,d}(F)$ induced by a polarity on the underlying projective space of Witt index at most $d-5$ such that $(P, \mathscr{L}) \cong A_{2d-1,d}(F)/\langle\sigma\rangle$.*

This theorem is proved in Section 6.

Applications. *Suppose $(P, \mathscr{L})$ is an incidence system with thick lines.*

(a) *$(P, \mathscr{L})$ is a Grassmann space whose singular subspaces have finite rank and in which $x^\perp \cap L^\perp \neq \varnothing$ for any $x \in P$ and $L \in \mathscr{L}$ iff $(P, \mathscr{L})$ is either a nondegenerate polar space of rank 3 or isomorphic to $A_{a,2}(F)$ for some $a > 4$ and some division ring F.*

(b) *$(P, \mathscr{L})$ is a Grassmann space in which for any two intersecting lines $L_1, L_2 \in \mathscr{L}$ and any point $z \in P$ there exists $u \in z^\perp$ with $u^\perp \cap L_1 \neq \varnothing$ and $u^\perp \cap L_2 \neq \varnothing$ iff $(P, \mathscr{L})$ is either a nondegenerate polar space of rank 3 or isomorphic to one of $A_{4,2}(F)$, $A_{5,3}(F)$ for some division ring F.*

These applications are treated in Section 7.

The following example, kindly supplied by Professor Shult, shows that the main theorem no longer holds if in the definition of Grassmann spaces the requirement that lines are thick is dropped. Let q be a nondegenerate quadratic form on $\mathbb{F}_2^6$ of Witt index 2, and set P for the set of nonzero nonsingular points (with respect to q) and $\mathscr{L}$ for the set of (unordered) pairs of mutually perpendicular vectors from P (with respect to the bilinear form defined by q). Then $(P, \mathscr{L})$ is a connected incidence system on 36 points satisfying the axioms (P1), (P2), (P3), (P4) of Grassmann spaces. However, neither (a), (b) nor (c) of the main theorem holds for $(P, \mathscr{L})$.

3. Preliminary Results

Throughout this section, $(P, \mathscr{L})$ will be a Grassmann space.

Some useful properties of generalized quadrangles and polar spaces can be found in [2], [7]. Some facts shall first be recalled from [2] whose proofs do not depend on any finiteness assumption.

Lemma 3.1. *Let $(P, \mathscr{L})$ be a Grassmann space. Then $(P, \mathscr{L})$ is linear and is determined by its collinearity graph in the sense that for any two distinct collinear $x, y \in P$, $\{x, y\}^{\perp\perp}$ is the unique line on x, y. Moreover, we have*

(a) *maximal cliques are singular subspaces;*

(b) *for any clique X of P, the subspace $\langle X\rangle$ is singular;*

(c) *if X is a subset of P, then $X^{\perp}$ is a subspace;*

(d) *if x, y, z form a clique of P not contained in a line, then $\{x, y, z\}^{\perp}$ is a maximal singular subspace.*

Proposition 3.2 (Cooperstein). *Let $(P, \mathscr{L})$ be a Grassmann space. For any $x, y \in P$ with $d(x, y) = 2$, the subset $S(x, y)$ defined by*

$$S(x, y) = \{z \in P \mid (\forall L \in \mathscr{L})(L \subseteq \{x, y\}^{\perp} \Rightarrow z^{\perp} \cap L \neq \varnothing)\}$$

is a subspace isomorphic to a polar space of rank 3 with the property that $z^{\perp} \cap S$ is a singular subspace (possibly empty) for any $z \in P \backslash S$. Moreover, $S(x, y) = \langle \{x, y\} \cup \{x, y\}^{\perp}\rangle$.

As a matter of fact, (P4) is not needed for Lemma 3.1 and Proposition 3.2.

The family of all $S(x, y)$ obtained as described above will be denoted by $\mathscr{S}$, and the family of all maximal cliques will be denoted by $\mathscr{M}$. A member of $\mathscr{S}$ will be called a *symp*; a maximal singular subspace will often be called *max space* for short.

Corollary 3.3.

(a) *Each singular subspace of rank ≤ 2 is contained in a symp. Hence, it is a point, a line or a projective plane.*

(b) *If M is a singular subspace and M properly contains a line, then M is a projective space.*

We shall denote the family of singular subspaces of rank 2 by $\mathscr{V}$ and call its members *planes*.

Remark 3.4. *Axiom* (P4) *can be replaced by*

(P4)′ $$(\forall S \in \mathscr{S})(\forall x \in P\backslash S)(|x^{\perp} \cap S| > 1 \Rightarrow x^{\perp} \cap S \in \mathscr{V})$$

Proof. (P4)$\Rightarrow$(P4)′. Let $|x^{\perp} \cap S| > 1$ for $S \in \mathscr{S}$ and $x \in P\backslash S$. By the above proposition, $x^{\perp} \cap S$ is a singular subspace of S and hence of rank 1 or 2. Take $z \in x^{\perp} \cap S$ and $y \in S\backslash(x^{\perp} \cup z^{\perp})$. Apply (P4) to the point y and the line $L = xz$. Since $y^{\perp} \cap (x^{\perp} \cap S) \neq \varnothing$, as S is a polar space and $x^{\perp} \cap S$ contains a line, we have $y^{\perp} \cap L^{\perp} \neq \varnothing$. Moreover, $u \in y^{\perp} \cap L$ would yield $u \in y^{\perp} \cap z^{\perp}$; hence $u \in S\backslash\{z\}$ and $x \in uz$, so $x \in S$, which is absurd. Therefore $y^{\perp} \cap L = \varnothing$, so that $y^{\perp} \cap L^{\perp}$ is a line contained in $x^{\perp} \cap S$ but not on z. It follows that $x^{\perp} \cap S$ is a plane.

(P4)$\Leftarrow$(P4)$'$. Suppose $x \in P$ and $L \in \mathscr{L}$ are such that $x^\perp \cap L = \emptyset$ and $x^\perp \cap L^\perp \neq \emptyset$. Take $y \in L$ and consider $S = S(x, y)$. Since $\langle y, x^\perp \cap L^\perp \rangle$ is a singular subspace of S of rank ≥ 1, it is a plane by (P4)$'$. It follows that $x^\perp \cap L^\perp$ is a line, as wanted.

LEMMA 3.5. *If S is a symp and $x, y \in P\backslash S$ are collinear, while $x^\perp \cap S \in \mathscr{V}$ and $y^\perp \cap S \neq \emptyset$, then either $y^\perp \cap S \subseteq x^\perp \cap S$ or $y^\perp \cap S \in \mathscr{V}$ and $x^\perp \cap y^\perp \cap S$ is a singleton.*

PROOF. Suppose $z \in y^\perp \cap S \backslash x^\perp$. First of all we show that $y^\perp \cap S$ is a plane, too. As $x^\perp \cap S \in \mathscr{V}(S)$ and S is a polar space, $z^\perp \cap x^\perp \cap S$ is a line in S. Now both $z^\perp \cap x^\perp \cap S$ and y are in the generalized quadrangle $x^\perp \cap z^\perp$, so there is $u \in x^\perp \cap S$ with $\{u\} = x^\perp \cap z^\perp \cap S \cap y^\perp$. Since $uz \subseteq y^\perp \cap S$, Remark 3.4 implies that $y^\perp \cap S$ is a plane. Finally, $x^\perp \cap y^\perp \cap S = z^\perp \cap x^\perp \cap y^\perp \cap S = \{u\}$.

COROLLARY 3.6. *If $S \in \mathscr{S}$ and $M \in \mathscr{M}$ satisfy $|M \cap S| > 1$, then $M \cap S \in \mathscr{V}(S)$.*

PROOF. For any $w \in M\backslash S$, we have $w^\perp \cap S \in \mathscr{V}(S)$ by Remark 3.4. If $z, w \in M\backslash S$, then $z^\perp \cap S = w^\perp \cap S$ by Lemma 3.5. If $M \subseteq S$, there is nothing to prove; so assume $M\backslash S \neq \emptyset$. Taking $z \in M\backslash S$, we get $z^\perp \cap S = \bigcap_{w \in M\backslash S} w^\perp \cap S = \bigcap_{w \in M} w^\perp \cap S = M^\perp \cap S = M \cap S$. In particular, $M \cap S = z^\perp \cap S \in \mathscr{V}(S)$.

Let S be a symp. On the set of planes $\mathscr{V}(S)$ a graph $(\mathscr{V}(S), \approx)$ is defined by $V_1 \approx V_2$ iff rk $(V_1 \cap V_2) = 0$ $(V_1, V_2 \in \mathscr{V}(S))$. It is well known that $(\mathscr{V}(S), \approx)$ has either one or two connected components. In the latter case, each line is in precisely two members of $\mathscr{V}(S)$, one of each connected component, and the connected components are complete graphs.

COROLLARY 3.7. *Let $S \in \mathscr{S}$ and let $\mathscr{K}$ be a union of connected components of $(\mathscr{V}(S), \approx)$. Then*

$$H(\mathscr{K}, S) = \bigcup_{K \in \mathscr{K}} K^\perp$$

is a subspace containing S.

PROOF. As $S = \bigcup_{K \in \mathscr{K}} K$, the subset $H(\mathscr{K}, S)$ clearly contains S. We need only show that if $x, y \in P\backslash S$ are collinear and $x^\perp \cap S$, $y^\perp \cap S \in \mathscr{K}$, then any $z \in xy$ is contained in $H(\mathscr{K}, S)$. If $x^\perp \cap y^\perp \cap S = x^\perp \cap S$, then clearly $z^\perp \cap S = x^\perp \cap S \in \mathscr{K}$, so we are done. Therefore (cf. Lemma 3.5), we may assume $x^\perp \cap y^\perp \cap S = \{u\}$ for some $u \in P$. Consequently, $z \in P\backslash S$. Take $v \in x^\perp \cap S\backslash\{u\}$ and $w \in v^\perp \cap y^\perp \cap S\backslash\{u\}$ (notice that w exists because v, $y^\perp \cap S$ are in the polar space S). Now x, y, w, v is a 4-circuit and $z \in xy$, so that there is $z_1 \in z^\perp \cap vw$. Notice that $z_1 \neq u$, for otherwise $v \in uw$, whence $v \in y^\perp \cap S$ conflicting $v \neq u$. Thus $|z^\perp \cap S| > 1$ as $z_1, u \in z^\perp \cap S$, and we are done by Remark 3.4 and Lemma 3.5.

LEMMA 3.8. (a) *If $M \in \mathscr{M}$ and $x \in P\backslash M$ satisfy $x^\perp \cap M \neq \emptyset$, then $x^\perp \cap M \in \mathscr{L}$.*
(b) *If $M \in \mathscr{M}$ and $L \in \mathscr{L}$ with* $\mathrm{rk}(L \cap M) = 0$, *then there is a unique $N \in \mathscr{M}_L$ with $M \cap N \in \mathscr{L}$.*

PROOF. (a) Suppose $z \in x^\perp \cap M$. Take $y \in M\backslash x^\perp$ and consider $S = S(x, y)$. If $M \subseteq S$, there is nothing to prove. Otherwise, $M \cap S$ contains z and y, so $M \cap S \in \mathscr{V}(S)$ by Corollary 3.6. It results that $x^\perp \cap M = x^\perp \cap (M \cap S)$ is a line.
(b) By (a), $L^\perp \cap M$ is a line. Thus $N = \langle L, L^\perp \cap M \rangle^\perp$ is the unique max space containing L with $M \cap N \in \mathscr{L}$.

Notice that Lemma 3.8(b) can be reformulated as: *$(\mathcal{L}_x, \mathcal{M}_x)$ is a generalized quadrangle for each $x \in P$.*

LEMMA 3.9. *The graph $(\mathcal{V}, \approxeq)$ defined by $V_1 \approxeq V_2$ iff $V_1 \not\subseteq V_2^\perp$ and $V_1 \cap V_2 \in \mathcal{L}$ for $V_1, V_2 \in \mathcal{V}$ is connected.*

PROOF. Notice that the subgraph induced on $\mathcal{V}(S)$ is connected for any $S \in \mathcal{S}$. Let $V \in \mathcal{V}$. By connectedness of $(P, \mathcal{L})$, it suffices to prove that any plane W with $V \cap W \neq \emptyset$ is joined to X by a path in $(\mathcal{V}, \approxeq)$. Let $W \in \mathcal{V} \backslash \{V\}$ with $V \cap W \neq \emptyset$. Suppose there are $v \in V \backslash W$ and $w \in W \backslash V$ with $v \notin w^\perp$. Consider $S(v, w)$. There are planes M, N in $S(v, w)$ such that $\langle v, V \cap W\rangle \subseteq M$ and $\langle w, V \cap W\rangle \subseteq N$. Now $\mathrm{rk}(M \cap V) > \mathrm{rk}(V \cap W)$ and $\mathrm{rk}(N \cap W) > \mathrm{rk}(V \cap W)$, so by the induction we are reduced to the case where $V \subseteq W^\perp$. It suffices to treat the case where $V \cap W \in \mathcal{L}$.

Because of Corollary 3.3(a) there is a symp S containing V. Let U be a plane in S with $V \cap U = V \cap W$. Again, take $v \in V \backslash W$, $w \in W \backslash V$ and $u \in U \backslash V$. Then $u \notin v^\perp$ and $w \in v^\perp$. If $w \in u^\perp$, then $W = \langle w, V \cap W\rangle \subseteq \langle u^\perp \cap v^\perp, U \cap V\rangle \subseteq S$. So we may assume $w \notin u^\perp$. But then $W \approxeq U \approxeq \mathcal{V}$, finishing the proof of the lemma.

COROLLARY 3.10. *The graph $(\mathcal{M}, \approxeq)$ defined by $M_1 \approxeq M_2$ iff $\mathrm{rk}(M_1 \cap M_2) = 1$, is connected.*

PROOF. Note that M_1 and M_2 are adjacent in $(\mathcal{M}, \approxeq)$ iff there are planes $V \subseteq M_1$ and $W \subseteq M_2$ with $V \not\subseteq W^\perp$ and $V \cap W \in \mathcal{L}$. Thus there is a surjective morphism $(\mathcal{V}, \approxeq) \to (\mathcal{M}, \approxeq)$ of graphs given by $V \mapsto V^\perp$ (cf. Lemma 3.1(d)). The desired result is therefore a consequence of the above lemma.

LEMMA 3.11. *The graph $(\mathcal{L}, \sim)$ defined by $L_1 \sim L_2$ iff $\mathrm{rk}(L_1 \cap L_2) = 0$ and $L_1 \not\subseteq L_2^\perp$, is connected.*

PROOF. As before, the proof comes down to the case where $L_1 \subset L_2^\perp$ and $\mathrm{rk}(L_1 \cap L_2) = 0$. But then $\langle L_1, L_2\rangle \in \mathcal{V}$, so the lemma results from the analogous statement for polar spaces with thick lines.

LEMMA 3.12. *Let $L_1, L_2 \in \mathcal{L}$. There is a bijection between $\mathcal{M}_{L_1}$ and $\mathcal{M}_{L_2}$.*

PROOF. By connectedness of $(\mathcal{L}, \sim)$ as defined in Lemma 3.11, we need only prove the lemma for $L_1, L_2 \in \mathcal{L}$ with $L_1 \not\subseteq L_2^\perp$ and $L_1 \cap L_2$ is a point. Take $x \in L_1 \backslash L_2$ and $y \in L_2 \backslash L_1$ and let $\alpha: \mathcal{M}_{L_1} \to \mathcal{M}_{L_2}$ be given by $\alpha(M) = \langle y, M \cap y^\perp\rangle^\perp$. It is not hard to verify that α is a bijection.

LEMMA 3.13. *Let $M, N \in \mathcal{M}$ satisfy $\mathrm{rk}(M \cap N) = 0$. Then $\mathrm{rk}(M) = \mathrm{rk}(N)$.*

PROOF. $M \cap N = \{u\}$ for some $u \in P$. In view of Lemma 3.8, the map $\phi: \mathcal{L}_u(M) \to \mathcal{L}_u(N)$ given by $\phi(X) = X^\perp \cap N$ is well defined. Moreover, it is an isomorphism of projective spaces. Hence the result.

Consider the graph $(\mathcal{M}, \approx)$ defined by $M_1 \approx M_2$ iff $\mathrm{rk}(M_1 \cap M_2) = 0$. The above lemma states that the members of a connected component of $(\mathcal{M}, \approx)$ all have the same rank. Lemma 3.8(b) and connectedness of $(P, \mathcal{L})$ yield that for any line L and each connected component $\mathcal{K}$ of $(\mathcal{M}, \approx)$ there is a member of $\mathcal{K}$ on L. The following lemma shows that in fact $(\mathcal{M}, \approx)$ cannot have more than two connected components.

LEMMA 3.14. *Suppose a line is contained in at least three max spaces. Then* $(\mathcal{M}, \approx)$ *is connected. In particular, all max spaces have the same rank.*

PROOF. By Lemma 3.12, any line is contained in at least three max spaces. Let M, N be two max spaces with $M \cap N \in \mathcal{L}$. We claim the existence of $K \in \mathcal{M}$ with $K \cap M = K \cap N$ a singleton. In view of Corollary 3.10 this yields that $(\mathcal{M}, \approx)$ is connected. The last statement is then a direct consequence of Lemma 3.13.

To show the existence of K as described choose $x \in M \cap N$ and $y \in (M \cap N)^{\perp} \backslash (M \cup N)$. Notice that y exists because of the assumption that $M \cap N$ is in at least three members of $\mathcal{M}$. By Lemma 3.9, $\langle M \cap N, y\rangle$ is contained in a symp, so there is $z \in P$ with $z^{\perp} \cap \langle M \cap N, y\rangle = \langle x, y\rangle$. Now $K = \langle x, y, z\rangle^{\perp} \in \mathcal{M}$ and $\{x\} \subseteq K \cap M = z^{\perp} \cap (y^{\perp} \cap M) = z^{\perp} \cap (M \cap N) = \{x\}$ by Lemma 3.8. So $K \cap M = \{x\}$. Similarly, $K \cap N = \{x\}$. Therefore, the claim holds.

LEMMA 3.15. *If* $\operatorname{rk}(M) = 2$ *for some* $M \in \mathcal{M}$, *then for any* $x \in P$ *and* $L \in \mathcal{L}$ *we have* $x^{\perp} \cap L^{\perp} \neq \varnothing$. *In particular, the diameter of* $(P, \mathcal{L})$ *is* 2.

PROOF. We may assume that $x^{\perp} \cap L = \varnothing$. By induction with respect to $d(x, L)$, it suffices to prove the first statement in the case where $d(x, L) = 2$. Let $y, z \in P$ be such that $x \in y^{\perp}$ and $z \in y^{\perp} \cap L$, and take $w \in L \backslash \{z\}$. The hypothesis implies that there is a max space N of rank 2 on yz. Since $x^{\perp} \cap N$ and $w^{\perp} \cap N$ are lines in N, they intersect in a point, say u. Since $u \in x^{\perp} \cap w^{\perp} \cap N \subseteq x^{\perp} \cap w^{\perp} \cap z^{\perp} = x^{\perp} \cap L^{\perp}$, we have shown $x^{\perp} \cap L^{\perp} \neq \varnothing$ as wanted.

COROLLARY 3.16. *If all max spaces have rank* 2, *then* $(P, \mathcal{L})$ *is a polar space of rank* 3.

PROOF. Let $x \in P$ and $L \in \mathcal{L}$. We prove the Buekenhout–Shult axiom $x^{\perp} \cap L \neq \varnothing$, cf. [1]. Suppose the contrary. Then, since the above lemma yields $x^{\perp} \cap L^{\perp} \neq \varnothing$, axiom (P4) implies that $x^{\perp} \cap L^{\perp}$ is a line disjoint from L. Thus $\operatorname{rk}(\langle L, x^{\perp} \cap L^{\perp}\rangle) = 3$, which conflicts with the hypothesis.

LEMMA 3.17. *If* $S \in \mathcal{S}$ *and* $x \in P$ *satisfy* $\mathcal{L}_x \subseteq \mathcal{L}(S)$, *then* $(P, \mathcal{L})$ *is a polar space of rank* 3.

PROOF. We prove that $P = S$. In view of the connectedness of $(P, \mathcal{L})$ it suffices to show that for any $y \in x^{\perp}$ all $z \in y^{\perp}$ are contained in S. Let y, z be as described. If $z \in x^{\perp} \backslash \{x\}$ we must have $zx \in \mathcal{L}(S)$, so $z \in S$. Suppose $z \notin x^{\perp}$. Then $S(x, z)$ is a symp on x. But since symps are geodesically closed, S is the only symp on x. We obtain $S(x, z) = S$, and $z \in S$ as wanted.

4. A PROPERTY OF CLASSICAL GENERALIZED QUADRANGLES

Throughout this section, $(P, \mathcal{L})$ *is a generalized quadrangle with thick lines.* $(P, \mathcal{L})$ is called *classical* whenever it occurs as the residue of a point in a nondegenerate polar space of rank 3 whose lines are thick. Since polar spaces of this rank are classified [7], the list of all classical generalized quadrangles is known. The result is quoted in Theorem 4.1. For the duration of this section, we shall adopt terminology from [7], without recalling all definitions. The aim of this section is to prove Proposition 4.2.

THEOREM 4.1. (Buekenhout–Shult, Veldkamp, Tits.) *Let $(P, \mathscr{L})$ be a classical generalized quadrangle. Then $(P, \mathscr{L})$ is one of the following*:

(a) *A polar space $Q(\pi)$ of a projective space over a division ring F where π is a polarity determined by a nondegenerate trace-valued (σ, ε)-hermitian form of Witt index 2 for some antiautomorphism σ of F with $\sigma^2 = 1$ and some $\varepsilon \in \{1, -1\}$.*

(b) *A polar space $Q(\kappa)$ of a projective space over a division ring F where κ is a projective pseudo-quadratic form represented by a nondegenerate σ-quadratic form of Witt index 2 for some antiautomorphism σ of F with $\sigma^2 = 1$.*

(c) *The dual of the generalized quadrangle $Q(\kappa_0)$ in a projective space over a field F defined in* (b) *where κ_0 is represented by the quadratic form $q: E \times F^4 \to F$ over F defined by*

$$(x_0, x_1, x_2, x_3, x_4) \mapsto N(x_0) - x_1x_3 + x_2x_4$$

for E a Cayley division algebra over F and $N: E \to F$ the quadratic norm form of this algebra.

(d) $\{x, y\}^{\perp}$ *for two noncollinear points x, y of $A_{3,2}(F)$.*

A *grid* is by definition a generalized quadrangle in which each point is in precisely two lines. Clearly the generalized quadrangles in (d) are grids. In Lemma 4.5 we shall find all grids occurring in the list.

We recall that a family $\mathscr{R}$ of lines in $(P, \mathscr{L})$ is called a *spread* in $(P, \mathscr{L})$ if the members of $\mathscr{R}$ partition P (i.e. $P = \bigcup_{L \in \mathscr{R}} L$ and for any two distinct $L_1, L_2 \in \mathscr{R}$ we have $L_1 \cap L_2 = \varnothing$).

A grid has precisely two spreads, they are also called the *parallel classes* of the grid. If L_1, L_2 are disjoint lines of $(P, \mathscr{L})$ such that the subspace $\langle L_1, L_2 \rangle$ is a grid, then L_1L_2 denotes the parallel class of the grid containing L_1 and L_2.

PROPOSITION 4.2. *Let $(P, \mathscr{L})$ be a nondegenerate generalized quadrangle with thick lines which is either finite or classical. Suppose it admits a spread $\mathscr{R}$ in which for any two distinct $L_1, L_2 \in \mathscr{R}$ the subspace $\langle L_1, L_2 \rangle$ is a grid and the family L_1L_2 is contained in $\mathscr{R}$ while $(\mathscr{R}, \{L_1L_2 \mid L_1, L_2 \in \mathscr{R};\ L_1 \neq L_2\})$ is a projective space. Then the rank of $\mathscr{R}$ as a projective space is* 1 *and $(P, \mathscr{L})$ is a grid.*

The remainder of this section is devoted to the proof of this proposition. Thus, *from now on until the end of this section we assume that $\mathscr{R}$ is a spread of the generalized quadrangle* $(P, \mathscr{L})$ as described in the hypothesis of the above proposition. In the next lemma, the finite case is dealt with by a standard computational argument.

LEMMA 4.3. *If $(P, \mathscr{L})$ is finite, then* $\mathrm{rk}(\mathscr{R}) = 1$.

PROOF. Suppose $\mathrm{rk}(\mathscr{R}) > 1$. Then $(P, \mathscr{L})$ is not a grid. In particular, it is then a regular generalized quadrangle, i.e. there is a constant number, say $1 + t$, of lines through each point, and a constant number of points, say $1 + s$, on each line. By well-known theory [3], we have $t \leq s^2$. On the other hand, $1 + st = |\mathscr{R}| = (s^{m+1} - 1)/(s - 1)$ if the rank of $\mathscr{R}$ is m. It follows that $t = 1 + s$ (and $m = 2$). A straightforward computation on multiplicities of eigenvalues of the adjacency matrix of the collinearity graph (cf. [3]) leads to integrality conditions which are only satisfied if $s = 1$. But this is excluded by the requirement that the lines are thick.

The assumption that lines are thick is necessary, since the regular complete bipartite graph on 6 points provides a counterexample.

The classical case depends on the classification of classical generalized quadrangles as stated in Theorem 4.1. If $(P, \mathscr{L})$ is as in (d) of this theorem, there is nothing to prove.

LEMMA 4.4. *$(P, \mathscr{L})$ is not isomorphic to a generalized quadrangle as described in* 4.1(c).

PROOF. Suppose $(P, \mathscr{L})$ satisfies (c) of Theorem 4.1. Then the dual of $\langle L_1, L_2\rangle$ is a bipartite graph in the dual of $(P, \mathscr{L})$. On the other hand, according to 10.7 of [7], the dual of $\langle L_1, L_2\rangle$ is the dual of $(P, \mathscr{L})$ itself. This yields the absurdity that $Q(\kappa_0)$ of Theorem 4.1(c) is a bipartite graph.

If X is a subset of a projective space we denote by $[X]$ the projective subspace of this projective space spanned by X.

LEMMA 4.5. *Let F be a division ring, let σ be an antiautomorphism of F such that $\sigma^2 = 1$ and let $\varepsilon \in \{1, -1\}$. Suppose ξ is either a polarity π determined by a nondegenerate trace valued (σ, ε)-hermitian form f of Witt index 2 or a projective pseudo-quadratic form κ represented by a nondegenerate σ-quadratic form q of Witt index 2. If L_1, L_2 are lines of $Q(\xi)$ with $L_1 \cap L_2 = \varnothing$ such that $\langle L_1, L_2\rangle$ is a grid, then $\sigma = 1$ and (if $\xi = \pi$) $\varepsilon = 1$. Thus, F is a field. Furthermore, $\langle L_1 \cup L_2\rangle = [L_1 \cup L_2] \cap Q(\xi)$ unless $\xi = \pi$ and F has characteristic 2.*

PROOF. Take distinct points e_1, e_3 in L_1 and e_2, e_4 in L_2 such that $\{e_2\} = e_3^\perp \cap L_2$ and $\{e_4\} = e_1^\perp \cap L_2$. Put $F_{\sigma,\varepsilon} = \{t - t^\sigma \varepsilon \mid t \in F\}$. As in (8.10) of [7], choose E_1, E_2, E_3, E_4, points of the vector space underlying the projective space in which $Q(\xi)$ is defined, such that E_i represents e_i (i.e. such that the ray through E_i is e_i for $i = 1, 2, 3, 4$) and such that

$$f\left(\sum_{i=1}^{4} E_i x_i, \sum_{i=1}^{4} E_i y_i\right) = x_1^\sigma y_2 + \varepsilon x_2^\sigma y_1 + x_3^\sigma y_4 + \varepsilon x_4^\sigma y_3 \qquad \text{if } \xi = \pi$$

and

$$q\left(\sum_{i=1}^{4} E_i x_i\right) = x_1^\sigma x_2 + x_3^\sigma x_4 + F_{\sigma,\varepsilon} \qquad \text{if } \xi = \kappa.$$

Now take $a \in F_{\sigma,\varepsilon}$ (where $\varepsilon = 1$ if $\xi = \kappa$). Then the calculation performed in (8.10) of [7] shows that the projective point $p(a)$ represented by $(1, a, 0, 0)$ on the basis E_1, E_2, E_3, E_4 is in $\langle L_1, L_2\rangle$. But $p(a)$ is collinear with both e_3 and e_4 and hence in $\{e_1, e_2\}$ as $\langle L_1, L_2\rangle$ is a grid. It follows that $a = 0$, and the conclusion is that $F_{\sigma,\varepsilon} = \{0\}$.

If $\varepsilon = -1$, this reads $t + t^\sigma = 0$ for all $t \in F$, so that F has characteristic 2 and $\varepsilon = 1$.

It results that $\varepsilon = 1$ and $t - t^\sigma = 0$ for all $t \in F$, whence $\sigma = 1$. Since σ is an antiautomorphism, F must be commutative and therefore a field.

The final statement of the lemma now results from (8.10) of [7].

LEMMA 4.6. *Let F be a field and let ξ be either a polarity π determined by a nondegenerate symmetric form f of Witt index 2 or a projective quadratic form κ represented by a nondegenerate quadratic form q of Witt index 2. Suppose L_1, L_2, L_3 are distinct lines of $Q(\xi)$ such that for each $i \in \{1, 2, 3\}$ the subspace $\langle L_i \cup L_{i+1}\rangle$ of $Q(\xi)$ is a grid and $L_i \cap \langle L_{i-1} \cup L_{i+1}\rangle = \varnothing$ (indices modulo 3). Then there are lines $N_1, M_1 \in L_1L_2 \backslash \{L_1\}$ and $N_2, M_2 \in L_1L_3 \backslash \{L_1\}$ such that if $\langle N_1 \cup N_2\rangle$ and $\langle M_1 \cup M_2\rangle$ are grids, $N_1N_2 \cap M_1M_2$ does not contain a line of $Q(\xi)$ which is disjoint with L_1.*

PROOF. Let e_1, e_2, e_3, e_4 and E_1, E_2, E_3, E_4 be as in the proof of Lemma 4.5. Thus $e_1, e_3 \in L_1$; $e_1 \neq e_3$; $e_2, e_4 \in L_2$ and $e_1 \in e_4^\perp$, $e_2 \in e_3^\perp$; furthermore the vector E_i represents

e_i $(i=1, 2, 3, 4)$ and

$$\left.\begin{aligned} f\left(\sum_{i=1}^{4} E_i x_i, \sum_{i=1}^{4} E_i y_i\right) &= x_1y_2+x_2y_1+x_3y_4+x_4y_3, && \text{if } \xi=\pi \\ q\left(\sum_{i=1}^{4} E_i x_i\right) &= x_1x_2+x_3x_4 && \text{if } \xi=\kappa \end{aligned}\right\}.$$

Next, take $e_5'\in L_3$ with $\{e_5'\}=e_1^\perp\cap L_3$ and $e_5\in e_1e_5'$ with $\{e_5\}=e_2^\perp\cap e_1e_5'$. Then $e_5\in\langle L_1\cup L_3\rangle$ so there is a line $L_3'\in\mathcal{R}$ on e_5 contained in $\langle L_1\cup L_3\rangle$. Since $e_1\neq e_5$, we may replace L_3 by L_3' without harming generality, so as to obtain $e_5\in e_1^\perp\cap e_2^\perp\cap L_3$. Let $e_6\in L_3$ be such that $\{e_6\}=e_3^\perp\cap L_3$ and let $e_4'\in L_2$ be such that $\{e_4'\}=e_6^\perp\cap L_2$. The projective space $A=[L_1\cup L_2\cup L_3]$ has rank 3, 4 or 5.

Let us first consider the case where $\mathrm{rk}(A)=5$. If $\xi=\pi$, then $\mathrm{char}(F)\neq 2$ as otherwise the Witt index would be strictly larger than 2. So we may assume that $\xi=\kappa$. Consider $q|_{[L_2\cup L_3]}$. Let $\gamma\in F$ and E_4' a vector representing e_4' be such that $E_4'=E_4+E_2\gamma$ (notice that $e_4'\neq e_2$). It is easily derived that there are vectors E_5, E_6 representing e_5, e_6 such that

$$q(E_2x_2+E_4x_4+E_5x_5+E_6x_6)=x_2x_6+x_4x_5+\gamma x_4x_6 \qquad (x_i\in F).$$

Considering $q|_{[L_1\cup L_3]}$, we obtain $\alpha, \beta\in F\backslash\{0\}$ such that

$$q(E_1x_1+E_3x_3+E_5x_5+E_6x_6)=\alpha x_1x_6+\beta x_3x_5 \qquad (x_i\in F).$$

The foregoing restrictions describe $q|_A$ fully:

$$q\left(\sum_{i=1}^{6} E_i x_i\right)=x_1x_2+x_3x_4+x_2x_6+x_4x_5+\alpha x_1x_6+\beta x_3x_5+\gamma x_4x_6 \qquad (x_i\in F)$$

Now let n_1 (n_2, n_3, n_4, respectively) be the point of $Q(\kappa)\cap A$ whose homogeneous coordinates with respect to $E_1, E_2, \ldots, E_6$ are $(1,0,0,0,1,0)$ $((0,0,-\alpha,0,0,\beta)$, $(1,0,0,1,0,0)$, $(0,-1,1,0,0,0)$, respectively). Then $N_1=n_1n_2$ is a line of L_1L_3 and $N_2=n_3n_4$ is a line of L_1L_2.

Notice that $N_1\cap N_2=\varnothing$ as $L_1L_2\cap L_1L_3=\{L_1\}$. Now suppose $\langle N_1\cup N_2\rangle$ is a grid with $N\in N_1N_2\cap L_2L_3$ for a line N of $Q(\kappa)$. Then clearly $N\neq N_1, N_2$. Moreover e_2e_5 is a line of $\langle L_2\cup L_3\rangle$ not parallel to L_2, so $e_2e_5\cap N\neq\varnothing$. But a point of $N\backslash\langle L_1\cup L_2\rangle$ has homogeneous coordinates of the form $v+\lambda u$ for $\lambda\in F$, where $v=(\zeta,0,-\alpha,0,\zeta,\beta)$ and $u=(1,-\eta,\eta,1,0,0)$ for $\zeta,\eta\in F$ are homogeneous coordinates of a point in N_1, N_2, respectively. Thus $e_2e_5\cap N\neq\varnothing$ implies the existence of $\zeta, \eta, \lambda, \mu, \nu\in F$ such that

$$(1+\lambda\zeta,\ -\eta,\ \eta-\lambda\alpha,\ 1,\ \lambda\zeta,\ \lambda\beta)=(0,\mu,0,0,\nu,0).$$

The equation leads to a classical contradiction in the fourth coordinate. This proves the lemma in the case where $\mathrm{rk}(A)=5$.

Next, assume that $\mathrm{rk}(A)\leqslant 4$. Then $L_3\cap[L_1\cup L_2]\neq\varnothing$, so $L_3\cap([L_1\cup L_2]\cap Q(\xi)\backslash\langle L_1\cup L_2\rangle)\neq\varnothing$. According to Lemma 4.5, this implies that F has characteristic 2 and that $\xi=\pi$. In particular, π is a symplectic form.

If $\mathrm{rk}(A)=4$, then π is degenerate and has a kernel consisting of a single (projective) point z. Clearly $z\notin A\cap Q(\xi)$, so we may consider the quotient by $[z]$ so as to reduce the proof to the case where $\mathrm{rk}(A)=3$.

Thus, for the rest of the proof, we have that F has characteristic 2, that $\mathrm{rk}(A)=3$ and that $\xi=\pi$ is a polarity determined by the symplectic form whose restriction to A is given by

$$f\left(\sum_{i=1}^{4} E_i x_i, \sum_{i=1}^{4} E_i y_i\right)=x_1y_2+x_2y_1+x_3y_4+x_4y_3 \qquad (x_i, y_i\in F).$$

A straightforward computation using $e_5 \in \{e_1, e_2\}^{\perp}$ yields the existence of $\alpha \in F\backslash\{0\}$ such that E_5 given by $(0, 0, \alpha, 1)$ on the basis E_1, E_2, E_3, E_4 represents e_5.

Also, $e_6 \in \{e_3, e_5\}^{\perp}$ leads to the existence of $\beta \in F\backslash\{0\}$ such that the vector E_6 given by $(1, \beta, 0, 0)$ on the same basis, represents e_6.

Now let n_1 (n_2, n_3, n_4, respectively) be the point of $Q(\pi)$ whose homogeneous coordinates with respect to E_1, E_2, E_3, E_4 are $(1, 0, \alpha, 1)$ ($(1, \beta, \beta, 0)$, $(1, 0, 0, 1)$, $(0, 1, 1, 0)$, respectively). Then $N_1 = n_1n_2$ is a line in L_1L_3 and $N_2 = n_3n_4$ is a line in L_1L_2. Now $\langle N_1 \cup N_2\rangle$ is a grid. Put $N = \langle N_1 \cup N_2\rangle \cap \langle L_2 \cup L_3\rangle$. Let x, y be the point of $Q(\pi)$ whose homogeneous coordinates with respect to E_1, E_2, E_3, E_4 are

$$
\begin{aligned}
X &= (0, \alpha, \alpha, 1), && \text{if } \alpha = \beta,\\
X &= (0, (\zeta+1)\alpha, \zeta\alpha, \zeta), && \text{if } \alpha \neq \beta, \text{ and where } \zeta^2 = \alpha(\alpha+\beta),\\
Y &= (\alpha, \alpha\beta, 0, \alpha + \eta\beta), && \text{where } \eta^2 = \alpha(\alpha+\beta)/\beta^2,
\end{aligned}
$$

respectively. Then x, y are distinct collinear points of N and $X\eta\beta^2 + Y(\alpha + \eta\beta) = (\alpha^2 + \eta\alpha\beta, 0, \alpha^2\beta, 0)$ ($= X\alpha^2 + Y\alpha$ if $\alpha = \beta$) represents a point of xy on L_1.

It follows that $\{xy\} = N_1N_2 \cap L_2L_3$, so that $N_1N_2 \cap L_2L_3$ does not contain a line of $Q(\pi)$ which is disjoint with L_1. This settles the lemma.

The classical case of Proposition 4.2 is dealt with by the following lemma.

LEMMA 4.7. *If $(P, \mathcal{L})$ is classical, then* $\mathrm{rk}(\mathcal{R}) = 1$, *whence $(P, \mathcal{L})$ is a grid.*

PROOF. In view of Lemma 4.4 and the observation, made before, that $(P, \mathcal{L})$ is a grid in case (d) of Theorem 4.1, we need only consider cases (a) and (b). Let L_1, L_2 be two lines from $\mathcal{R}$. Then $L_1 \cap L_2 = \emptyset$ and $\langle L_1 \cup L_2\rangle$ is a grid, so by Lemma 4.5 we may assume that $(P, \mathcal{L}) = Q(\xi)$ for ξ as described in the hypotheses of Lemma 4.6. Suppose we have $L_3 \in \mathcal{R}\backslash L_1L_2$. Then $\langle L_i \cup L_{i+1}\rangle$ is a grid and $L_{i-1} \cap \langle L_i \cup L_{i+1}\rangle = \emptyset$ for each $i \in \{1, 2, 3\}$ (indices taken modulo 3). By Lemma 4.6, however, there are $N_1, M_1 \in L_1L_2\backslash\{L_1\}$ and $N_2, M_2 \in L_1L_3\backslash\{L_1\}$ such that $N_1N_2 \cap M_1M_2$ does not contain a member of $\mathcal{R}$. This means that Pasch's axiom is not satisfied, contradicting that $(\mathcal{R}, \{L_1L_2 | L_1, L_2 \in \mathcal{R}; L_1 \neq L_2\})$ is a projective space. The conclusion is that $\mathcal{R} = L_1L_2$, in other words, that $\mathrm{rk}(\mathcal{R}) = 1$.

5. THE POINT RESIDUE OF A GRASSMANN SPACE

We continue the study of Grassmann spaces. Proposition 4.2 will be used in Lemma 5.7 to derive Proposition 5.9, the main goal of this section. For the duration of this section, *$(P, \mathcal{L})$ is a connected Grassmann space*. Furthermore, ∞ is a fixed point of P and $P^{\infty}, \mathcal{L}^{\infty}, \mathcal{S}^{\infty}, \mathcal{M}^{\infty}$ stand for $\mathcal{L}_{\infty}, \mathcal{L}_{\infty}(\mathcal{V}_{\infty}), \mathcal{L}_{\infty}(\mathcal{S}_{\infty}), \mathcal{L}_{\infty}(\mathcal{M}_{\infty})$, respectively. Moreover, if $V \in \mathcal{V}_{\infty}$, then V^{∞} denotes $\mathcal{L}_{\infty}(V)$. Similarly for members of $\mathcal{L}, \mathcal{S}$ and $\mathcal{M}$.

It is straightforward to check that the residue $(P^{\infty}, \mathcal{L}^{\infty})$ on ∞ is a connected incidence system of diameter 2 satisfying axioms (P1) and (P2). By Lemma 3.1, the members of $\mathcal{M}^{\infty}$ are maximal singular subspaces of $(P^{\infty}, \mathcal{L}^{\infty})$ isomorphic to projective spaces and of the form $L^{\perp}$ for any line $L \in \mathcal{L}^{\infty}$ contained in them. Moreover, $(P^{\infty}, \mathcal{M}^{\infty})$ is a generalized quadrangle by the remark following Lemma 3.8, which is easily seen to be nondegenerate. Members of $\mathcal{S}$ lead to generalized quadrangles in $(P^{\infty}, \mathcal{L}^{\infty})$. We shall call them *quads*. Any two noncollinear points are in a unique quad. Also, if $S \in \mathcal{S}^{\infty}$ and $x \in P^{\infty}\backslash S$, then $x^{\perp} \cap S$ is either empty or a line of $(P^{\infty}, \mathcal{L}^{\infty})$. This is immediate from (P4)$'$.

LEMMA 5.1. *Suppose $M \in \mathcal{M}^\infty$ and $S, T \in \mathcal{S}^\infty$ satisfy $S \cap T \neq \varnothing$, $M \cap S \neq \varnothing$ and $M \cap T \neq \varnothing$. Then $M \cap S \cap T \neq \varnothing$.*

PROOF. Let $x \in S \cap T$ and $u \in M \cap S$, $w \in M \cap T$. If $x \in M$ or $u = w$, we are done. So assume that $x \notin M$ and $u \neq w$. Now $x \in w^\perp$ would imply $w \in u^\perp \cap x^\perp \subseteq S$ if $u \notin x^\perp$ and $x \in (uw)^\perp = M$ otherwise; similarly $x \in u^\perp$ can be settled. Assume $x \notin u^\perp \cup w^\perp$. There is a unique point y in $x^\perp \cap M$. We have $y \in \{x, u\}^\perp \cap \{x, w\}^\perp \subseteq S \cap T$, so $y \in M \cap S \cap T$.

LEMMA 5.2. *Assume that for any $M \in \mathcal{M}^\infty$ and $S \in \mathcal{S}^\infty$, we have $M \cap S \neq \varnothing$. Then $\mathcal{M}^\infty = \mathcal{L}^\infty$ and $|\mathcal{S}^\infty| = 1$, so that $(P, \mathcal{L})$ is a polar space of rank 3.*

PROOF. Fix $x \in P^\infty$. Suppose S, T are distinct quads on x. Write $L = S \cap T$. We shall first show that L is a line. Indeed, it is a singular subspace on x, so L is either a point or a line. Choose $M \in \mathcal{M}^\infty$ not on x. By Lemma 5.1, there must be a point y in $M \cap S \cap T$, so that $xy \subseteq L$. It follows that $L = xy$ is a line. If $N \in \mathcal{M}^\infty$ is disjoint from $L^\perp$, we get a contradiction with $N \cap L = \varnothing$. Since such N exist, it follows that S is the only quad on x. Therefore, S contains all points in P^∞ noncollinear with x. But for each point $z \in x^\perp \backslash \{x\}$, there is a point $u \in z^\perp \backslash x^\perp$, so that $z \in x^\perp \cap u^\perp \subseteq S$. This shows that $P^\infty = S$. Thus the maximal cliques are members of $\mathcal{L}^\infty$, i.e. $\mathcal{M}^\infty = \mathcal{L}^\infty$. Finally, by Lemma 3.17, the Grassmann space $(P, \mathcal{L})$ must be a polar space of rank 3.

LEMMA 5.3. *If $\mathrm{rk}(M_0) = 2$ for some $M_0 \in \mathcal{M}$, then $x^\perp \cap M \neq \varnothing$ for any $x \in P$ and any $M \in \mathcal{M}$ of rank > 2.*

PROOF. Suppose $M \in \mathcal{M}$ is of rank > 2 and $x \in P \backslash M$. In view of the connectedness of $(P, \mathcal{L})$, we may restrict attention to the case where there are $z \in P$ and $y \in M$ such that $z \in x^\perp \cap y^\perp$. As $y \in z^\perp \cap M$, we have $L = z^\perp \cap M \in \mathcal{L}$ by Lemma 3.8(a). If $x^\perp \cap L \neq \varnothing$, we are done. So assume $x^\perp \cap L = \varnothing$. Now $z \in x^\perp \cap L^\perp$, so $x^\perp \cap L^\perp \in \mathcal{L}$ by (P4), and $\langle x^\perp \cap L^\perp, L\rangle$ is a projective space of rank 3 on L. But M is the unique space on L of rank > 2 by Lemmas 3.13 and 3.14, so $z \in x^\perp \cap L^\perp \subseteq M$; in particular $z \in x^\perp \cap M$, terminating the proof.

LEMMA 5.4. *Suppose $\mathrm{rk}(M_0) = 2$ for some $M_0 \in \mathcal{M}$. If both $M_1, M_2 \in \mathcal{M}$ have rank > 2, then $|M_1 \cap M_2| = 1$.*

PROOF. We only need to establish $M_1 \cap M_2 \neq \varnothing$ in view of Lemma 3.14. Suppose $M_1 \cap M_2 = \varnothing$. Take $x \in M_1$. By the previous lemma and Lemma 3.8(a), $L = x^\perp \cap M_2$ is a line. Take $v, w \in L$ with $v \neq w$ and consider $B = v^\perp \cap M_1$ and $C = w^\perp \cap M_1$. If $B = C$, then $\langle B, L\rangle$ is a projective space of rank 3 on L so is contained in M_2, which conflicts $M_1 \cap M_2 = \varnothing$. Thus $B \neq C$. Now B, C are lines on x in M_1, so $\mathrm{rk}(\langle B, C\rangle) = 2$ and there is $y \in M_1 \backslash \langle B, C\rangle$. But $y^\perp \cap L = \varnothing$ so $A = y^\perp \cap L^\perp \in \mathcal{L}$ as $x \in A$ by (P4). Consequently, $\langle A, L\rangle$ has rank 3 and contains L, so is in M_2. It results that A is in M_2, whence $x \in M_1 \cap M_2$.

COROLLARY 5.5. *Suppose there are $M_1, M_2 \in \mathcal{M}$ with $\mathrm{rk}(M_1) = 2$ and $\mathrm{rk}(M_2) = m > 2$. Let $\mathcal{M}^+$ ($\mathcal{M}^-$, respectively) be the connected component of $(\mathcal{M}, \approx)$ whose members have rank m (2, respectively). Then $(\mathcal{M}^+, P^{\mathcal{M}^+})$, where $P^{\mathcal{M}^+} = \{\mathcal{M}_x^+ \mid x \in P\}$, is a projective space of rank $m+1$ such that the points and lines of $(P, \mathcal{L})$ correspond to the lines and pencils of $(\mathcal{M}^+, P^{\mathcal{M}^+})$ respectively. In other words, $(P, \mathcal{L})$ is isomorphic to $A_{m+1,2}(F)$ for some division ring F.*

PROOF. We verify Tallini's axioms in [6]. First of all, it is obvious that no line is a maximal singular subspace.

(a) *Any two members of* $\mathcal{M}^+$ *meet in exactly one point.* (This is the content of Lemma 5.4.)

(b) *If* $M \in \mathcal{M}^+$ *and* $M_1 \in \mathcal{M}^-$ *then* $M \cap M_1$ *is either empty or a line.* (This follows from Lemmas 3.13 and 3.1(d).)

(c) *For any line* L *there is exactly one* $M \in \mathcal{M}^+$ *and one* $M_1 \in \mathcal{M}^-$ *such that* $L = M \cap M_1$. (This results from the remarks preceding Lemma 3.14.)

The corollary now follows from Proposition I in [6].

Instead of referring to [6], a direct proof could have been given, but this would have lengthened the paper by another few pages.

LEMMA 5.6. *Assume that each line is in at least three max spaces. If* $M \cap S$ *is empty for* $M \in \mathcal{M}^\infty$ *and* $S \in \mathcal{S}^\infty$, *then* $\{x \in M \mid x^\perp \cap S \in \mathcal{L}^\infty\}$ *contains a subspace which is a projective plane.*

PROOF. Take $x \in S$. It has a unique neighbor y in M. As $L_1 = y^\perp \cap S$ contains x, it must be a line on x. Let L be another line in S on x, and take $x_2 \in L \backslash \{x\}$. There is $y_2 \in x_2^\perp \cap M$. Notice that $y \neq y_2$ for $y^\perp \cap S$ is a clique and $x_2 \notin L_1^\perp$. Write $L_2 = y_2^\perp \cap S$. This is a line disjoint from L_1 (cf. Lemma 3.5). Suppose L^1 is a third line on x, not in $L_1^\perp \cup L^\perp$ (such a line exists by assumption). Take $w \in L^1 \backslash \{x\}$. If $w \notin S$, then $w^\perp \cap S$ contains x, so must be a line in S distinct from L_1 and L. Therefore there is a point $x_3 \in x^\perp \cap S \backslash (L_1 \cup L)$. Again, take $y_3 \in x_3^\perp \cap M$ and consider $y_3^\perp \cap S$. It has a line on x_3 not in $\langle L_1, L_2\rangle$. Thus $y_3 \notin yy_2$ and $\langle y, y_2, y_3\rangle$ is a subspace of the desired kind.

We recall from Corollary 3.7 that for $S \in \mathcal{S}$ and $M \in \mathcal{M}$, the subset $S \cup \{z \in M \backslash S \mid z^\perp \cap S \in \mathcal{V}\}$ is denoted by $H(\mathcal{V}(S), S)$. We shall also write $H(S)$ instead of $H(\mathcal{V}(S), S)$.

LEMMA 5.7. *Suppose there are* $M \in \mathcal{M}$ *and* $S \in \mathcal{S}$ *with* $M \cap S = \{\infty\}$. *Then* $M \cap H(S)$ *is a subspace of* M *of rank at most* 2.

PROOF. Set $V = M \cap H(S)$. It follows from Corollary 3.7 that V is a subspace of M. Recall that M^∞, S^∞, V^∞ denote the subspaces of $(P^\infty, \mathcal{L}^\infty)$ induced by M, S, V, respectively. Let $\mathcal{R}$ be the subfamily of $\mathcal{L}^\infty$ whose members occur as $z^\perp \cap S^\infty$ for some $z \in V^\infty$. Then $\mathcal{R}$ is a spread of the quad S^∞, for any two members of $\mathcal{R}$ are disjoint (in P^∞) by Lemma 3.5 and if $x \in S^\infty$, then $x^\perp \cap M^\infty = \{y\}$ for some $y \in V^\infty$ by Lemma 3.8(a), whence $y^\perp \cap S^\infty$ is a member of $\mathcal{R}$ on x. Now let L be a line of $(P^\infty, \mathcal{L}^\infty)$ in V^∞. Then $U = \bigcup_{x \in L} x^\perp \cap S^\infty$ is a grid in S^∞. For suppose there are $x_1, y_1 \in U$ with $x_1 \in y_1^\perp \backslash \{y_1\}$. Then there are unique $x, y \in L$ with $x_1 \in x^\perp \cap S^\infty$ and $y_1 \in y^\perp \cap S^\infty$. If $z_1 \in x_1y_1$, then either $x = y$ and $z_1 \in x^\perp \cap S^\infty$ or $x \neq y$. In the latter case x, y, y_1, x_1 is a 4-circuit, so there is $z \in xy$ with $z_1 \in z^\perp \cap S^\infty$. So U is a subspace. Proceeding with $x \neq y$, we see that x_1y_1 and $x^\perp \cap S^\infty$ are the only two lines on x_1 in U, so U is indeed a grid in S^∞. Moreover, one parallel class of lines in U is entirely contained in $\mathcal{R}$. Denoting by L_1L_2 for $L_1, L_2 \in \mathcal{R}$ the parallel class of lines in $\langle L_1, L_2\rangle$ belonging to $\mathcal{R}$, we obtain a surjective morphism

$$\alpha\colon (V^\infty, \mathcal{L}^\infty(V^\infty)) \to (\mathcal{R}, \{L_1L_2 \mid L_1, L_2 \in \mathcal{R}; L_1 \neq L_2\})$$

of projective spaces given by $\alpha(x) = x^\perp \cap S^\infty (x \in V^\infty)$. If $x_1, x_2 \in V^\infty$ satisfy $x_1^\perp \cap S^\infty = x_2^\perp \cap S^\infty$, then $N = \langle x_1, x_2, x_1^\perp \cap S^\infty\rangle$ is a singular subspace with $\mathrm{rk}(M^\infty \cap N) \geq \mathrm{rk}(\langle x_1, x_2\rangle)$.

Since $N \cap S^\infty = x_1^\perp \cap S^\infty \neq \emptyset$, we have $M^\infty \neq N$, whence $\mathrm{rk}(M^\infty \cap N) \leq 0$. It results that $\mathrm{rk}(\langle x_1, x_2 \rangle) = 0$, i.e. $x_1 = x_2$. This shows that u is bijective, so that $\mathrm{rk}(\mathcal{V}) = \mathrm{rk}(\mathcal{V}^\infty) + 1 = \mathrm{rk}(\mathcal{R}) + 1 \leq 2$ by Proposition 4.2.

COROLLARY 5.8. *Each line of $(P, \mathcal{L})$ is in precisely two max spaces, unless $(P, \mathcal{L})$ is a polar space of rank* 3.

PROOF. Suppose there is a line in strictly more than two max spaces. Let $S \in \mathcal{S}$ and $M \in \mathcal{M}$ satisfy $M \cap S = \{\infty\}$ and consider $V = M \cap H(S)$ (cf. Corollary 3.7 and Lemma 5.7). By Lemma 5.7, $\mathrm{rk}(V) \leq 2$ and by Lemma 5.6, $\mathrm{rk}(V) \geq 3$, contradiction. It results that the conditions of Lemma 5.2 are satisfied, so that $(P, \mathcal{L})$ is a polar space of rank 3.

We summarize the results obtained in this section.

PROPOSITION 5.9. *Let $(P, \mathcal{L})$ be a connected Grassmann space whose max spaces have finite rank. Assume $(P, \mathcal{L})$ is not isomorphic to a polar space (of rank* 3) *or $A_{n,2}(F)$ for some $n \geq 4$ and some division ring F. Then for each point $x \in P$, the residue $(\mathcal{L}_x, \mathcal{M}_x)$ is a grid. In particular, each line is in precisely two max spaces. Moreover, the rank of any max space is* >2.

6. COOPERSTEIN'S THEOREM A

Until further notice, $(P, \mathcal{L})$ is a connected Grassmann space such that any line is in precisely two max spaces, each of them of rank >2. We fix a point ∞ of P and maintain the notation of Sections 3 and 5.

LEMMA 6.1. *Let $M, N \in \mathcal{M}$ and $S \in \mathcal{S}$ with $M \cap S, N \cap S \in \mathcal{V}$. Then* $\mathrm{rk}\,(M \cap N \cap S) = 0$ *iff $M \approx N$.*

PROOF. The assertion follows from the fact that $x^\perp \cap S$ is a singular subspace for any $x \in P \backslash S$.

We supply the graph $(\mathcal{M}, \approx)$ with the natural family of lines that turns $\mathcal{M}$ into a Gamma space whose collinearity graph is $(\mathcal{M}, \approx)$. To avoid confusion, we denote by $M^\top$ for $M \in \mathcal{M}$ (rather than $M^\perp$ which has a distinct interpretation) the set of vertices in $(\mathcal{M}, \approx)$ at distance at most 1 to M. For $M_1, M_2 \in \mathcal{M}$ with $M_1 \approx M_2$, the *line* M_1M_2 is defined by $M_1M_2 = \{M_1, M_2\}^{\top\top}$. A priori, it is not clear that this turns $\mathcal{M}$ into a linear incidence system, but it will follow from 6.3 that it does. By $\mathcal{C}$ we denote the family of all such lines, i.e.

$$\mathcal{C} = \{M_1M_2 | M_1, M_2 \in \mathcal{M}, M_1 \approx M_2\}.$$

We need some more notation. For $x \in P$, $L \in \mathcal{L}$, $V \in \mathcal{V}$ and $M \in \mathcal{M}$ with $x \in L \subseteq V \subseteq M$, denote by $p(L, M)$ or $p(L, V)$ the unique member of $\mathcal{M}$ containing L, and distinct from M. Furthermore, put $l(x, V) = \{p(L^1, V) | L^1 \in \mathcal{L}(V)_x\}$, $m(x, M) = \{p(L^1, M) | L^1 \in \mathcal{L}(M)_x\}$ and

$$n(x, V) = \{p(L', W) \mid W \in \mathcal{V}_x;\ W \cap p(L'', V) \in \mathcal{L}, \text{for each } L'' \in \mathcal{L}(V)_x;\ L' \in \mathcal{L}(W)\}.$$

LEMMA 6.2. *Two distinct max spaces M_1, M_2 are at distance* 2 *in $(\mathcal{M}, \approx)$ iff $M_1 \cap M_2 = \emptyset$ and there is $M \in \mathcal{M}$ with $M \cap M_1$, $M \cap M_2 \in \mathcal{L}$. Moreover, connected components of $(\mathcal{M}, \approx)$ are not complete.*

PROOF. Suppose M_1, M_2 are at distance 2 in $(\mathcal{M}, \approx)$. Then $M_1 \cap M_2 = \varnothing$, for if $M_1 \cap M_2 \in \mathcal{L}$ then $\{M_1, M_2\}^\top = \varnothing$ by Lemma 3.8(b). Let $H \in \{M_1, M_2\}^\top$. There are $x_i \in P$ with $H \cap M_i = \{x_i\}$ for $i = 1$, 2. Consider $L_1 = x_2^\perp \cap M_1$ and $L_2 = x_1^\perp \cap M_2$. By Lemma 3.8 we know $L_i \in \mathcal{L}$. Now $\langle x_1, x_2, L_i\rangle^\perp$ $(i = 1, 2)$ and H are three max spaces on x_1x_2, while H differs from the first two as it intersects M_1 and M_2 in a point. So $M = \langle x_1, x_2, L_1, L_2\rangle^\perp$ is a max space with $M \cap M_i = L_i$.

Conversely, let M be a max space with $M \cap M_i \in \mathcal{L}$ for $i = 1, 2$ and suppose $M_1 \cap M_2 = \varnothing$. Take $x_i \in M \cap M_i$ and consider $H = p(x_1x_2, M)$. By Lemma 3.8, $H \cap M_i = \{x_i\}$ so that $H \in \{M_1, M_2\}^\top$. Thus M_1, M_2 have distance $\leqslant 2$; but their distance is $\geqslant 2$ as $M_1 \cap M_2 = \varnothing$. This establishes that M_1, M_2 are at distance 2.

Finally, let $M \in \mathcal{M}$ and let $L_1, L_2 \in \mathcal{L}$ be disjoint lines (they exist as rk $(M) \geqslant 3$). Then $p(L_1, M)$ and $p(L_2, M)$ are not joined as they have distance 2 by the above criterion. This shows that connected components of $(\mathcal{M}, \approx)$ are not complete.

LEMMA 6.3. *Suppose $M_1, M_2 \in \mathcal{M}$ satisfy $M_1 \cap M_2 = \{\infty\}$. Let $V \in \mathcal{V}_\infty$ be such that $V \cap M_1$, $V \cap M_2 \in \mathcal{L}$. Then $\{M_1, M_2\}^\top = m \cup n$, where $m = m(^\infty, V^\perp)$ and $n = n(\infty, V)$ are maximal cliques in $(\mathcal{M}, \approx)$ with $m \cap n = l(\infty, V)$. Moreover, if for $Y \in m$ and $Y' \in n$ we have $Y \approx Y'$, then at least one of Y, Y' is in $l(\infty, V)$. In particular, $M_1M_2 = l(\infty, V)$.*

PROOF. Clearly $l(\infty, V) = l(\infty, V')$ for any $V' \in \mathcal{V}_\infty$ with $V' \cap M_1$, $V' \cap M_2 \in \mathcal{L}$, as V^∞ and V'^∞ are parallel lines of a grid in $(P^\infty, \mathcal{L}^\infty)$ on the 4-circuit $(V' \cap M_1)^\infty$, $(V \cap M_1)^\infty$, $(V \cap M_2)^\infty$, $(V' \cap M_2)^\infty$.

Let us now determine $\{M_1, M_2\}^\top$. By Lemma 3.8, any two distinct members X, X' of m satisfy $X \cap X' = \{\infty\}$, so m is a clique contained in $\{M_1, M_2\}^\top$.

If $M \in n$, there are $W \in \mathcal{V}_\infty$ with $W \cap p(L', V) \in \mathcal{L}$ for each $L' \in \mathcal{L}(V)_\infty$ and $L \in \mathcal{L}$ with $L = M \cap W$. Notice that $M_i \cap W \in \mathcal{L}$ since $(W^\perp)^\infty$ is parallel to $(V^\perp)^\infty$ in $(P^\infty, \mathcal{M}^\infty)$. As $l(\infty, V) = l(\infty, W)$, we may replace W by V without loss of generality. If $M \in m$, we have $M \in l(\infty, V)$ as before. Assume $\infty \notin L$, and let $i \in \{1, 2\}$. Since $V \cap M_i$ and L are lines in V, there is a point $x_i \in V$ such that $\{x_i\} = L \cap M_i$. It follows by Lemma 3.8(b) that $M \cap M_i = \{x_i\}$ (notice that M and $V^\perp$ meet in xx_i). Thus $M \in \{M_1, M_2\}^\top$.

Retain $M \in l(\infty, V) \cap n$ with $L = M \cap V$. Suppose that $N \in n \backslash \{M\}$. We shall show that $M \approx N$. As before, there are $U \in \mathcal{V}_\infty$ with $U \cap M_i \in \mathcal{L}$ and $K \in \mathcal{L}$ with $K = N \cap U$. If $U = V$, then $M \cap N = K \cap L$ is a singleton as $M \neq N$, so $M \approx N$, indeed. Suppose $U \neq V$. Consider the symp S containing both U and V (it exists as $\mathcal{U}^\infty$ and $\mathcal{V}^\infty$ are parallel lines of $(P^\infty, \mathcal{L}^\infty)$).

Since $M \cap S \supseteq L$ and $N \cap S \supseteq K$ we have $M \cap S$, $N \cap S \in \mathcal{V}$. Moreover, they belong to the same connected component of $(\mathcal{V}(S), \approx)$ (see the text preceding Corollary 3.7), as $U \approx V$. Since the two connected components of $(\mathcal{V}(S), \approx)$ are complete graphs, we have $M \cap S \approx N \cap S$, i.e. $\mathrm{rk}(M \cap N \cap S) = 0$. Thus $M \approx N$ by Lemma 6.1. The conclusion is that $M \in N^\top$ for any two $M, N \in n$. In other words, n is a clique.

We have seen that $m \cup n \subseteq \{M_1, M_2\}^\top$. The converse is straightforward: each member M of $\{M_1, M_2\}^\top$ is in m whenever $M \cap M_1 = \{\infty\}$ and in n otherwise. Let us now consider $m \cap n$. The inclusion $l(\infty, V) \subseteq m \cap n$ is obvious. To prove the opposite inclusion, let $M \in m \cap n$. Since $M \in n$, there are $W \in \mathcal{V}_\infty$ with $W \cap M_1$, $W \cap M_2 \in \mathcal{L}$ and $L' \in \mathcal{L}(W)$ such that $M \cap W = L'$; but $\infty \in M$ as $M \in m$, so $\infty \in L'$ and $M \in l(\infty, W)$. By the first paragraph of this proof, this is equivalent to $M \in l(\infty, V)$. This proves $m \cap n = l(\infty, V)$. Next, let $Y \in m$ and $Y' \in n \backslash l(\infty, V)$ with $Y \approx Y'$. Let x, y_1, y_2 be such that $\{x\} = Y \cap Y'$, $Y' \cap M_1 = \{y_1\}$, $Y' \cap M_2 = \{y_2\}$. Then $\{x, y_1, y_2\}$ is a clique (in Y'). Note that $y_1 \neq y_2$ as $Y' \notin l(\infty, V)$. Set $L = Y \cap \langle \infty, y_1, y_2\rangle$. This is a line. If $x \notin y_1y_2$, then $\langle x, y_1, y_2\rangle \in \mathcal{V}$ and $\langle x, y_1, y_2\rangle^\perp = Y'$. But $L \subseteq \langle y_1, y_2, x\rangle^\perp$, so $L \subseteq Y'$ contradicting that $Y \cap Y'$ is a singleton.

Hence $x \in y_1 y_2$, so that $L = \infty x$ is in V and $Y \in l(\infty, V)$. This establishes the one but last claim of the lemma. The last one follows directly as well as the maximality of m and n as cliques.

COROLLARY 6.4. *$(\mathcal{M}, \mathcal{C})$ is a Gamma space with thick lines, where $\mathcal{C} = \{l(x, V) | x \in P, V \in \mathcal{V}_x\}$.*

PROOF. It follows from the previous lemma that $(\mathcal{M}, \mathcal{C})$ is a Gamma space and that the definition of $\mathcal{C}$ coincides with the one given in the text preceding Lemma 6.2. Thickness of the lines is a consequence of the bijection $L \to l(x, V)$ for fixed $L \in \mathcal{L}(V)$ with $x \notin L$ given by $y \mapsto p(xy, V^{\top})(y \in L)$.

LEMMA 6.5. *If $X_1, X_2 \in \mathcal{M}$ are at distance 2 in $(\mathcal{M}, \mathcal{C})$, then $\{X_1, X_2\}^{\top}$ is a grid.*

PROOF. By 6.2 we have $X_1 \cap X_2 = \varnothing$ and there is $K \in \mathcal{M}$ with $L_i = H \cap X_i \in \mathcal{L}$ for each $i \in \{1, 2\}$. We claim that any $Y \in \{X_1, X_2\}^{\top}$ meets X_i in a point of L_i. For let $y_1 \in X_1 \backslash L_1$. Then $y_1^{\perp} \cap L_2 = \varnothing$ as otherwise $y_1^{\perp} \cap K$ would contain more than the line L_1, leading to $y_1 \in K$ and $K = X_1$, conflicting $X_1 \cap X_2 = \varnothing$.

Thus $L_1 = y_1^{\perp} \cap L_2^{\perp}$ by (P4), so that $y_1^{\perp} \cap X_2 \neq \varnothing$ implies $X_1 \cap X_2 \neq \varnothing$, which is absurd. We conclude that $y_1^{\perp} \cap X_2 = \varnothing$, so that no max space on y_1 is in $X_2^{\top}$. This implies that any $Y \in \{X_1, X_2\}^{\top}$ meets X_1 in a point of L_1. Similarly the claim is proved for $i = 2$. The claim yields the following description of the subspace under study:

$$\{X_1, X_2\}^{\top} = \{p(L, K) \,|\, L \in \mathcal{L}(K), L \cap L_1 \neq \varnothing, L \cap L_2 \neq \varnothing\}.$$

The lines in $\{X_1, X_2\}^{\top}$ are of the form $l(x, \langle x, L_i \rangle)$ for $x \in L_j$, where $\{i, j\} = \{1, 2\}$.

In particular, $\{X_1, X_2\}^{\top}$ is isomorphic to the geometry on the lines intersecting two disjoint given lines L_1, L_2 in a projective space of rank 3, in which two members are collinear whenever they intersect. This geometry is well known and easily checked to be that of a grid (cf. Section 4).

LEMMA 6.6. *Suppose $l \in \mathcal{C}$ and $M \in \mathcal{M}$ satisfy $M^{\top} \cap l = \varnothing$ and $M^{\top} \cap l^{\top} \neq \varnothing$. Then $M^{\top} \cap l^{\top} \in \mathcal{C}$.*

PROOF. It suffices to show that $M^{\top} \cap L^{\top}$ contains at least two points.

Let $x \in P$ and $V \in \mathcal{V}_x$ be such that $l = l(x, V)$. Suppose $H \in M^{\top} \cap l^{\top}$. Then by Lemma 6.3 we have $H \in m \cup n$, where $m = m(x, V^{\perp})$ and $n = n(x, V)$. Notice that $M \cap V = \varnothing$. Let $y \in P$ be such that $H \cap M = \{y\}$. Suppose $H \in m$. Then $x \in H$. Write $L = x^{\perp} \cap M$. This is a line on y, so $W = \langle x, L \rangle$ is a plane. Take $L' \in \mathcal{L}(W)_x$ with $y \notin L'$, then $p(L', W)$ is a member of $M^{\top} \cap l^{\top}$ distinct from H.

Suppose $H \in n \backslash m$. Then $d(x, y) = 2$. Consider $S = S(x, y)$. Notice that $y^{\perp} \cap V$ is a line of $x^{\perp} \cap y^{\perp}$. Let L be a line of $x^{\perp} \cap y^{\perp}$ parallel (and distinct) to $y^{\perp} \cap V$, and define $H' = \langle y, L \rangle^{\perp}$. Then $H' = p(L, \langle x, L \rangle) \in n$ so $H' \in M^{\top} \cap n \subseteq M^{\top} \cap l^{\top}$. As $H' \neq H$, we are done.

COROLLARY 6.7. *$(\mathcal{M}, \mathcal{C})$ is a Grassmann space.*

PROOF. Axiom (P1) is proved in Corollary 6.4 (where it is also stated that lines are thick), (P2) in Lemma 6.2, (P3) in Lemma 6.5 and (P4) in Lemma 6.6.

LEMMA 6.8. *Take $M \in \mathcal{M}_{\infty}$ of rank i. Then $m(\infty, M)$ is a projective space in $(\mathcal{M}, \mathcal{C})$ of rank $i - 1$. If $V \in \mathcal{V}_{\infty}$ is such that $V \cap M \in \mathcal{L}$, then $n(\infty, V)$ is a projective space of rank $i + 1$.*

PROOF. Write $m = m(\infty, M)$ and $n = n(\infty, V)$. By Corollary 3.3, both m and n are projective spaces. By construction of m, there is a bijective map $\mu: M^\infty \to m$ given by $\mu(L^\infty) = p(L, M)$ for $L \in \mathscr{L}_\infty(M)$. Given $\mathscr{V}' \in \mathscr{V}(M)_\infty$, we have $\mu(\mathscr{L}(\mathscr{V}')_\infty) = \{p(L', V') \mid \infty \in L' \in \mathscr{L}(V')\} = l(\infty, V')$, so that μ maps lines of $(P^\infty, \mathscr{L}^\infty)$ in M^∞ onto lines of $(\mathcal{M}, \mathscr{C})$ in m. As $\mathrm{rk}(M^\infty) = i-1$, this shows that $\mathrm{rk}(m) = i-1$.

Next, consider n. Choose $L_1, L_2 \in \mathscr{L}(V)_\infty$ distinct and write $M_i = p(L_i, V)$. Furthermore let H_i be a hyperspace of M_i disjoint from ∞. Given $x_1 \in H_1$, the line $x_1^\perp \cap M_2$ on ∞ intersects H_2 in a point x_2.

This leads to a map $\psi: H_1 \to n$ given by $\psi(x_1) = p(x_1x_2, \langle \infty, x_1x_2 \rangle)$. This map is easily seen to be injective. Moreover, if L_1' is a line of H_1, then $\psi(L_1')$ is a line of $(\mathcal{M}, \mathscr{C})$:

Let $L_2' = \bigcup_{x \in L_1'} x^\perp \cap H_2$ and take $x_1, y_1 \in L_1'$, $x_1 \neq y_1$. Then there are unique $x_2, y_2 \in L_2'$ collinear with x_1, y_1 respectively. Consider the generalized quadrangle $x_1^\perp \cap y_2^\perp$. It contains the lines ∞x_2 and ∞y_1, so there is a point $\infty' \in x_1^\perp \cap y_2^\perp \cap x_2^\perp \cap y_1^\perp \backslash \infty^\perp$. Now $\psi(L_1') = l(\infty', \langle \infty', L_1' \rangle)$ is a line in n.

As a consequence, $\psi(H_1)$ is a singular subspace of n of rank $i-1$ (note that M_1 is of rank i). But $l(\infty, V)$ is a line of n completely disjoint from $\psi(H_1)$. We conclude that $\mathrm{rk}\, n \geq i+1$. We finish by showing that any member N of n is on a line in $(\mathcal{M}, \mathscr{C})$ from a member of $l(\infty, V)$ to a member of $\psi(H_1)$. By analogous arguments to what we have seen before, we are easily led to the case where $N = p(y_1y_2, V)$ for distinct y_i in $N \cap M_1 \backslash \{\infty\}$ $(i = 1, 2)$. Let $x_i \in N \cap H_i$, so that $x_i\infty = y_i\infty$. If $x_1 = y_1$ and $x_2 = y_2$ then $N \in \psi(H_1)$, so we may assume that $x_1x_2 \neq y_1y_2$. Since both lines are in V, there is $z \in V$ with $x_1x_2 \cap y_1y_2 = \{z\}$. Now $N = p(y_1y_2, V)$ is on the line $l(z, V)$ which has member $p(\infty z, \mathscr{V})$ in $l(\infty, V)$ and member $p(x_1x_2, V)$ in $\psi(H_1)$.

We conclude that n is spanned by $l(\infty, V)$ and $\psi(H_1)$. Thus $\mathrm{rk}(n) \leq i+1$, and equality holds.

Recall that in Section 2 the quotient of an incidence system by an automorphism group is defined as well as the incidence system $A_{a,d}(F)$.

We are now ready for the proof of the main result of this paper.

We drop the assumptions on $(P, \mathscr{L})$ made at the beginning of this section.

6.9. PROOF OF THE MAIN THEOREM. First of all, notice that if $(P, \mathscr{L})$ is as described in (a) or (b) of the main theorem, then $(P, \mathscr{L})$ is a connected Grassmann space whose max spaces have finite ranks. Furthermore, if $(P, \mathscr{L})$ is a connected Grassmann space and σ is an involutory automorphism of $(P, \mathscr{L})$ satisfying $d(x, x^\sigma) \geq 5$ for all $x \in P$, then $(P, \mathscr{L})/\langle \sigma \rangle$ is readily seen to be a connected Grassmann space, too. This yields that if $(P, \mathscr{L})$ is as in (c) of the theorem, it is a Grassmann space of the desired kind. This proves the "if" part of the theorem.

As for the "only if" part, let $(P, \mathscr{L})$ be a connected Grassmann space whose max spaces have finite rank. We claim that one of the following holds:

(a) *$(P, \mathscr{L})$ is a nondegenerate polar space of rank* 3 *with thick lines.*

(b) *$(P, \mathscr{L}) \cong A_{a,d}(F)$ for some $a \geq 4$, $d \leq (a+1)/2$ and some division ring F.*

(c) *There is a natural number $d \geq 5$, a division ring F and an involutory automorphism σ of $A = A_{2d-1,d}(F)$, interchanging the connected components of the graph $(\mathcal{M}, \approx)$ on the max spaces, with $d(x, x^\sigma) \geq 5$ for all points x of A such that $(P, \mathscr{L}) \cong A/\langle \sigma \rangle$.*

In case (c) above, σ is induced by a polarity of the projective space over F of rank $2d-1$ such that $x \cap x^\sigma$ has codimension at least 5 in x for any subspace x of rank $d-1$. By the classification of such polarities, cf. [3], it follows that F must be infinite. Therefore, establishing the claim suffices for the proof of the main theorem.

By Lemmas 3.13 and 3.14, $\mathrm{rk}(M)$ for $M \in \mathcal{M}$ attains at most two values. Let d be the minimal of these and let b be the other one if it exists, let $b = d$ otherwise. The proof runs by induction on d. The case $d = 2$ has been settled in Proposition 5.9.

Assume $d>2$, and suppose $(P, \mathscr{L})$ is not a polar space of rank 3. By Proposition 5.9 we have that each line is in exactly two max spaces. By Corollary 6.7 and Lemma 6.8, there is a connected component $\mathscr{M}^+$ of $(\mathscr{M}, \approx)$ such that the induced subgraph $(\mathscr{M}^+, \approx)$ is the collinearity graph of the connected Grassmann space $(\mathscr{M}^+, \mathscr{C}^+)$ where $\mathscr{C}^+ = \{l \in \mathscr{C} \mid l \cap \mathscr{M}^+ \neq \emptyset\}$, whose max spaces have ranks $d-1$, $b+1$. The induction hypothesis then yields that (b) occurs, so that $(\mathscr{M}^+, \mathscr{C}^+) \cong A_{d+b-1,d-1}(F)$ for some division ring F. Now $A_{d+b-1,d}(F)$ can be thought of as the incidence system obtained from $A_{d+b-1,d-1}(F)$ by taking the max spaces of rank $d-1$ from one connected component under $\approx$ in $A_{d+b-1,d-1}(F)$ for points and the relation $\approx$ (i.e. $M \approx N$ iff M and N meet in a point) for collinearity. Remember that this determines $A_{d+b-1,d}(F)$ as any Grassmann space is determined by its collinearity graph (cf. Lemma 3.1).

Let $(P', \mathscr{L}')$ be the incidence system that can be obtained from $(\mathscr{M}^+, \mathscr{C}^+)$ in just the way $A_{d+b-1,d}(F)$ is obtained from $A_{d+b-1,d-1}(F)$ (notice that this makes sense as $(\mathscr{M}^+, \mathscr{C}^+) \cong A_{d+b-1,d-1}(F)$).

If $m \in P'$, then m is a projective space in $(\mathscr{M}, \mathscr{C})$ of rank $d-1$, so $m = m(x, M)$ for a unique $x \in P$ and some $M \in \mathscr{M}\backslash m$. Thus there is a map $\mu: P' \to P$ sending $m \in P'$ to the unique $x \in P$ for which there is $M \in \mathscr{M}\backslash m$ with $m = m(x, M)$. This map is clearly surjective and is either $2:1$ or $1:1$ according as $\mathscr{M} = \mathscr{M}^+$ or not, i.e. according as $(\mathscr{M}, \approx)$ has one or two connected components. We claim that μ is a morphism of graphs. For if m, n are collinear in $(P', \mathscr{L}')$, the points $\mu(m)$ and $\mu(n)$ are both contained in the max space M for which $m \cap n = \{M\}$. Consequently, $\mu(m)$ and $\mu(n)$ are collinear in $(P, \mathscr{L})$.

If $(\mathscr{M}, \approx)$ is disconnected, the inverse map is a morphism, too, and we have $(P, \mathscr{L}) \cong (P', \mathscr{L}') \cong A_{d+b-1,d}(F)$.

Let from now on $(\mathscr{M}, \approx)$ be connected. Now μ is a surjective $2:1$ morphism. Also $b = d$ in view of Lemmas 3.13 and 3.14 so $(P', \mathscr{L}') \cong A_{2d-1,d}(F)$ for some $d \geqslant 3$. Choose $m \in P'$. We shall show that μ is bijective when restricted to the neighborhood $m^\perp$ of m in $(P', \mathscr{L}')$. Let x, y be distinct collinear points of P, suppose $\mu(m) = x$ and let $m_1, m_2 \in P'$ both be collinear with m and such that $\mu(m_1) = \mu(m_2) = y$.

As before, we may assume that $m = m(x, M)$ for $M \in \mathscr{M}$ with $xy \subseteq M$. Similarly we may take $M_i \in \mathscr{M}$ with $xy \subseteq M_i$ such that $m_i = m(y, M_i)$, for each $i \in \{1, 2\}$. Suppose now that $M_1 \neq M_2$. Since each of M, M_1, M_2 contains xy, it follows that M coincides with M_1 or M_2. Without loss of generality we may assume that $M = M_1$. Since m_2 is collinear with m, there is $Y \in \mathscr{M}$ such that $Y \in m(x, M) \cap m(y, M_2)$.

Thus Y contains xy, so either $Y = M$ or $Y = M_2$. But $Y = M$ conflicts $M \cap Y \in \mathscr{L}$ and $Y = M_2$ conflicts $M_2 \cap Y \in \mathscr{L}$. It results that $M_1 = M_2$, so that $m_1 = p(y, M_1) = p(y, M_2) = m_2$. We have established that the restriction of μ to the members of P' collinear with a given point is injective.

Our next step is to show that the restriction of μ to the subset $m^\perp$ of P' of members collinear with m is an isomorphism of graphs. Thus for $m_1, m_2 \in P'\backslash\{m\}$ collinear with m such that $x_1 = \mu(m_1)$, $x_2 = \mu(m_2)$ are collinear in P, we have to derive that m_1 is collinear with m_2 in $(P', \mathscr{L}')$. Let $V = \langle x, x_1, x_2\rangle$, where $x = \mu(m)$.

Since $m \perp m_i$, there are $X_i \in m \cap m_i$ for $i = 1, 2$. Thus X_i contains xx_i. If $x_1 \in xx_2$, then $X_1 \cap X_2 = xx_2 \in \mathscr{L}$, conflicting $X_1 \approx X_2$. It follows that V is a plane. Since $V \cap X_i = xx_i \in \mathscr{L}$, we have $m_i = m(x_i, V^\perp)$ so that $p(x_1x_2, V) \in m_1 \cap m_2$. Hence $m_1 \perp m_2$.

Next, define $\sigma: (P', \mathscr{L}') \to (P', \mathscr{L}')$ to be the unique map such that $\mu^{-1}(\mu(m)) = \{m, m^\sigma\}$ for each $m \in P'$. Clearly, σ is an involution. Also, σ is an automorphism of $(P', \mathscr{L}')$. For if $m \perp n$ for $m, n \in P'$, then $\mu(m) \perp \mu(n)$ and $n^\sigma \notin m^\perp$ since μ is bijective on $m^\perp$. But then $m^\sigma \perp n^\sigma$ since μ is bijective on $(n^\sigma)^\perp$. So indeed, σ is an automorphism of $(P', \mathscr{L}')$ and $d(m, m^\sigma) \geqslant 3$ for any $m \in P'$.

But if there is $m \in P'$ with $d(m, m^\sigma) = 3$, then there are $m_1, m_2 \in P'$ with $m \perp m_1 \perp m_2 \perp m^\sigma$, so that $m \perp m_2^\sigma \perp m_1^\sigma \perp m^\sigma$. Since μ is an isomorphism on the subgraph induced on

$m^\perp$, and $\{\mu(m_1), \mu(m_2), \mu(m)\}$ is a clique, this yields that $m_1 \perp m_2^\sigma$. This is in contradiction with $m_1 \perp m_2$.

We have shown that $d(m, m^\sigma) \geq 4$ for any $m \in P'$. Suppose $d(m, m^\sigma) = 4$ for some $m \in P'$. Then there is a minimal path $m, m_1, m_2, m_3, m^\sigma$ with $m_i \in P'$ $(i = 1, 2, 3)$, so that $m_1 \in m^\perp \cap m_2^\perp$ and $m_3^\sigma \in m^\perp \cap (m_2^\sigma)^\perp$. This leads to two connected components $\mu(m^\perp \cap m_2^\perp)$ and $\mu(m^\perp \cap (m_2^\sigma)^\perp)$ in $\mu(m)^\perp \cap \mu(m_2)^\perp$. Indeed, if there are $n_1 \in m^\perp \cap m_2^\perp$ and $n_2 \in m^\perp \cap (m_2^\sigma)^\perp$ with $\mu(n_1) \perp \mu(n_2)$, then $n_1 \perp n_2$ as $n_1, n_2 \in m^\perp$, so $m_2, n_1, n_2, m_2^\sigma$ is a path of length 3 contradicting $d(m_2, m_2^\sigma) \geq 4$. But this contradicts the fact that $\mu(m)^\perp \cap \mu(m_2)^\perp$ is connected (as it is a generalized quadrangle by assumption). We conclude that $d(m, m^\sigma) \geq 5$ for all $m \in P'$. Finally, since $(P', \mathscr{L}') \cong A_{2d-1,d}(F)$ has diameter d, the existence of σ implies that $d \geq 5$. This ends the proof of the theorem.

We conclude this section by mentioning that $A_{2d-1,d}(\mathbb{R})/\langle\sigma\rangle$ for $d \geq 5$, where σ is the polarity associated with the quadratic form $\sum_{i=1}^{2d} x_i^2$ (or any other nondegenerate form of Witt index at most $d-5$), provides an example of a Grassmann space of the type occurring in (c) of the main theorem.

7. Applications

In this section $(P, \mathscr{L})$ is a connected Grassmann space. Consider the following two axioms, each of them stronger than (P4).

(Q4) If $x \in P$ and $L \in \mathscr{L}$ with $x^\perp \cap L = \varnothing$, then $x^\perp \cap L^\perp \in \mathscr{L}$.

(R4) If $L_1, L_2 \in \mathscr{L}$ with $L_1 \cap L_2 \neq \varnothing$ and $z \in P$, then there is $u \in z^\perp$ with $u^\perp \cap L_1 \neq \varnothing$ and $u^\perp \cap L_2 \neq \varnothing$.

It is an easy exercise to show that (Q4) holds for $(P, \mathscr{L})$ iff (Q4)′ holds, where

(Q4)′ If $S \in \mathscr{S}$ and $x \in P \backslash S$, then $x^\perp \cap S$ is either empty or a maximal clique in S.

Also, (R4) is easily shown to be equivalent to (R4)′:

(R4)′ If $S \in \mathscr{S}$ and $x \in P \backslash S$, then $x^\perp \cap S$ is either a singleton or a maximal clique in S.

We note that $(P, \mathscr{L})$ has diameter 2 if (Q4) holds and diameter at most 3 if (R4) holds.

Lemma 7.1. *Suppose $(P, \mathscr{L})$ satisfies (Q4). Let S, T be distinct symps on ∞. If $S \cap T \supset \{\infty\}$, then $S \cap T \in \mathscr{V}$.*

Proof. Consider the residue of ∞. Suppose $x \in S^\infty \cap T^\infty$. Take $y \in T^\infty \backslash x^\perp$. Notice that $y \notin S^\infty$ as $S^\infty \cap T^\infty$ is a clique. Since $L = y^\perp \cap S^\infty$ must be a line in $S^\infty \backslash \{x\}$, there is $z \in x^\perp \cap L \backslash \{x\}$. This implies $z \in x^\perp \cap y^\perp \subseteq T^\infty$, so that $xz \subseteq S^\infty \cap T^\infty$. Thus $S \cap T$ contains a plane, hence coincides with a plane, and we are done.

Theorem 7.2. *If $(P, \mathscr{L})$ is a connected Grassmann space all of whose max spaces have finite rank and in which (Q4) holds, then $(P, \mathscr{L})$ is either a polar space of rank 3 or isomorphic to $A_{a,2}(F)$ for some $a \geq 4$ and some division ring F.*

Proof. Suppose $M_1, M_2 \in \mathscr{M}$ have rank >2 and $M_1 \cap M_2 \in \mathscr{L}$. In order to apply Proposition 5.9, we verify that $L = M_1 \cap M_2$ is in at least three max spaces. The hypotheses on the ranks of M_1, M_2 imply the existence of points $x_1, x_2 \in M_1 \backslash M_2$, and $y_1, y_2 \in M_2 \backslash M_1$ such that $\mathrm{rk}(\langle x_1, x_2, L\rangle) = \mathrm{rk}(\langle y_1, y_2, L\rangle) = 3$. Clearly $x_i \notin y_i^\perp$. Consider $S_i = S(x_i, y_i)$ for $i = 1, 2$. As $M_1 \cap M_2 \in \mathscr{L}(S_1 \cap S_2)$, Lemma 7.1 yields that $S_1 \cap S_2 \in \mathscr{V}$. Thus, if $S_1 \cap S_2 \subseteq M_1$, then $S_1 \cap S_2 = S_1 \cap S_2 \cap M_1 = S_1 \cap M_1 = S_2 \cap M_2$ by consideration of ranks, so $\langle x_1, x_2, L\rangle \subseteq S_1 \cap S_2$ and $3 = \mathrm{rk}(\langle x_1, x_2, L\rangle) \leq \mathrm{rk}(S_1 \cap S_2) = 2$, a contradiction. Hence $S_1 \cap S_2 \nsubseteq M_1$. Similarly, one can prove $S_1 \cap S_2 \nsubseteq M_2$. Now $(S_1 \cap S_2)^\perp$ is a third max space on L, and we can finish by Proposition 5.9.

The only incidence systems among $A_{a,2}(F)$ for $a \geq 4$ and F a division ring in which (R4) holds for those for which $a=4$. Thanks to Proposition 5.9, we may therefore, and shall, restrict attention to the case where *any line is in precisely two max spaces, each of them of rank* >2.

LEMMA 7.3. *Let* x_1, x_2, x_3, x_4, x_5 *be a minimal* 5*-circuit* (i.e. $x_i^\perp \cap x_{i+2}x_{i+3} = \varnothing$ *for all i, indices taken modulo* 5). *If* $x_1^\perp \cap S(x_2, x_4) \in \mathscr{V}$, *then* $x_i^\perp \cap S(x_{i+1}, x_{i+3})$ *and* $x_i^\perp \cap S(x_{i-1}, x_{i-3})$ *are in* $\mathscr{V}$ *for all* $i(1 \leq i \leq 5)$.

PROOF. Notice that $x_1^\perp \cap S(x_2, x_4) \in \mathscr{V}$ iff $\{x_1, x_2, x_3, x_4\}^\perp \neq \varnothing$. Thus $x_4^\perp \cap S(x_1, x_3) \in \mathscr{V}$ follows. Also for $u \in \{x_1, x_2, x_3, x_4\}^\perp$, we have $ux_4 \subseteq x_3^\perp \cap S(x_1, x_4)$, so $x_3^\perp \cap S(x_1, x_4) \in \mathscr{V}$. Similarly $x_2^\perp \cap S(x_1, x_4) \in \mathscr{V}$. The argument is easily completed.

LEMMA 7.4. *Suppose* $(P, \mathscr{L})$ *satisfies* (R4). *If* $x_1, x_2, \ldots, x_5$ *is a minimal* 5*-circuit in* P, *then*

(a) $x_i^\perp \cap S(x_{i+1}, x_{i+3}) \in \mathscr{V}$ *for each* i $(1 \leq i \leq 5$; *indices taken modulo* 5).

(b) $\{x_1, x_2, \ldots, x_5\}^\perp = \varnothing$.

PROOF. (a) Suppose $x_1, x_2, \ldots, x_5$ is a minimal 5-circuit which is a counterexample to the statement. By Lemma 7.3, it is a circuit with $x_i^\perp \cap S(x_{i+1}, x_{i+3})$ a singleton for each i. Let M be a max space on x_3x_4 and take $M_1 \in \mathscr{M}$ on x_2x_3 with $M_1 \cap M = \{x_3\}$ and $M_2 \in \mathscr{M}$ on x_4x_5 with $M_2 \cap M \in \mathscr{L}$. Now $L_1 = x_1^\perp \cap M_1$, $L_2 = x_1^\perp \cap M_2$, $L_3 = x_3^\perp \cap M_2 = M \cap M_2$, $L_4 = x_4^\perp \cap M_1$ are lines on x_2, x_5, x_4, x_3, respectively.

Since $L_1 \cap L_4 \subseteq \{x_1, x_2, x_3. x_4\}^\perp$ and $L_2 \cap L_3 \subseteq \{x_1, x_5, x_4, x_3\}^\perp$ we have by the assumption that $L_1 \cap L_4 = L_2 \cap L_3 = \varnothing$. Take $u \in L_3 \backslash \{x_4\}$ and $v \in L_4 \backslash \{x_3\}$. Then $u \notin v^\perp$. For $u \in v^\perp$ would imply $L_3 \subseteq L_4^\perp$ and $\langle L_3, L_4 \rangle^\perp = M$ so that $M \cap M_1$ would contain the line L_4, which conflicts $M \cap M_1 = \{x_3\}$.

Consider $S = S(u, v)$. Notice that $V = x_5^\perp \cap S$ contains L_3 and must therefore be a plane in S. Similarly for $W = x_2^\perp \cap S$. Also, $x_1 \notin S$, for else $x_1^\perp \cap x_3x_4 \neq \varnothing$.

Now $x_1^\perp \cap S \neq \varnothing$ by (R4)$'$. As $x_1 \in x_2^\perp \cup x_5^\perp$, Lemma 3.5 implies that $x_1^\perp \cap x_2^\perp \cap S$ and $x_1^\perp \cap x_5^\perp \cap S$ are nonempty. If $z \in \{x_1, x_2, x_5\}^\perp \cap S$, then $z \in \{x_1, x_2, x_3, x_4, x_5\}^\perp \cap S$, as $x_5^\perp \cap S$ is a clique on x_4 and $x_2^\perp \cap S$ is a clique on x_3. So we may assume $\{x_1, x_2, x_5\}^\perp \cap S = \varnothing$. Thus $|x_1^\perp \cap S| \geq |x_1^\perp \cap x_2^\perp \cap S| + |x_1^\perp \cap x_5^\perp \cap S| \geq 2$, so that $x_1^\perp \cap S \in \mathscr{V}$. Write $U = x_1^\perp \cap S$. Since U, x_3, x_4 are in S, there is $w \in x_3^\perp \cap x_4^\perp \cap U$. But now wx_3 is a line in $x_4^\perp \cap S(x_1, x_3)$; this settles (a).

(b) Assume $u \in \{x_1, x_2, \ldots, x_5\}^\perp$. Put $L = x_1^\perp \cap (x_3x_4)^\perp$. Since $x_1^\perp \cap x_3x_4 = \varnothing$ by minimality of the circuit, $L \in \mathscr{L}$. Now $\langle x_1, x_5, u \rangle^\perp$, $\langle x_1, x_2, u \rangle^\perp \in \mathscr{M}_L$, so $\langle x_1, L \rangle \subseteq \langle x_1, x_5, u \rangle^\perp$ or $\langle x_1, L \rangle \subseteq \langle x_1, x_2, u \rangle^\perp$.

Without loss of generality, assume $\langle x_1, L \rangle \subseteq \langle x_1, x_5, u \rangle^\perp$. Then $\langle x_4, L \rangle \subseteq x_3^\perp \cap x_5^\perp$ conflicting ranks.

LEMMA 7.5. *Let* $(P, \mathscr{L})$ *satisfy* (R4). *If* S, T *are distinct symps, then* $S \cap T$ *is not a singleton.*

PROOF. Suppose S, T are symps such that $S \cap T = \{x\}$ for some $x \in P$. Take $z \in S \backslash x^\perp$. By axiom (R4)$'$, there is $y \in z^\perp \cap T$. Now $y \in T \backslash x^\perp$, for else $y \in x^\perp \cap z^\perp \subseteq S$, so $y \in S \cap T = \{x\}$ and $y = x$, which conflicts with $z \notin x^\perp$. Choose $v_1, v_2 \in x^\perp \cap y^\perp$ with $v_1 \notin v_2^\perp$, and take $u \in x^\perp \cap z^\perp$.

Let $i \in \{1, 2\}$. Now u, x, v_i, y, z is a 5-circuit with $u \notin y^\perp$ (for else $u \in x^\perp \cap y^\perp \subseteq T$), $x \notin z^\perp \cup y^\perp$ and $v_i \notin z^\perp$ (for otherwise $v_i \in x^\perp \cap z^\perp \subseteq S$). Hence, by Lemma 7.4, either $u \in v_i^\perp$ and $v_i^\perp \cap S \supseteq xu$, or $v_i^\perp \cap S \in \mathscr{V}$. At any rate, $v_i^\perp \cap S \in \mathscr{V}$ for each $i \in \{1, 2\}$. Put $V_i = v_i^\perp \cap S$

and consider $W = y^{\perp} \cap S$. As $z \in W\backslash(V_1 \cup V_2)$, we must have $W \in \mathscr{V}$ by Lemma 3.5. But then $x^{\perp} \cap W$ is a line (as both x, W are in S) contained in $x^{\perp} \cap y^{\perp}$, hence in T.

LEMMA 7.6. *Suppose* (R4) *holds for* $(P, \mathscr{L})$. *Then* $\mathrm{rk}(M) \leq 3$ *for any* $M \in \mathscr{M}$.

PROOF. Suppose M is a max space of rank ≥ 4. Pick $x \in M$ and $V, W \in \mathscr{V}(M)$ with $V \cap W = \{x\}$, and let S, T be symps on V, W respectively. Since $x \in S \cap T$, we know by Lemma 7.5 that there is a line L on x in $S \cap T$. Now $V \subseteq L^{\perp}$ would imply $L \subseteq V^{\perp} \cap T = W$; but also $L \subseteq V$, as $\langle V, L\rangle$ is a singular subspace of S, so that $L \subseteq \mathscr{V} \cap \mathscr{W} = \{x\}$ which is absurd. Hence there is $z \in L\backslash\{x\}$ with $z \notin V^{\perp}$. Since z, V are in S, we obtain that $L_1 = z^{\perp} \cap V$ is a line on x. Similarly, $L_2 = z^{\perp} \cap W$ is a line on x. But now $z \in L_1^{\perp} \cap L_2^{\perp} = \langle L_1, L_2\rangle^{\perp} = M$, so $L \subseteq M$ and $V \subseteq L^{\perp}$, which has just been excluded.

It follows that no max space of rank ≥ 4 exists.

THEOREM 7.7. *If* $(P, \mathscr{L})$ *is a connected Grassmann space in which* (R4) *holds, then* $(P, \mathscr{L})$ *is either a polar space of rank* 3 *or isomorphic to one of* $A_{4,2}(F)$, $A_{5,3}(F)$ *for some division ring* F.

PROOF. This is a direct consequence of the main theorem and Lemma 7.6.

ACKNOWLEDGEMENTS

The problem dealt with in this paper was raised in a discussion with W. M. Kantor. Both his and E. E. Shult's interest were a great support. Also, H. A. Wilbrink's contributions to the proof of a property on generalized quadrangles (Proposition 4.2) are gratefully acknowledged.

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Received 22 March 1982

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