On Almost Nilpotent-by-Abelian Lie Algebras

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ABSTRACT

We examine Lie algebras all of whose proper subalgebras are nilpotent-by-abelian but which themselves are not nilpotent-by-abelian. We study the existence and structure of these algebras.

Let $C$ be a class of Lie algebras other than abelian. A Lie algebra $L$ is said to be almost $C$ if all of the subalgebras of $L$ except $L$ itself belong to $C$. Note that some authors have used the terminology minimal non-$C$ rather than almost $C$. If $L$ is simple and every proper subalgebra is abelian, then we say $L$ is simple semiabelian. (We reserve the phrase "$L$ is almost abelian" for a different class of algebras.)

Simple semiabelian Lie algebras have been studied by Gein in [8] and [7], Farnsteiner in [5] and [6], Elduque in [2], and Varea in [18]. Almost nilpotent Lie algebras have been studied by Stitzinger in [12], Gein and Kuznecov in [9], Towers in [15], and Farnsteiner in [4]. Almost supersolvable Lie algebras have been studied by Towers in [17] and Elduque and Varea in [3]. Finally, almost solvable Lie algebras have been studied by Towers in [17].

Let $\mathcal{X}$ and $\mathcal{Y}$ be two classes of Lie algebras (not necessarily different). We say a Lie algebra $L$ is $\mathcal{X}$-by-$\mathcal{Y}$ if $L$ has an ideal $H \in \mathcal{X}$ such that $L/H \in \mathcal{Y}$. The aim of this paper is to study the almost nilpotent-by-abelian Lie algebras.

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Throughout this paper \( L \) will denote a finite-dimensional Lie algebra over a field \( F \). Vector space direct sums will be denoted by \( \oplus \). We denote the subspace (respectively subalgebra) of \( L \) generated by \( x_1, \ldots, x_n \in L \) by \( \bigoplus_{i=1}^{n} Fx_i \) (respectively by \( \langle x_1, \ldots, x_n \rangle \)). The Frattini subalgebra of \( L \), denoted by \( F(L) \), is the intersection of all the maximal subalgebras of \( L \), and the Frattini ideal of \( L \), denoted by \( \phi(L) \), is the largest ideal of \( L \) contained in the Frattini subalgebra of \( L \). If \( \phi(L) = 0 \), we say \( L \) is \( \phi \)-free.

If \( U \) and \( V \) are subsets of \( L \), we shall write \( C_U(V) \) for \( \{ x \in U : xu = 0 \text{ for all } u \in V \} \). The center of \( L \) will be denoted by \( Z(L) \). Let \( x \in L \); then the adjoint mapping of \( x \) will be denoted by \( \text{ad}_Lx \). The derived algebra of \( L \) will be denoted by \( L^2 \). We shall denote the finite field with \( q \) elements by \( F_q \). Standard results in the theory of Lie algebras are taken from [10].

It is easy to see the following

**Lemma 1.** Let \( L \) be any Lie algebra. Then \( L \) is nilpotent-by-abelian if and only if \( L^2 \) is nilpotent.

It follows from standard Lie theory that there are no solvable almost nilpotent-by-abelian Lie algebras over fields of characteristic zero. We can now easily show

**Theorem 2.** Let \( L \) be any Lie algebra over a field \( F \) of characteristic zero. Then the following are equivalent:

(i) \( L \) is almost nilpotent-by-abelian;
(ii) \( L \) is simple semiabelian or else \( L = \text{sl}_2(F) \).

**Proof.** Let \( L \) be almost nilpotent-by-abelian. By Levi's theorem and the above remarks we see that \( L \) must be simple and every proper subalgebra solvable. Consequently, by Theorem 2.2 of [17] and the remarks following it, we have that \( L \) is simple semiabelian or \( \text{sl}_2(F) \). It is easy to see that the converse is true. \( \square \)

We note that if \( F \) is algebraically closed of characteristic zero, then there are no simple semiabelian algebras, and if \( F \) is the real field, then there is only one: the three-dimensional nonsplit simple algebra (see [4]). For examples of simple semiabelian Lie algebras over more general fields see [11].

We now turn to the case of characteristic \( p > 0 \). This situation is quite different from the characteristic zero case owing to the existence of solvable almost nilpotent-by-abelian algebras. A great many results that hold for solvable algebras over fields of characteristic zero hold for solvable algebras
with nilpotent derived algebra over arbitrary fields. This makes solvable almost nilpotent-by-abelian algebras extremely useful in trying to extend such results by using them as minimal counterexamples. As we shall see, they form a very interesting class of algebras.

Obvious examples of solvable almost nilpotent-by-abelian algebras are solvable almost supersolvable algebras. In fact, there is a very close relationship between these two classes of algebras, and so we include the relevant structure of the latter type. This was given by Elduque and Varea in [3] as Theorem 1.2. Note that the notation has been altered slightly.

**Theorem 3.** Let $L$ be a solvable, $\phi$-free Lie algebra over an arbitrary field $F$ such that the derived algebra, $L^2$, is not nilpotent. Then $L$ is almost supersolvable if and only if $F$ has characteristic $p > 0$ and one of the following holds:

(i) $L = (\bigoplus_{i=0}^{p-1} Fe_i) \oplus Fs \oplus Fx$ with $e_i x = (\alpha + i)e_i$, where $\alpha$ is some fixed scalar in $F$, $e_is = e_{i+1}$ (indices modulo $p$), $sx = s$, and all other products zero, with the condition $F = \{t^p - t : t \in F\}$. Moreover, two such algebras, corresponding to scalars $\alpha$ and $\bar{\alpha}$, are isomorphic if and only if $\alpha - \bar{\alpha} \in F^p$.

(ii) $L = (\bigoplus_{i=0}^{p-1} Fe_i) \oplus Fc \oplus Fs \oplus Fx$ with $e_i c = e_i$, $e_is = e_{i+1}$ for $i = 0, \ldots, p-2$, $e_{p-1}s = 0$, $e_ix = ie_{i-1}$ for $i = 0, 1, \ldots, p-1$ and $e_{-1} = 0$, $sx = c$, and all other products zero. Also, $F$ is a perfect field whenever $p = 2$.

We now give the structure of certain solvable almost nilpotent-by-abelian algebras.

**Theorem 4.** Let $L$ be solvable and $\phi$-free. Then $L$ is almost nilpotent-by-abelian if and only if $F$ has characteristic $p > 0$ and $L = A \oplus B$ is a semidirect sum, where $A$ is the unique minimal ideal of $L$, $\dim A \geq 2$, $A^2 = 0$, and either $B = M \oplus Fx$ is a semidirect sum, where $M \subseteq B$ is a minimal ideal of $B$ such that $M^2 = 0$ (type I), or $B$ is the three-dimensional Heisenberg algebra (type II). Moreover, if $p \geq 3$ then $\dim A$ is divisible by $p$.

**Proof.** $\Rightarrow$: Suppose that $L$ is $\phi$-free and almost nilpotent-by-abelian. Then clearly we have $\text{char } F = p > 0$ by remarks following Lemma 1. Since $L$ is $\phi$-free, it follows from Theorem 7.3 of [14] that we have

$$L = (A_1 \oplus \cdots \oplus A_n) \oplus B,$$

where $A_i$ is a minimal abelian ideal of $L$ for $1 \leq i \leq n$ and $B$ is a subalgebra of $L$. In particular, each $A_i$ is an irreducible $B$-module and $B^2A_i \in \{A_i, 0\}$.
for $1 \leq i \leq n$. Note that $B^2$ is nilpotent and that

$$L^2 = \sum_{i=1}^{n} A_i B + B^2 \subset \sum_{i=1}^{n} A_i + B^2.$$  

Consequently, the assumption $B^2 A_i = 0$ for $1 \leq i \leq n$ implies that $L^2$ is nilpotent, a contradiction. We may thus assume that $A_1 B^2 = A_1$. This entails that $(A_1 + B)^2 = A_1 + B^2$ is not nilpotent. As a result $A_1 + B$ is not nilpotent-by-abelian and therefore coincides with $L$. Hence $L = A \oplus B$, and $A$ is the unique minimal ideal of $L$, because it is self-centralizing.

Now let $M$ be any maximal ideal of $B$ that contains $B^2$. Then $A + M$ is an ideal of $L$ strictly contained in $L$, and so $(A + M)^2$ is nilpotent. Now let $N(L)$ denote the nilradical of $L$. Then

$$M^2 \subseteq (A + M)^2 \subseteq N(L) = A$$

by Theorem 7.4 of [14], and so $M^2 = 0$. Since $B^2 \subseteq M$ and $B$ is solvable, $M$ has codimension one in $B$. Thus $B = M \oplus Fx$.

Let $C = C_M(x)$. Since $B^2 \neq 0$, $C$ is properly contained in $M$. Let $I \subset M$ be an ideal of $B$ that is properly contained in $M$. Then $A + I$ is an ideal of $L$, and so is $AI = A(A + I)$. According to Lemma 1.4 of [16] we have $C_L(A) = A$, implying $AI = A$ whenever $I \neq 0$.

Consider a proper ideal $0 \neq D \subset M$ of $B$. Then we have that $AD = A$ and that $Dx = DB \subset M$ is an ideal of $B$. Moreover, since $A + D + Fx$ is a proper subalgebra of $L$, $A + Dx = (A + D + Fx)^2$ is nilpotent. Consequently, $A(Dx) \neq A$, whence $Dx = 0$.

Hence $C$ is the unique largest ideal of $B$ strictly contained in $M$. Now either $C = 0$ or $C \neq 0$.

Suppose that $C \neq 0$. Let $c_1, \ldots, c_r$ be a basis for $C$ and let $y \in M \setminus C$. Put $s_1 = y$, $s_2 = yx$, $s_3 = (yx)x$, $s_4 = ((yx)x)x$, and so on. Now there exists an $n \geq 1$ such that $c_1, \ldots, c_r, s_1, \ldots, s_n$ are linearly independent but $c_1, \ldots, c_r, s_1, \ldots, s_n, s_{n+1}$ are linearly dependent. Write

$$s_{n+1} = s_n x = c + \sum_{i=1}^{n} \lambda_i s_i$$

where $c = \sum_{i=1}^{r} \mu_i c_i$. Now $\langle c, s_1, \ldots, s_n \rangle = Fc \oplus (\bigoplus_{i=1}^{n} Fs_i)$ is an ideal of $B$, by construction and because $Fc \oplus (\bigoplus_{i=1}^{n} Fs_i) \subseteq M$. If $c = 0$ then $\bigoplus_{i=1}^{n} Fs_i \subset M$. But $(\bigoplus_{i=1}^{n} Fs_i)x \neq 0$, since $s_1 x \neq 0$, which is a contradiction. Thus $c \neq 0$. Moreover $[Fc \oplus (\bigoplus_{i=1}^{n} Fs_i)]x \neq 0$ and so $Fc \oplus (\bigoplus_{i=1}^{n} Fs_i) = M$. Hence $r = 1$ and $C = Fc$.

We now claim that $\phi(B) = C$. First note that $\phi(B)$ is a nilpotent ideal of $B$ strictly contained in every maximal subalgebra of $B$, by Theorem 6.1.
of [14]. So in particular $\phi(B) \subseteq M$. Since the only ideals of $B$ strictly contained in $M$ are $C$ and $0$, we must have $\phi(B) \subseteq C$. Moreover, $C \subseteq Z(B)$ and $0 \neq B^2 \subseteq M$, so $C \subseteq B^2$. Now $Z(B) \cap B^2 \subseteq \phi(B)$ and so $C \subseteq \phi(B)$. Hence $\phi(B) = C$ as claimed.

Assume first that $n = 1$. If $\lambda_1 \neq 0$, then $F(c + \lambda_1 s_1)$ is an ideal of $B$ and
\[(c + \lambda_1 s_1)x = cx + \lambda_1 s_1 x = \lambda_1 (c + \lambda_1 s_1),\]
which belongs to $F(c + \lambda_1 s_1)$. But $Fc \neq F(c + \lambda_1 s_1) \subseteq M$, which is a contradiction. If $\lambda_1 = 0$, then $s_1 x = c$, and in this case $B$ is the three-dimensional Heisenberg algebra.

Assume now that $n \geq 2$. If $\lambda_1 \neq 0$ then
\[K = \left( \bigoplus_{i=2}^{n} F s_i \right) \oplus F(c + \lambda_1 s_1) \oplus Fx\]
is a maximal subalgebra of $B$. Thus $Fc \subseteq K$, since $Fc = \phi(B)$. But $Fc \subseteq K$ if and only if $\lambda_1 = 0$, which is a contradiction. If $\lambda_1 = 0$, then
\[s_{n+1} = s_{n} x = c + \lambda_2 s_2 + \cdots + \lambda_n s_n.\]
Now let $K = \left( \bigoplus_{i=2}^{n} F s_i \right) \oplus Fc \oplus Fx$. Then $K$ is a subalgebra of $B$. Moreover
\[KB = K(M + Fx) \subseteq K,\]
and so $K$ is an ideal of $B$. Thus $K^2$ is an ideal of $B$. Now $K^2 \subseteq M$, since $s_1 \notin K^2$; thus $K^2 x = 0$ and so $K^2 = Fc$ or $K^2 = 0$. But clearly $K^2 \neq 0$, since $c \in K^2$ and $c \neq 0$; thus $K^2 = Fc$. However, $A + K$ is an ideal of $L$, and so $AK$ is an ideal of $L$ contained in $A$. As $A$ is self-centralizing, we have $AK \neq 0$, whence $AK = A$. Now $A + K \subseteq L$, and so $(A + K)^2$ is nilpotent, and
\[(A + K)^2 = A + K^2 = A + Fc.\]
Also $Ac = A$ (by a similar argument to that above), and so we have
\[(A + Fc)^2 = Ac = A.\]
Thus $A + Fc$, and hence $A + K^2$, is not nilpotent, which is a contradiction.

If $C = 0$ then $M$ is a minimal ideal of $B$, since any ideal of $B$ properly contained in $M$ is contained in $C = 0$.

Also, since $A$ is self-centralizing, it is a faithful $B$-module and we have an embedding $D \hookrightarrow \text{End}_F(A)$. Thus, $(\dim A)^2 \geq \dim B \geq 2$, proving that $\dim A \geq 2$. 

Finally, suppose that \( p \geq 3 \). We have that \( A \) is a faithful irreducible module for the solvable algebra \( B \). Let \( \overline{F} \) be the algebraic closure of \( F \), and put \( \overline{B} = B \otimes_F \overline{F} \), \( \overline{A} = A \otimes_F \overline{F} \). Then \( \overline{B} \) is an \( \overline{A} \)-module and decomposes into weight spaces \( \overline{A} = \bigoplus_{\alpha} \overline{A}_\alpha(\overline{B}^2) \) relative to \( \overline{B}^2 \), since \( \overline{B}^2 \) is nilpotent. Moreover, each weight space is an \( \overline{B} \)-submodule, because \( \overline{B}^2 \) operates nilpotently on \( \overline{B} \). Now suppose that \( \overline{A}_0(\overline{B}^2) \neq 0 \). Then this implies that the Fitting null component of \( A \) relative to \( B^2 \), \( A_0(B^2) \), is nonzero. Thus, since \( A_0(B^2) \) is a \( B \)-submodule of \( A \), we have \( A = A_0(B^2) \). However, \( B^2 \) operates on \( A \) by nilpotent transformations. Thus, by the Engel-Jacobson theorem, \( B^2A = 0 \), a contradiction. Thus \( \overline{A}_0(\overline{B}^2) = 0 \).

Now let \( V \) be a \( \overline{B} \)-composition factor of \( \overline{A}_\alpha(\overline{B}^2) \). If \( \dim_F V < p \), then Lie's theorem holds, and \( \overline{B} \) operates on \( V \) by upper triangular matrices. As a result, \( \overline{B}^2 \) acts on \( V \) by nilpotent transformations. Since \( \alpha(x) \) is the only eigenvalue of \( x \in \overline{B}^2 \) on \( V \), we obtain \( \alpha = 0 \). It now follows from Corollary 8.5 on p. 239 of [13] that \( p \) divides \( \dim_F V \). Consequently, \( p \) divides \( \dim_F \overline{A} = \dim_F A \).

\( \Leftarrow \): For the converse it suffices to show that every maximal subalgebra of \( L \) is such that its derived algebra is nilpotent.

Let \( K \subset L \) be a maximal subalgebra. Then either \( K = B \) or \( K = A + N \) for a maximal subalgebra \( N \subset B \). In particular, \( L \) is \( \phi \)-free. If \( L \) is of type I, then, using the fact that \( M \) is ad \( x \)-irreducible and \( M^2 = 0 \), we see that \( N \) is abelian. For type II this follows directly. Consequently, \( K^2 = B^2 \) or \( K^2 \subseteq A \), implying that \( K \) is nilpotent-by-abelian. That completes the proof.

We shall refer to the algebras described in the above theorem as type I and type II respectively.

It is well known that over algebraically closed fields, if \( L \) is a Lie algebra such that \( L^2 \) is nilpotent, then \( L \) is supersolvable (see for example [1]). Thus we have the following

**Theorem 5.** Let \( L \) be a solvable almost nilpotent-by-abelian algebra over an algebraically closed field \( F \).

If \( L/\phi(L) \) is of type I then \( L/\phi(L) \) is of type I of Theorem 3.

If \( L/\phi(L) \) is of type II then \( L/\phi(L) \) is of type II of Theorem 3.

**Proof.** This follows easily from seeing that \( L^2 \) is nilpotent if and only if \( (L/\phi(L))^2 \) is nilpotent, by applying Theorem 2.5 of [1] and the above remarks.

This raises the question whether algebras of type I and type II are always almost supersolvable. This is not the case. Clearly we could always take
a basis and multiplication as given in Theorem 3 but choose a field which does not allow the algebra to be almost supersolvable. For type I take the field $\mathbb{F}_p$. Then it is easily seen by Fermat's little theorem that $t^p - t + 1$ has no roots in $\mathbb{F}_p$.

More interestingly, we can construct algebras of type I which have a different basis and multiplication from those of Theorem 3(i). We now give such an example.

Let $L = (\bigoplus_{i=0}^{2} \mathbb{F}_3e_i) \oplus (\bigoplus_{i=1}^{2} \mathbb{F}_3s_i) \oplus \mathbb{F}_3x$ over the field $\mathbb{F}_3$. Put

$$e_i s_1 = -s_1 e_i = e_{i+1} \quad \text{(indices modulo 3)} \quad \text{for} \quad 0 \leq i \leq 2,$$

$$e_0 s_2 = -s_2 e_0 = 2e_0 + 2e_1 + e_2, \quad e_1 s_2 = -s_2 e_1 = e_0 + 2e_1 + 2e_2,$$

$$e_2 s_2 = -s_2 e_2 = 2e_0 + e_1 + 2e_2,$$

$$e_0 x = -xe_0 = e_0, \quad e_1 x = -xe_1 = 2e_0 + e_2,$$

$$e_2 x = -xe_2 = 2e_0 + e_1 + 2e_2,$$

$$s_1 x = -xs_1 = s_2, \quad s_2 x = -xs_2 = 2s_1,$$

and all other products zero. Then $L$ is solvable $\phi$-free almost nilpotent-by-abelian but not almost supersolvable.

Notice that in classifying all the algebras of type I we want to find all the non-equivalent irreducible representations of the algebra $(\bigoplus_{i=1}^{n} \mathbb{F}_s) \oplus \mathbb{F}x$ with multiplication as described in Theorem 4. Recall the following result of Jacobson, which can be found as Theorem 2 on p. 205 of [10].

**Theorem 6.** Every finite-dimensional Lie algebra over a field of characteristic $p > 0$ has a faithful finite-dimensional representation which is not completely reducible and a faithful finite-dimensional completely reducible representation.

This immediately gives us

**Theorem 7.** Let $F$ be a field of characteristic $p > 0$. For each irreducible polynomial in $F[t]$ there exists an algebra of type I.

We now consider algebras of type II. If we choose $L$ to have the same basis and multiplication as in Theorem 3(ii), then $L$ will be almost nilpotent-by-abelian but not almost supersolvable precisely when our field $F$ has characteristic two and is nonperfect [for example, $\mathbb{F}_2(t)$, the field of rational expressions in the indeterminate $t$ over the field $\mathbb{F}_2$]. However, over fields other than characteristic two there are type II algebras which are not almost supersolvable, as the following example shows.
Let \( L = (\bigoplus_{i=0}^{2} \mathbb{F}_3 e_i) \oplus \mathbb{F}_3 s \oplus \mathbb{F}_3 c \oplus \mathbb{F}_3 x \) over the field \( \mathbb{F}_3 \), where

\[
\begin{align*}
e_i s &= -s e_i = e_{i+1} \quad \text{(indices modulo 3)} \quad \text{for} \quad 0 \leq i \leq 2, \\
e_i c &= -c e_i = e_i \quad \text{for} \quad 0 \leq i \leq 2, \\
e_0 x &= -x e_0 = e_0 + e_1 + e_2, \\
e_1 x &= -x e_1 = 2e_0 + e_1 + e_2, \\
e_2 x &= -x e_2 = e_0 + e_2,
\end{align*}
\]

and all other products zero. It can be checked that \( L \) is of type II. However, it is not almost supersolvable, since the minimum polynomial of \( \text{ad}_L x |_{(\bigoplus_{i=0}^{2} \mathbb{F}_3 e_i)} \) is \( t^3 + 1 \), which does not split over \( \mathbb{F}_3 \), and so \((\bigoplus_{i=0}^{2} \mathbb{F}_3 e_i) \oplus \mathbb{F}_3 x \) is a proper subalgebra of \( L \) which is not supersolvable.

In classifying all the algebras of type II we want to know all the irreducible representations of the three-dimensional Heisenberg algebra.

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