The forbidden subgraph characterization of directed vertex graphs

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Abstract

A graph is called a directed vertex (DV) graph if it is the intersection graph of a family of directed paths in a directed tree, i.e., a tree in which each edge is oriented, with one or more vertices of indegree zero. In this paper we present the forbidden subgraph characterization of DV graphs. © 1999 Elsevier Science B.V. All rights reserved

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1. Introduction

A class \( \mathcal{S} \) of graphs is said to admit a forbidden subgraph characterization if there exists a class \( \mathcal{F}_\mathcal{S} = \{ H \mid H \notin \mathcal{S} \text{ but } H - v \in \mathcal{S} \text{ for every } v \in V(H) \} \) such that \( G \in \mathcal{S} \) iff \( G \) does not contain any member of \( \mathcal{F}_\mathcal{S} \) as an induced subgraph. If \( \mathcal{S} \) admits forbidden subgraph characterization, then \( \mathcal{F}_\mathcal{S} \) is called the class of minimal forbidden subgraphs. It is well known that a class \( \mathcal{S} \) admits forbidden subgraph characterization iff \( \mathcal{S} \) is closed under vertex-induced subgraphs. For many classes of graphs, it follows from the definition that the class is closed under vertex-induced subgraphs; in such cases, there must be a forbidden subgraph characterization. However, finding the explicit forbidden subgraph characterization can be difficult. An old and well-known example of this phenomenon is the class of planar graphs.

Let \( F \) be a finite family of non-empty sets. An undirected graph \( G \) is an intersection graph for \( F \) if there is a one-to-one correspondence between the vertices of \( G \) and the sets in \( F \) such that two vertices in \( G \) are adjacent iff the corresponding sets have non-empty intersection. If \( F \) is a family of paths in an undirected tree \( T \), then \( G \) is called an undirected vertex (UV) or a path graph. If \( F \) is a family of directed paths in a directed tree \( T \), i.e., a tree in which each edge is oriented, then \( G \) is called a directed

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vertex (DV) or a directed path graph. Note that a directed tree may have more than one vertex of indegree zero. A rooted directed tree is a directed tree having exactly one vertex of indegree zero. If \( F \) is a family of directed paths in a rooted directed tree, then \( G \) is called a rooted directed vertex (RDV) graph.

A graph \( G \) is chordal if every cycle in \( G \) of length at least four has a chord, i.e. an edge joining two non-consecutive vertices of the cycle. In fact, Walter [18], Gavril [6], and Buneman [2] have shown that \( G \) is a chordal graph iff \( G \) is the intersection graph of a family of subtrees of a tree. So DV, RDV and UV graphs are subclasses of chordal graphs. A graph \( G \) is called perfect if the chromatic number equals the clique number of each of its induced subgraphs (see [1,9]). It is well known [9] that the class of chordal graphs is a subclass of perfect graphs. Polynomial recognition algorithms for DV, RDV, UV, and chordal graphs are known; whereas, the problem of designing a polynomial recognition algorithm for perfect graphs is still open (see [1,9]). Chordal graphs can be recognized in linear time (see [15,17]). Gavril [8] found the first polynomial time algorithm to recognize UV graphs. Schaffer [16] found a better algorithm, based on the work of Monma and Wei [11], to recognize UV graphs. Gavril [7] first reported a polynomial algorithm to recognize RDV graphs. This was improved to linear time by Deitz [3]. DV graphs can also be recognized in polynomial time (see [11]).

Since the classes of chordal graphs, DV graphs, RDV graphs, UV graphs and perfect graphs are closed under vertex-induced subgraphs, they admit a forbidden subgraph characterization. In fact, Berge’s strong perfect graph conjecture (see [1,9]) says that odd holes (odd cycle of length at least five) and odd antiholes (complement of odd holes) are the only forbidden subgraphs for perfect graphs, which is yet to be settled. Renz [13] posed the problem of finding the forbidden subgraph characterization of UV graphs. The problem of finding the forbidden subgraph characterization for DV graphs, RDV graphs, and UV graphs are open.

In this paper, we solve the problem of finding the forbidden subgraph characterization of DV graphs.

2. Definitions and preliminaries

Throughout the paper we use ‘iff’ for if and only if, ‘w.r.t.’ for with respect to, ‘s.t.’ for such that, and ‘wlg’ for without loss of generality. Throughout the discussion our graph is assumed to be connected.

For a graph \( G = (V,E) \), let \( N(v) = \{w \mid vw \in E(G)\} \) denote the set of neighbours of \( v \) and let \( N[v] = N(v) \cup \{v\} \). A subset \( C \) of \( V(G) \) is called a clique if the induced subgraph \( G[C] \) is a maximal complete subgraph of \( G \). Let \( C(G) \) be the set of all cliques of \( G \), and for each \( v \in V(G) \), \( C_v(G) \) denote the set of all cliques of \( G \) containing \( v \). A vertex \( v \) is a simplicial vertex of \( G \) if \( N[v] \) is a clique of \( G \). An ordering \( \alpha = (v_1, v_2, \ldots, v_n) \) is a perfect elimination ordering (PEO) of \( G \) if \( v_i \) is a simplicial vertex of \( G_i = G[\{v_i, v_{i+1}, \ldots, v_n\}] \), for all \( i, 1 \leq i \leq n \).
As DV graphs are subclasses of chordal graphs, we first present some known results about chordal graphs.

**Theorem 2.1** (Deitz [4], Lekkerkerker and Boland [10]). A chordal graph has a simplicial vertex. Moreover, if $G$ is non-complete, then $G$ has two non-adjacent simplicial vertices.

**Theorem 2.2** (Fulkerson and Gross [5], Rose [14]). $G$ is chordal iff $G$ has a PEO. Moreover, any simplicial vertex can be the starting vertex of some PEO of $G$.

Though in the definition of DV graphs the tree is arbitrary, there exists a tree satisfying a nice property, which is given in the following theorem:

**Theorem 2.3** (Monma and Wei [11], Clique Tree Theorem). A graph $G = (V,E)$ is a DV graph iff there exists a directed tree $T$ with vertex set $C(G)$ s.t. for each $v \in V(G)$, $T[C_v(G)]$ is a directed path in $T$.

A tree satisfying Theorem 2.3 is called a **clique tree** for the graph it characterizes. In Fig. 1, a DV graph $G$ and clique tree $T$ for $G$ are given.

Next, we present the characterization of DV graphs due to Monma and Wei [11]. To this end, we need to introduce some new concepts.

If $G - C$ is disconnected by a clique $C$ into components $H_i = (V_i, E_i)$, $1 \leq i \leq r$, then $C$ is said to be a **separating clique** and $G_i = G[V_i \cup C]$, $1 \leq i \leq r$, $r \geq 2$, is said to be a **separated graph of $G$ w.r.t. $C$**. Let $C$ be a separating clique of $G$. Cliques which intersect $C$ but not equal to $C$ are called **relevant cliques** w.r.t. $C$.

In the following definitions, only relevant cliques are considered.

Let $C_1$ and $C_2$ be two cliques of $G$. We say (1) $C_1$ and $C_2$ are **unattached**, denoted $C_1 \perp C_2$, if $C_1 \cap C \cap C_2 = \emptyset$; otherwise, they are **attached**. (2) $C_1$ dominates $C_2$,

$$
C = \{1, 2, 3, 4\} \\
C_1 = \{1, 3, 8\} \\
C_2 = \{3, 4, 5\} \\
C_3 = \{2, 4, 6\} \\
C_4 = \{1, 2, 7\}
$$

Fig. 1. A $DV$ graph $G$ and its clique tree $T$. 

[Diagram of DV graph G and clique tree T]
denoted $C_1 \geq C_2$, if $C_1 \cap C \supseteq C_2 \cap C$, (3) $C_1$ properly dominates $C_2$, denoted $C_1 > C_2$, if $C_1 \cap C \supseteq C_2 \cap C$, (4) $C_1$ and $C_2$ are congruent, denoted $C_1 \sim C_2$, if they are attached and $C_1 \cap C = C_2 \cap C$, and (5) $C_1$ and $C_2$ are antipodal, denoted $C_1 \Leftrightarrow C_2$, if they are attached and neither dominates the other.

Let $G_1$ and $G_2$ be two separated graphs of $G$ w.r.t. $C$. We say (1) $G_1$ and $G_2$ are unattached, denoted $G_1 \not\supseteq G_2$, if $C_1 \not\supseteq C_2$ for every clique $C_1$ in $G_1$ and for every clique $C_2$ in $G_2$; otherwise, they are attached, (2) $G_1$ dominates $G_2$, denoted $G_1 \supseteq G_2$, if they are attached and for every clique $C_1$ in $G_1$, $C_1 \supseteq C_2$ for all cliques $C_2$ in $G_2$ or $C_1 \not\supseteq C_2$ for all cliques $C_2$ in $G_2$, (3) $G_1$ properly dominates $G_2$, denoted $G_1 > G_2$, if $G_1 \supseteq G_2$ but not $G_2 \supseteq G_1$, (4) $G_1$ and $G_2$ are congruent, denoted $G_1 \sim G_2$, if $G_1$ dominates $G_2$ and $G_2$ dominates $G_1$; in this case, $C_1 \sim C_2$ for all $C_1$ in $G_1$ and for all $C_2$ in $G_2$, and (5) $G_1$ and $G_2$ are antipodal, denoted $G_1 \Leftrightarrow G_2$, if they are attached and neither dominates the other.

Consider the graph $G = (V, E)$ separated by the clique $C = \{a, b, c, d\}$ into $G_1$ through $G_5$ as shown in Fig. 2. We have $G_3 \Leftrightarrow G_i$, for $i = 1, 2, 4, 5$, $G_4 \sim G_5$, $G_2 > G_1$, and $G_i \not\supseteq G_j$, for $i = 1, 2$ and for $j = 4, 5$.

The relation 'congruent to' is an equivalence relation on $S_G$, the set of all separated graphs w.r.t. $C$. The equivalence classes are called congruence classes. For graph theoretic concepts not defined here and for a rich collection of results about various subclasses of chordal graphs, we refer to Golumbic [9].

For a separated subgraph $G_i$, let $W(G_i) = \{v \in C \mid$ there is a vertex $w \in (V(G_i) - C)$ s.t. $vw \in E(G_i)\}$. Let $G$ be a chordal graph, and $G_i$, $1 \leq i \leq r$, $r \geq 2$ be the separated graphs of $G$ w.r.t. some separating clique $C$ of $G$. Relevant cliques of $G_i$ which contain $W(G_i)$ are called principal cliques of $G_i$.

The existence of a principal clique of any separated graph of a chordal graph is assured by the following result due to Panda and Mohanty [12].

**Proposition 2.4** (Panda and Mohanty [12]). Every separated graph $G_i$ of a chordal graph $G$ has a principal clique.

The following result characterizes the antipodality of two separated graphs of a chordal graph.

**Lemma 2.5.** Two separated graphs $G_1$ and $G_2$ of a chordal graph $G$ are antipodal iff (1) $G_1 \Leftrightarrow G_2$, or (2) $G_1 > G_2'$, $G_2 > G_1'$, for some cliques $C_1$, $C_1'$ in $G_1$ and $C_2$, $C_2'$ in $G_2$. (The cliques $C_1$ and $C_2$ in condition (2) are principal cliques of $G_1$ and $G_2$, respectively.)

**Proof.** Sufficiency follows from the definition of antipodality of separated graphs.

**Necessity:** Assume that $G_1 \Leftrightarrow G_2$. So, $W(G_1) \cap W(G_2) \neq \emptyset$. Let $C_i$ be a principal clique of $G_i$, $i = 1, 2$.

**Case 1:** $W(G_1) = W(G_2)$.
Fig. 2. Example of the separation of $G$ into $G_1$, $G_2$, $G_3$, $G_4$ and $G_5$ by $C = \{a, b, c, d\}$.

Then each of $G_1$ and $G_2$ has at least two relevant cliques. Again, $G_i$, $1 \leq i \leq 2$ has a relevant non-principal clique; otherwise, one separated graph dominates the other. Let $C_i'$ be some relevant non-principal clique of $G_i$, $1 \leq i \leq 2$. Now $C_1$, $C_1'$, $C_2$, and $C_2'$ satisfy condition (2).

Case 2: $W(G_1)$ and $W(G_2)$ are comparable but $W(G_1) \neq W(G_2)$.

Wlg, $W(G_1) \subset W(G_2)$. So, $C_2 > C_1$. Since $G_1 \leftrightarrow G_2$, there exists a relevant clique $C_i'$ of $G_2$ s.t. $C_i'$ is attached to $C_1$ but does not dominate $C_1$; otherwise, $G_2 > G_1$. So, either $C_1 > C_2'$ or $C_1 \leftrightarrow C_2'$. If $C_1 \leftrightarrow C_2'$, then condition (1) holds. If $C_1 > C_2'$, then condition (2) holds.

Case 3: $W(G_1)$ and $W(G_2)$ are incomparable.

Then $C_1$ and $C_2$ are incomparable. Now, $C_1$ and $C_2$ are attached and neither dominates the other. So, $C_1 \leftrightarrow C_2$. So, condition (1) holds.

This completes the proof of the necessity. ☐
The proof of the following corollary follows from the proof of the necessity of Lemma 2.5.

**Corollary 2.6.** Let $G_1 \leftrightarrow G_2$ and $C_1$ and $C_2$ be some principal cliques of $G_1$ and $G_2$, respectively. Then the following are true.

1. If $W(G_1) = W(G_2)$, then there exists a relevant non-principal clique $C_i'$ of $G_i$, $i \leq i \leq 2$, s.t. $C_1 > C_2'$, and $C_2 > C_1'$.
2. If $W(G_1) \subseteq W(G_2)$, then there exists a relevant non-principal clique $C'_2$ of $G_2$ s.t. either $C_1 > C'_2$ or $C_1 \leftrightarrow C'_2$.
3. If $W(G_1)$ and $W(G_2)$ are incomparable, then $C_1 \leftrightarrow C_2$.

Our main result depends on the following characterization of DV graphs due to Monna and Wei [11].

**Theorem 2.7 (Monna and Wei [11], Separator Theorem).** Assume that $C$ separates $G = (V, E)$ into separated graphs $G_i = G[V_i \cup C], 1 \leq i \leq r, r \geq 2$. $G$ is a DV graph iff

1. each $G_i$ is DV, and
2. the $G_i$s can be two colored s.t. no antipodal pairs have the same color.

Let $H_i, 1 \leq i \leq r, r \geq 2$ be the separated graphs of a chordal graph $H$ w.r.t. a separating clique $C$ of $H$. Define the *antipodal graph* $\mathcal{A}(H, C)$ of $H$ as follows. $V(\mathcal{A}) = \{H_i\}_{i=1}^r$ and $E(\mathcal{A}) = \{H_iH_j s.t. H_i \leftrightarrow H_j\}$.

A separated graph $G_i$ is said to be a *strong separated graph* if there exists an induced odd cycle $\alpha = G_1, G_2, \ldots, G_j, \ldots, G_{2k+1}, k > 1$, in $\mathcal{A}(G, C)$ s.t. $G_i$ dominates $G_j$ for all $j$ except $j = i - 1$ and $j = i + 1$ (operations on the indices are under modulo $(2k + 1)$).

**Lemma 2.8.** Let $\alpha = G_1, G_2, \ldots, G_{2k+1}, k > 1$ be a chordless odd cycle of $\mathcal{A}(G, C)$, where $G$ is a chordal graph. Then the following conditions hold.

1. If $G_i$ dominates some $G_j$, then $G_i$ is a strong separated graph.
2. There exist at most two strong separated graphs in $\alpha$. Moreover, if there are two, then they appear consecutively in $\alpha$.

**Proof.** (i) Wlg, $i = 1$. Now $G_1 > G_j$. We claim that $G_1 > G_m, 3 \leq m \leq 2k$. We prove this by contradiction. Assume that $G_1$ does not dominate some $G_r$ seeking to establish a contradiction to the fact that $\alpha$ is a chordless cycle. Wlg, $3 \leq r \leq j - 1$; otherwise, one can reverse the cycle indices. Let $r_1, 3 \leq r_1 \leq j - 1$ be the largest index s.t. $G_1$ does not dominate $G_{r_1}$. Since $G_{r_1+1} \leftrightarrow G_{r_1}$ and $G_1$ dominates $G_{r_1+1}$, $G_1$ is attached to $G_{r_1}$. If $G_{r_1}$ dominates $G_1$, then $G_{r_1}$ also dominates $G_{r_1+1}$, as $G_1$ dominates $G_{r_1+1}$ (the relation ‘domination’ is a transitive relation on the set of separated graphs), contradicting the fact that $G_{r_1+1} \leftrightarrow G_{r_1}$. So $G_{r_1}$ does not dominate $G_1$. Since, by assumption $G_1$ does not dominate $G_{r_1}, G_1 \not\leftrightarrow G_{r_1}$. Hence $\alpha$ is not a chordless cycle, which is a contradiction. So Lemma 2.8(i) holds.
(ii) The proof is by contradiction. Assume that there are three strong separated graphs, say \( G_i, G_j, G_k \) seeking to establish a contradiction of the fact that \( \alpha \) is a chordless cycle. Since \( k > 1 \), wlg we assume that \( G_i \) is not antipodal to \( G_j \). So \( G_i \sim G_j \) and \( G_i \) and \( G_k \) do not occur consecutively in \( \alpha \). Then the separated graphs \( G_i, G_{i+1}, \ldots, G_{j-1} \) form a cycle. This contradicts the fact that \( \alpha \) is a chordless cycle. So there are at most two strong separated graphs. Again, if two such separated graphs exist, then they must occur consecutively in \( \alpha \); otherwise, using a similar analysis it can be shown that \( \alpha \) is not a chordless cycle.  

3. Forbidden subgraph characterization of DV graphs

In this section we present the forbidden subgraph characterization of DV graphs. Let \( \mathcal{C} \) be the class of DV graphs. A forbidden subgraph for DV-graphs is said to be a critical non-DV graph. Let \( G \in \mathcal{C} \). Then \( G \) has at least three cliques. If \( G \) is not chordal, then it must be isomorphic to \( C_n, n \geq 4 \). So assume that \( G \) is a chordal graph. \( G \) has a separating clique as it has more than two cliques. Let \( C \) separate \( G \) into \( G_i \), \( i < r, r > 2 \).

The following lemma gives the structure of \( \mathcal{A}(G, C) \).

**Lemma 3.1.** \( \mathcal{A}(G, C) \) is isomorphic to \( C_{2k+1} \) for some \( k \geq 1 \).

**Proof.** Since \( G \in \mathcal{C} \), each \( G_i \) is a DV graph. So by Theorem 2.7, the \( G_i \)'s cannot be 2-coloured in such a way that antipodal pairs receive different colors. So \( \mathcal{A}(G, C) \) is not bipartite and hence contains an odd cycle. Since a graph containing an odd cycle also contains an induced odd cycle, \( \mathcal{A}(G, C) \) contains an induced odd cycle, say \( C_{2k+1} = G_1, G_2, \ldots, G_{2k+1} \), for some \( k \geq 1 \). Now the separated graphs \( G_1, G_2, \ldots, G_{2k+1} \) violate the condition of Theorem 2.7. Since \( G \in \mathcal{C} \), \( \mathcal{A}(G, C) \) is isomorphic to \( C_{2k+1} \).  

Next, we classify the odd cycle of \( \mathcal{A}(G, C) \) into three types and tackle each type separately. To this end, the indices are modulo \( 2k + 1 \).

Let \( \alpha = G_1, G_2, \ldots, G_{2k+1}, k \geq 1 \), be the induced odd cycle of \( \mathcal{A}(G, C) \). \( \alpha \) is said to be a comparable cycle if there exists \( i \) s.t. \( W(G_i) = W(G_{i+1}) \). \( \alpha \) is said to be a semicomparable cycle if it is not a comparable cycle and there exists \( i \) s.t. either \( W(G_{i+1}) \) or \( W(G_{i-1}) \) is properly contained in \( W(G_i) \). If \( \alpha \) is neither, it is said to be a noncomparable cycle.

We present below a series of lemmas which will be used in the main result.

**Lemma 3.2.** Let \( G \) be a chordal graph having an antipodal pair \( (G_1, G_2) \) w.r.t. some separating clique \( C \). If \( W(G_1) \) is a proper subset of \( W(G_2) \) and \( G_2 \) is a DV-graph, then \( G \) contains a subgraph isomorphic to one of the graphs in Fig. 3.
Proof. Assume that \( G_1 \iff G_2 \), and \( W(G_1) \subset W(G_2) \). So by Corollary 2.6(2), there exists a relevant non-principal clique \( C_2' \) of \( G_2 \) s.t. either \( C_1 \iff C_2' \) or \( C_1 > C_2' \), where \( C_1 \) is a principal clique of \( G_1 \).

**Case 1:** \( C_1 > C_2' \).

Since \( G_2 \) is a DV graph, there exists a clique tree \( T \) for \( G_2 \). Let \( T^* \) be the tree obtained from \( T \) by ignoring the direction. So, \( T^* \) is a clique tree for the chordal graph \( G_2 \). Since \( C_2 \cap C \cap C'_2 \neq \emptyset \), there exists a path, say, \( P = C, C_2, C_3, \ldots, C_n, C'_2 \) from \( C \) to \( C'_2 \) in \( T^* \). Wlg, assume that \( C_i \geq C_1, 2 \leq i \leq n \); otherwise, we can take \( C'_2 = C_i \), where \( i \) is the smallest index s.t. \( 2 < i \leq n \) and \( C_1 > C_i \). By Theorem 2.3, \( C_{i+1} \cap C \subseteq C_i \cap C, 2 \leq i \leq n - 1 \), as \( P \) is a path in \( T^* \). As \( |C_1 \cap C| \geq 2 \) and \( C_2 > C_1 \),
\[ |C| \geq 4. \] Let \( G' = G'[\{C, C_2, \ldots, C_{n+1}\}] \), where \( C_{n+1} = C_2 \). Then \( G' \) is a chordal graph. So by Theorem 2.1, \( G' \) contains at least two simplicial vertices. Since \( C \) and \( C_{n+1} \) are the only end vertices of the clique tree \( Q \) of \( G' \), there exist simplicial vertices \( z \) and \( v_1 \) s.t. \( z \in C_2' - C_n \), and \( v_1 \in C - C_2 \). Let \( x \in C_2' \cap C_1 \). Take \( y \in (C \cap C_1') - C_2' \). Then \( \{x, y\} \subseteq W(G_1) \cap W(G_2) \), \( xz \in E(G) \) but \( yz \notin E(G) \). Let \( v_2 \in W(G_2) - W(G_1) \) and \( x_1 \in C_1 - C \). Since \( C_1 > C_2' \), \( zv_2 \notin E(G) \). Let \( P = z, v_n, v_{n-1}, \ldots, v_2 \) be a shortest \( z - v_2 \) path in \( G_2 - (C - v_2) \). Then \( G'[\{v_1, v_2, \ldots, v_n, x, y, x_1, z\}] \) is isomorphic to \( H'_1 \), as \((v_1, v_2, \ldots, v_n, x, y, x_1, z)(\cdots)(v_1, v_2, \ldots, v_n, a, b, c, d) \) is an isomorphism.

**Case 2:** \( C_1 \iff C_2' \).

Then, let \( x_1 \in C_2' \cap C_1 \cap C, x_2 \in (C_1 \cap C) - C_2', x_3 \in (C_2' \cap C) - C_2, x_4 \in C - C_2, x_5 \in C_1 - C, x_6 \in (C_2' \cap C_2) - C, \) and \( x_7 \in C_2' - C_2 \). Now, \( G'[\{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}] \) is isomorphic to \( H'_2 \), as \((x_1, x_2, x_3, x_4, x_5, x_6, x_7)(\cdots)(a, b, e, d, c, f, g) \) is an isomorphism. □

A separating clique \( C \) of \( G \) is said to be a **maximal separating clique** if \( G \) has maximum number of separated graphs w.r.t. \( C \).

**Lemma 3.3.** Let \( \mathcal{A}(G, C) \) be isomorphic to an odd cycle \( \alpha = G_1, G_2, G_3, \) and \( C \) be a maximal separating clique. If \( W(G_1) \subseteq W(G_2) \), then \( W(G_2) \) and \( W(G_3) \) are incomparable.

**Proof.** The proof is by contradiction. Assume that \( W(G_2) \) and \( W(G_3) \) are comparable seeking a contradiction of the fact that \( C \) is a maximal separating clique.

First assume that \( W(G_3) \subseteq W(G_2) \). Since \( G_3 \not\subseteq G_2 \), \( G_2 \) has a non-principal clique \( C_2' \), otherwise, \( G_2 \geq G_3 \). Let \( C_i \) be a principal clique of \( G_i, 1 \leq i \leq 3 \). Let \( x_1 \in C_1 - C, x_2 \in C_2' - C_2, x_3 \in C - C_2, \) and \( x_4 \in C_1 - C \). Now, \( x_1, x_2, x_3, \) and \( x_4 \) lie in four different components of \( G - C_2 \). This contradicts the fact that \( C \) is a maximal separating clique. So \( W(G_3) \) is not a subset of \( W(G_2) \).

Similarly, if \( W(G_2) \subseteq W(G_3) \), then \( G - C_3 \) will have four components, where \( C_3 \) is any principal clique of \( G_3 \). This contradicts the fact that \( C \) is a maximal separating clique of \( G \). □

**Lemma 3.4.** Let \( \mathcal{A}(G, C') \) be isomorphic to a semicomparable odd cycle of length three for every maximal separating clique \( C' \) of \( G \). Then there exists a separating clique \( C \) of \( G \) s.t. \( \mathcal{A}(G, C) \) is isomorphic to a semicomparable odd cycle \( \alpha = G_1, G_2, G_3 \) s.t. \( W(G_1) \subseteq W(G_2) \), and \( C_1 \iff C_3 \), where \( C_i \) is a principal clique of \( G_i \) for \( i = 1, 3 \).

**Proof.** Choose a maximal separating clique \( C \) s.t. \( \mathcal{A}(G, C) \) is isomorphic to a semicomparable odd cycle \( \alpha = G_1, G_2, G_3 \) s.t. \( W(G_1) \subseteq W(G_2) \) and \( G_3 \) has minimum number of cliques. Let \( C_1 \) and \( C_3 \) be some principal cliques of \( G_1 \) and \( G_3 \), respectively. We will prove that \( C_1 \iff C_3 \). The proof is by contradiction. Since \( W(G_1) \subseteq W(G_2) \), by Lemma 3.3, \( W(G_3) \) is not a subset of \( W(G_1) \). So assume that \( W(G_1) \subseteq W(G_3) \)
seeking a contradiction to the choice of C. Since $G_1 \gg G_3$ and $W(G_1) \subset W(G_3)$, by Corollary 2.6(2), there exists a relevant non-principal clique $C'_1$ of $G_3$ s.t. either $C_1 \gg C'_1$ or $C_1 > C'_1$. Now, $C_2$ is a maximal separating clique of $G$. Let $x_1 \in C_1 - C$, $x_2 \in C_2 - C$, and $x_3 \in C'_1 - C_3$. Let $G'_1, G'_2, G'_3$ be the three separated graphs of $G$ w.r.t. $C_3$ s.t. $x_1 \in W(G'_1), x_2 \in W(G'_2)$, and $x_3 \in W(G'_3)$. Now, $\alpha' = G'_1, G'_2, G'_3$ is a semicomparable odd cycle s.t. $W(G'_1) \subset W(G'_2)$ and $G'_1$ has fewer cliques than $G_3$. This contradicts the choice of $C$. Hence $W(G_1)$ is not a subset of $W(G_3)$.

Since $G_1 \gg G_3$ and $W(G_1)$ and $W(G_3)$ are not comparable, by Corollary 2.6(3), $C_1 \gg C_3$. □

**Lemma 3.5.** Let $G$ be a critical non-DV graph and $\mathcal{A}(G, C)$ be isomorphic to a chordless odd cycle $\alpha = G_1, G_2, \ldots, G_{2k+1}$ for some $k \geq 1$. If $W(G_1) = W(G_2)$, then each of $G_1$ and $G_2$ has exactly two relevant cliques.

**Proof.** The proof is by contradiction. Assume w.l.g. that $G_1$ has more than two relevant cliques seeking a contradiction to the fact that $G$ is a critical non-DV graph. Let $C_i$ be a principal clique of $G_i$, $i = 1, 2$. Since $G_1 \gg G_2$, and $W(G_1) = W(G_2)$, by Corollary 2.6(1), there exists a non-principal clique $C'_i$ of $G_i$, $i = 1, 2$ s.t. $C_1 \gg C'_1$, and $C_2 \gg C'_2$. Now $G_1$ is a DV graph. Let $T_1$ be a clique tree for $G_1$. Let $P = C, C^*_1(1), C^*_2(1), \ldots, C^*_r(1), C'_1$ be the path from $C$ to $C'_1$ in $T_1$. Such a path exists in $T_1$, as $C \cap C^*_r(1) \cap C'_1 \neq \emptyset$. W.l.g., $C^*_1(1)$ is a principal clique of $G_1$, for $1 \leq j \leq r$; otherwise, we can take $C'_1 = C^*_1(1)$, where $i$ is the smallest index s.t. $C^*_i(1)$ is not a principal clique of $G_1$. Let $G'_1 = G[\{C \cup C^*_1(1) \cup C'_1\}]$. Then clearly $G'_1 \gg G_2$ and $\alpha' = G'_1, G_2, \ldots, G_{2k+1}$ is an odd cycle of $G' = G'_1 \cup G_2 \cup \cdots \cup G_{2k+1}$. So $G'$ is not a DV graph as $G'_1, G_2, \ldots, G_{2k+1}$ violate the condition (2) of Theorem 2.7. Since $G'$ is a proper subgraph of $G$, this is a contradiction to the fact that $G$ is a critical non-DV graph. □

**Lemma 3.6.** Let $C$ be a maximal separating clique of a critical non-DV graph $G$ s.t. $\mathcal{A}(G, C)$ is isomorphic to the cycle $\alpha = G_1, G_2, \ldots, G_3$. If $W(G_1) = W(G_2)$, then no clique of $G_1$ is antipodal to any clique of $G_2$.

**Proof.** Let $C_i$ be a principal clique of $G_i$, $i = 1, 2$. Since $G_1 \gg G_2$, and $W(G_1) = W(G_2)$, by Corollary 2.6(1), there exists a non-principal clique $C'_i$ of $G_i$, $i = 1, 2$ s.t. $C_1 > C'_1$, and $C_2 > C'_2$. Now by Lemma 3.5, $C_i$ and $C'_i$ are the only relevant cliques of $G_i$ for $i = 1, 2$. Since $W(G_1) \subseteq W(G_2)$, by Lemma 3.3, $W(G_3)$ and $W(G_2)$ are incomparable.

Let $C_3$ be a maximal clique of $G_3$. Let $x_1 \in C'_1 - C_3$, $x_2 \in (C'_1 \cap C_1) - C$, $y_1 \in C'_2 - C_3$, $y_2 \in (C'_2 \cap C_2) - C$, $z_1 \in C_3 - C$, and $z_2 \in W(G_1) - W(G_2)$.

We prove that $C'_1$ is not antipodal to $C'_2$. The proof is by contradiction. Assume that $C'_1 \gg C'_2$, seeking a contradiction to the fact that $G$ is a critical non-DV graph.

**Case 1:** $C'_1 \gg C'_2$, $i = 1, 2$.

Then, let $z_3 \in (C'_1 \cap C'_2) - C$, $z_4 \in W(G_2) \cap W(G_3)$, $z_5 \in (C'_1 \cap C) - C'_2$, and $z_6 \in (C'_2 \cap C) - C'_1$. Now, $G[\{x_1, x_2, y_1, y_2, z_1, z_2, z_3, z_4\}]$ is isomorphic to $A_2$ of Fig. 5 as $(x_1, x_2, y_1, y_2, z_1, z_2, z_3, z_4) \leftrightarrow (h, c, e, d, g, f, a, b)$ is an isomorphism. Since $\{z_5, z_6\} \subseteq V(G) - V(A_2)$,
$A_2$ is a subgraph of $G$. Again, $A_2$ is a non-DV graph because the separated graphs of $A_2$ w.r.t. $C = \{a, b, f\}$ violate the condition (2) of Theorem 2.7. So, we get a contradiction.

Case 2: Exactly one of $C_1'$ and $C_2'$ is attached to $C_3$.

Wlg, $C_1'$ is attached to $C_3$. Let $z_3 \in (C_1' \cap C_2' \cap C)$, $z_4 \in (C_1' \cap C_3)$, and $z_5 \in (C_2' \cap C) - C_1'$. Now, $G[\{x_1, x_2, y_1, y_2, z_1, z_2, z_3, z_4, z_5\}]$ is isomorphic to $A_2$ of Fig. 5 as $(x_1, x_2, y_1, y_2, z_1, z_2, z_4, z_5) \leftrightarrow (e, d, g, f, h, c, a, b)$ is an isomorphism. Since, $z_3 \in V(G) - V(A_2)$, $A_2$ is a proper induced subgraph of $G$. Again, as in case 1, $A_2$ is a non-DV graph. So, we get a contradiction.

Case 3: At least one of $C_1'$ and $C_2'$ is attached to $C_3$.

Subcase 3.1: $C_1' \cap C_3$ and $C_2' \cap C_3$ are comparable.

Wlg, $C_1' \cap C_3 \subseteq C_2' \cap C_3$ and $C_1' \cap C_3 \neq \emptyset$. Let $z_3 \in (C_1' \cap C_2' \cap C)$, $z_4 \in (C_1' \cap C) - C_2'$ and $z_5 \in (C_2' \cap C) - C_1'$. Then, $z_4 \in C_3$.

If $z_5 \in C_3$, then $G[\{x_1, x_2, y_1, y_2, z_1, z_2, z_3, z_4, z_5\}]$ is isomorphic to $A_2$ of Fig. 5 as $(x_1, x_2, y_1, y_2, z_1, z_2, z_4, z_5) \leftrightarrow (g, f, e, d, h, c, b, a)$ is an isomorphism. As in case 1, $A_2$ is a non-DV graph. Since $z_3 \in V(G) - V(A_2)$, $A_2$ is a proper induced subgraph of $G$. So, we have a contradiction.

If $z_5 \notin C_3$, then $G[\{x_1, y_1, z_1, z_2, z_3, z_4, z_5\}]$ is isomorphic to the graph $A_5$ of Fig. 5 as $(x_1, y_1, z_1, z_2, z_3, z_4, z_5) \leftrightarrow (e, g, c, a, b, d)$ is an isomorphism. Since $x_2 \in V(G) - V(A_5)$, $A_5$ is a proper induced subgraph of $G$. Again, $A_5$ is a non-DV graph because the separated graphs of $A_5$ w.r.t. $C = \{a, b, c\}$ violate condition (2) of Theorem 2.7. So, we get a contradiction.

Subcase 3.2: $C_1' \cap C_3$ and $C_2' \cap C_3$ are incomparable.

Let $z_4 \in (C_1' \cap C_3) - C_2'$ and $z_5 \in (C_2' \cap C_3) - C_1'$. If $(C_1' \cap C_2' \cap C)$ is not a subset of $(C_3 \cap C)$, then let $z_6 \in (C_1' \cap C_2' \cap C) - C_3$. Now, $G[\{x_1, y_1, z_1, z_2, z_3, z_4, z_5, z_6\}]$ is isomorphic to $A_7$ as $(x_1, y_1, z_1, z_2, z_3, z_4, z_5, z_6) \leftrightarrow (f, e, d, c, b, a)$ is an isomorphism. Since $x_2 \in V(G) - V(A_7)$, $A_7$ is a proper induced subgraph of $G$. Again, $A_7$ is a non-DV graph because the separated graphs of $A_7$ w.r.t. $C = \{a, b, c\}$ violate condition (2) of Theorem 2.7. So, we get a contradiction.

If $(C_1' \cap C_2' \cap C) \subseteq (C_3 \cap C)$, then let $z_7 \in (C_2' \cap C) - C_3$. Then, $z_7$ belongs to at most one of $C_1'$ and $C_2'$.

Assume that $z_7$ belongs to exactly one of $C_1'$ and $C_2'$. Wlg, $z_7 \in C_2'$. Now, $G[\{x_1, x_2, y_1, y_2, z_1, z_2, z_3, z_4, z_5\}]$ is isomorphic to $A_2$ of Fig. 3 as $(x_1, x_2, y_1, y_2, z_1, z_2, z_4, z_7) \leftrightarrow (e, d, g, f, h, c, a, b)$ is an isomorphism. Since $z_5 \in V(G) - V(A_2)$, $A_2$ is a proper induced subgraph of $G$. Again, as in case 1, $A_2$ is a non-DV graph. So, we have a contradiction.

If $z_7 \notin C_1' \cup C_2'$, then $G[\{x_1, x_2, y_1, y_2, z_1, z_2, z_4, z_5\}]$ is isomorphic to $A_6(n = 3)$ as $(x_1, x_2, y_1, y_2, z_1, z_4, z_5, z_7) \leftrightarrow (a_0, v_1, a_1, v_1, a_2, a, b, v_2)$ is an isomorphism. Since $z_2 \in V(G) - V(A_6(n = 3))$, $A_6(n = 3)$ is a proper induced subgraph of $G$. Again, $A_6(n = 3)$ is non-DV graph because the separated graphs of $A_6(n = 3)$ w.r.t. $C = \{a, b, v_1, v_2\}$ violate condition (2) of Theorem 2.7. So, we get a contradiction.

So, our lemma is proved by contradiction. \qed
$V(A_1(k > 1)) = \{v_1, v_2, \ldots, v_{2k+1}, u_1, u_2, \ldots, u_{2k+1}, x_1\}$, $k > 1$ s.t. \{v_1, v_2, \ldots, v_{2k+1}\} is a $K_{2k+1}$, $u_i$ is joined to $v_{i-1}$ and $v_i$, $2 \leq i \leq 2k + 1$, $u_1$ is joined to each of $v_1, v_2, \ldots, v_{2k}, x_1$, and $x_1$ is joined to $u_1$ and $v_1$.

$V(A_2(k > 1)) = \{v_1, v_2, \ldots, v_{2k+1}, u_1, u_2, \ldots, u_{2k+1}\}$ s.t. 
\{v_1, v_2, \ldots, v_{2k+1}\} is a clique of $A_1(k > 1)$ and \{u_1, u_2, \ldots, u_{2k+1}\} is an independent set of $A_2(k > 1)$ and $u_i$ is adjacent to only $v_i$ and $v_{i+1}$, for $1 \leq i \leq 2k - 1$, and $u_{2k+1}$ is adjacent to $v_{2k+1}$ and $v_1$.

$V(A_3(k > 1)) = \{v_1, v_2, \ldots, v_{2k+1}, u_1, u_2, \ldots, u_{2k+1}\}$ $k > 1$ s.t. 
\{v_1, v_2, \ldots, v_{2k+1}\} is a $K_{2k+1}$, $u_i$ is joined to $v_{i-1}$ and $v_i$, $2 \leq i \leq 2k$, $u_i$ is joined to each of $v_i$, $2 \leq i \leq 2k$, and $u_{2k+1}$ is joined to each of $v_1, \ldots, v_{2k-1}$.

The graphs $A_1^{(2)}, A_2^{(2)}, A_3^{(2)}$, and $A_4^{(2)}$ are given in Fig. 4.

The following theorem characterizes DV graphs in terms of forbidden subgraphs.

**Theorem 3.7.** $G$ is a DV graph iff it does not contain any of the graphs in Fig. 5 and any of the graphs $A_1(k > 1)$ to $A_4(k > 1)$ as an induced subgraph.

**Proof.** *Necessity:* The proof is by contrapositive, i.e., if $G$ contains any of the graphs mentioned in Theorem 3.7 as an induced subgraph, then we will show that $G$ is not a DV graph.

The separated graphs $G_1, G_2,$ and $G_3$ of $A_1$ w.r.t. the clique $C = \{a, b, c\}$ are pairwise antipodal. So these separated graphs cannot be two-colored in such a way that no antipodal pair receives the same color. So by Theorem 2.7, $A_1$ is not a DV graph. Again, it is straightforward to check using Theorem 2.7 recursively that every separated graphs of $A_1 - x$ w.r.t. every separating clique of $A_1 - x$ is a DV graph for every $x \in V(A_1)$ and the separated graphs of $A_1 - x$ can be two-colored in such a way that no antipodal pair receive the same color. So, by Theorem 2.7, $A_1 - x$ is a DV graph for every $x \in V(A_1)$. Hence, $A_1$ is critical non-DV graph. Using a similar analysis, it is a routine exercise to show that each of the graphs in Fig. 5 and each of the graphs $A_1(k > 1)$ to $A_4(k > 1)$ is a critical non-DV graph. Since the class of DV graphs is closed under vertex-induced subgraphs, no graph containing any of the graphs in Fig. 5 or any of the graph $A_1(k > 1)$ to $A_4(k > 1)$ as an induced subgraph is a DV graph.

* Sufficiency: The proof is by contrapositive, i.e, if $G$ is not a DV graph, then we will show that $G$ contains a subgraph isomorphic to one of the graphs mentioned in Theorem 3.7.

Assume that $G$ is not a DV graph. Wlg, assume that $G$ is a critical non-DV graph. If $G$ is not chordal, then it must be isomorphic to $C_n$, $n \geq 4$, which is $A_{15}(n \geq 4)$ of Fig. 5. So, assume that $G$ is a chordal graph. Let $C$ be a maximal separating clique
of $G$. Since $G \in \mathcal{F}_6$, by Lemma 3.1, $\mathcal{A}(G, C)$ is isomorphic to an induced odd cycle $\alpha = G_1, G_2, \ldots, G_{2k+1}$.

Case 1: $k = 1$, i.e., $\alpha = G_1, G_2, G_3$.

Case 1.1: $\alpha$ is a comparable cycle.

W.l.o.g., $W(G_1) = W(G_2)$. Let $C_i$ be a principal clique of $G_i$, $1 \leq i \leq 3$. Since $G_1 \cong G_2$ and $W(G_1) = W(G_2)$, by Corollary 2.6(1), there exists a relevant non-principal clique $C_i'$ in $G_i$, $i = 1, 2$. Now by Lemma 3.5, $C_i$ and $C_i'$ are the only relevant cliques of $G_i$, $i = 1, 2$. Since $W(G_1) = W(G_2)$, by Lemma 3.3, $W(G_2)$ and $W(G_3)$ are non-comparable. So, $C_2 \ncong C_3$, and $C_1 \ncong C_3$. Since $G$ is a critical non-DV graph and
Fig. 5. Forbidden subgraphs for $DV$ graphs.
If $C'_1 \cap C_3 = 0 = C'_2 \cap C_3$, then there exist $z_1, z_2, z_3, z_4$, and $z_5$ s.t. $z_1 \in C'_1 \cap C_1 \cap C$, $z_2 \in C'_2 \cap C_2 \cap C$, $z_3 \in C'_1 \cap C_1 \cap C_3$, $z_4 \in (C_3 \cap C_2) - C_1$, and $z_5 \in C_3 - C$. Now $G'[\{x_1, x_2, y_1, y_2, z_1, z_2, z_3, z_4, z_5\}]$ is isomorphic to $A_3$, as $(x_1, x_2, y_1, y_2, z_1, z_2, z_3, z_4, z_5) \leftrightarrow (f, e, g, b, d, c, a, h, i)$ is an isomorphism.

If $C'_1 \cap C_3 \neq \emptyset$ but $C'_2 \cap C_3 = \emptyset$, then there exist $z_1, z_2, z_3$, and $z_4$ s.t. $z_1 \in C_3 \cap C'_1$, $z_2 \in C'_2 \cap C_2 \cap C$, $z_3 \in (C_3 \cap C) - C_2$, and $z_4 \in C_3 - C$. Then $G'[\{x_1, x_2, y_1, y_2, z_1, z_2, z_3, z_4\}]$ is isomorphic to $A_2$ as $(x_1, x_2, y_1, y_2, z_1, z_2, z_3, z_4) \leftrightarrow (e, d, g, f, a, b, c, h)$ is an isomorphism.

Similarly, if $C'_1 \cap C_3 \neq \emptyset$ but $C'_2 \cap C_3 \neq \emptyset$, then $G$ will be isomorphic to $A_2$.

Case 1.1.2: $C'_1 \cap C'_2 \neq \emptyset$.

If $C'_1 \cap C_3 = 0$ and $C'_2 \cap C_3 = 0$, then there exist $z_1, z_2, z_3, z_4$, and $z_5$ s.t. $z_1 \in C'_1 \cap C_1 \cap C$, $z_2 \in (C_1 \cap C_2) - (C'_1 \cap C'_2)$, $z_3 \in (C_3 \cap C) - C_2$, and $z_4 \in C_3 - C$. Now, $G'[\{x_1, x_2, y_1, y_2, z_1, z_2, z_3, z_4\}]$ is isomorphic to $A_2$ as $(x_1, x_2, y_1, y_2, z_1, z_2, z_3, z_4) \leftrightarrow (h, c, e, d, a, b, f, g)$ is an isomorphism.

If $C'_1 \cap C_3 \neq \emptyset$ and $C'_2 \cap C_3 = 0$, then $|C'_1 \cap C| \geq 2$ as $C'_1 \cap C'_2 \neq \emptyset$. So, there exist $z_1, z_2, z_3, z_4$, and $z_5$ s.t. $z_1 \in C'_1 \cap C_2 \cap C$, $z_2 \in C'_1 \cap C_1 \cap C_3$, $z_3 \in (C_3 \cap C) - (C'_1 \cup C'_2 \cup C_1)$, $z_4 \in (C_1 \cap C) - C_2$, and $z_5 \in C_3 - C$. Since $C'_1 \cap C'_2 \neq \emptyset$, $C'_1 \cap C_3 \neq \emptyset$, $C'_2 \cap C'_2 = \emptyset$, and $C'_1$ is not antipodal to $C'_2$, $C'_1 > C'_2$. So, $(C_2 \cap C) - (C'_1 \cup C'_2 \cup C_3) \neq \emptyset$. So, the existence of $z_3$ is assured. Let $G' = G'[\{x_1, x_2, y_1, y_2, z_1, z_2, z_3, z_4\}]$. Now, $G' - x_2$ is isomorphic to $A_3(3)$ as $(x_1, x_2, y_1, y_2, z_1, z_2, z_3, z_4) \leftrightarrow (a_2, a_3, v_3, a, b, v_1, a_1)$ is an isomorphism. This contradicts the fact that $G$ is a critical non-DV graph as a proper induced subgraph of $G$ is isomorphic to $A_3(3)$, which is not a DV graph. So the condition $C'_1 \cap C_3 \neq \emptyset$ and $C'_1 \cap C'_2 = \emptyset$ does not arise.

Similarly, the condition $C'_2 \cap C_3 \neq \emptyset$ and $C'_3 \cap C'_1 = \emptyset$ does not exist.

So, finally assume that $C'_1 \cap C_3 \neq \emptyset$ and $C'_3 \cap C'_1 = \emptyset$.

If $C'_1 \cap C'_2 \cap C_3 = \emptyset$, then there exist $z_1, z_2, z_3, z_4, z_5$ s.t. $z_1 \in C'_1 \cap C'_2 \cap C$, $z_2 \in (C'_1 \cap C'_2 \cap C) - C'_2$, $z_3 \in (C'_2 \cap C'_3 \cap C) - C'_2$, $z_4 \in (C_3 \cap C) - C_2$, and $z_5 \in C_3 - C$. Let $G' = G'[\{x_1, y_1, z_1, z_2, z_3, z_4\}]$ is isomorphic to $A_7$ as $(x_1, y_1, z_1, z_2, z_3, z_4) \leftrightarrow (e, f, a, b, c, d)$ is an isomorphism. Since $G'$ is a proper induced subgraph of $G$, $(C'_1 \cap C'_2 \cap C_3) = \emptyset$ is not possible.

If $(C'_1 \cap C'_2 \cap C_3) \neq \emptyset$, then there exist $z_1, z_2, z_3$, and $z_4$ s.t. $z_1 \in C'_1 \cap C'_2 \cap C_3$, $z_2 \in (C_2 \cap C) - (C'_1 \cup C'_2 \cup C_1)$, $z_3 \in (C_3 \cap C) - C_2$, and $z_4 \in C_3 - C$. The existence of $z_2$ is assured from the fact that $C'_1$ is not antipodal to $C'_2$ and $C_2 \not\subset C_3$. Now, $G' = G'[\{x_1, x_2, y_1, y_2, z_1, z_2, z_3, z_4\}]$ is isomorphic to $A_1$ as $(x_1, x_2, y_1, y_2, z_1, z_2, z_3, z_4) \leftrightarrow (d, c, f, e, a, b, h, g)$ is an isomorphism.

Case 1.2: $x$ is a semicomparable cycle.

Wlg, $W(G_2) \subset W(G_1)$. Wlg, assume that the separating clique $C$ satisfies Lemma 3.4. So, $C_1 \not\subset C_2$, where $C_i$ is a principal clique of $G_i$, $i = 1, 2$. Now $G_2 \not\subset G_3$, $W(G_2) \subset$
$W(G_3)$, and $G_3$ is a DV graph. So by Lemma 3.2, $G_2 \cup G_3$ contains a subgraph isomorphic to either $H'_1$ ($n \geq 3$) or $H'_2$. Since $G$ is a critical non-DV graph, $G_2 \cup G_3$ is isomorphic to either $H'_1$ ($n \geq 3$) or $H'_2$.

First, assume that $G_2 \cup G_3$ is isomorphic to $H'_1$ ($n \geq 3$). Now $C_1 \not\subset C_2$. If every relevant clique of $G_3$ is attached to every relevant clique of $G_1$, then $G$ will be isomorphic to $A_5$ ($n \geq 3$); otherwise, $G$ will be isomorphic to $A_6$ ($n \geq 3$).

Next assume that $G_2 \cup G_3$ is isomorphic to $H'_2$. Let $C_3$ be the non-principal clique of $G_3$. If $C_3 \subset C_1$, then $G$ will be isomorphic to $A_4$. If $C_3$ is attached to $C_1$, then $C_1 \not\subset C_3$; otherwise, $C_3 \not\subset C_1$. So, $G[(C \cup C_3 \cup C_1 \cup C_2)]$ will be a non-DV graph contradicting the fact that $G$ is a critical non-DV graph. Now, $G$ will be isomorphic to $A_{10}$.

Case 1.3: $\alpha$ is a non-comparable cycle.

If $W(G_1) \cap W(G_2) \cap W(G_3) = \emptyset$, then there exist $x, y, z, x_1, x_2,$ and $x_3$ s.t. $x \in W(G_1) \cap W(G_2), y \in W(G_2) \cap W(G_3), z \in W(G_3) \cap W(G_1)$, and $x_i \in C_i - W(G_i)$, where $C_i$ is a principal clique of $G_i$, $1 \leq i \leq 3$. Then $G[\{x, y, z, x_1, x_2, x_3\}]$ is isomorphic to $A_7$ as $(x, y, z, x_1, x_2, x_3) \leftrightarrow (a, b, c, d, e, f, g)$ is an isomorphism.

If $W(G_1) \cap W(G_2) \cap W(G_3) \neq \emptyset$, then let $x \in W(G_1) \cap W(G_2) \cap W(G_3)$. Again $C' = C - x$ is a separating clique of $G' = G - x$. Let $G'_1 = G_1 - x$. Then $G'_1, G'_2, \text{ and } G'_3$ are the only separated graphs of $G'$ w.r.t. $C'$. Note that $G'_1 \Leftrightarrow G'_2 \text{ if } G'_2 \text{ is attached to } G'_1$. If $G'_1, G'_2$ and $G'_3$ are pair wise unattached, then $\leq i \leq 3$. Then $G[\{x, x_1, x_2, x_3, y_1, y_2, y_3\}]$ is isomorphic to $A_8$ as $(x, x_1, x_2, x_3, y_1, y_2, y_3) \leftrightarrow (a, b, c, d, e, f, g, h)$ is an isomorphism.

Next assume that $G'_1 \Leftrightarrow G'_2$, $G'_3 \Leftrightarrow G'_1$, and $G'_3 \not\subset G'_2$. Then there exist $x_1, x_2, x_3, x_4, y_1, y_2,$ and $y_3$ s.t. $x_1 \in W(G'_1) \cap W(G'_2), x_2 \in W(G'_2) \cap W(G'_3), x_3 \in (C'_1 \cap C'_2) - C'_2, x_4 \in (C'_1 \cap C'_2) - C'_2,$ and $y_i \in C'_i - C'$, where $C'_i$ is a principal clique of $G'_i$, $1 \leq i \leq 3$. Then $G[\{x, x_1, x_2, x_3, x_4, y_1, y_2, y_3\}]$ is isomorphic to $A_9$ as $(x, x_1, x_2, x_3, x_4, y_1, y_2, y_3) \leftrightarrow (e, a, b, c, d, f, g, h)$ is an isomorphism.

Case 2: $k > 1$.

Case 2.1: $\alpha$ is a comparable cycle.

Wlg, $W(G_1) = W(G_2)$. Now $W(G_2) \cap W(G_3) \neq \emptyset$. So $G_1$ is attached to $G_3$. Since $k > 1$, $G_1 \not\subset G_3$. So by Theorem 2.8(i), $G_1$ is a strong separated graph. So $W(G_i) \subseteq W(G_1)$ for all $i$, $2 \leq i \leq 2k + 1$. Since $G_1 \Leftrightarrow G_{2k+1}$, and $W(G_{2k+1}) \subset W(G_1)$, by Corollary 2.6(2), $G_1$ has a relevant non-principal clique. Let $C_1$ be a principal clique of $G_1$. Then clearly, $C_1$ is a separating clique of $G$, and $\mathcal{A}(G, C_1)$ has more vertices than
that of $\mathcal{A}(G, C)$. This contradicts the choice of $C$. Hence, $\alpha$ cannot be a comparable cycle.

Case 2.2: $\alpha$ is a semicomparable cycle.

Wlg, $W(G_2) \subset W(G_1)$. We claim that $G_1$ dominates $G_j$, $3 \leq j \leq 2k$. Now $G_3 \Rightarrow G_2$. As $k > 1$, $W(G_3)$ and $W(G_1)$ are comparable. If $W(G_1) \subseteq W(G_3)$, then since $G_2 \Rightarrow G_3$, $G_1 \Rightarrow G_3$. Since $k > 1$, it contradicts the fact that $\alpha$ is chordless. So $G_1$ dominates $G_3$. Hence, by Lemma 2.8(i), $G_1$ dominates $G_j$, $3 \leq j \leq 2k$. Again by Lemma 2.8(i), $W(G_i)$ is incomparable with $W(G_{i+1})$, $2 \leq i \leq 2k - 1$. Let $C_i$ be a principal clique of $G_i$, $1 \leq i \leq 2k + 1$. Now $G_1 \Rightarrow G_2$, and $W(G_2) \subset W(G_1)$. So, by Corollary 2.6(2), there exists $C_i'$ in $G_1$ s.t. either $C_2 > C_i'$ or $C_2 \subset C_i'$. Note that $W(G_{2k+1})$ is not a subset of $W(G_1)$; otherwise, $C_1$ will be a separating clique of $G$ s.t. $\mathcal{A}(G, C_1)$ will have more vertices than that of $\mathcal{A}(G, C)$ contradicting the choice of $C$. So $W(G_1)$ and $W(G_{2k+1})$ are incomparable.

Assume that $G_{2k+1}$ dominates some $G_i$, $i \not\in \{1, 2k\}$. Then $G_{2k+1}$ is a strong separated graph. Let $G' = G - (C - W(G_1))$. Then $C_i$ is a separating clique of $G'$ and $\mathcal{A}(G', C_i)$ has an induced odd cycle, namely $G'_1, G'_2, \ldots, G'_{2k+1}$, where $G'_i$ is the separated graph of $G'$ w.r.t. $C_i$ containing $C_i$, $2 \leq i \leq 2k + 1$. So $G'$ is not a DV graph, which is contrary to the fact that $G \in \mathcal{F}_k$. So $G_{2k+1}$ does not dominate any $G_i$, $1 \leq i \leq 2k$. So $W(G_{2k})$ is not a subset of $W(G_{2k+1})$. Hence $C_i \Rightarrow C_i'$, $2 \leq i \leq 2k$. Let $x_i \in (C_i \cap C_{i+1})$, $y_i \in C_i - C$, $2 \leq i \leq 2k$, $y_{2k+1} \in C_{2k+1} - C$, $x_i \in (C_{2k} \cap C_{i+1} \cap C)$, $x_{2k+1} \in (C_{2k+1} \cap C)$, $y_i \in (C_{2k} \cap C_{i+1}) - C$, and $y_{2k+1} \in C_{2k+1} - C$. Then $G_i \{x_1, x_2, \ldots, x_{2k+1}, y_1, y_2, \ldots, y_{2k+1}, y_{2k+1}\}$ is isomorphic to $A_{13}$ as $(v_1, v_2, \ldots, v_{2k+1}, v_{2k+1}, u_1, u_2, \ldots, u_{2k+1}, u_{2k+1})$ is an isomorphism.

Case 3: $\alpha$ is a non-comparable cycle.

Case 3.1: $G_i \Rightarrow G_j$ iff $i \neq j - 1$, $1 \leq i < j \leq 2k + 1$.

Let $C_i$ be a principal clique of $G_i$, $1 \leq i \leq 2k + 1$. Let $x_{i+1} \in C_i \cap C_{i+1}$, $y_{i+1} \in C_i - C$, $1 \leq i \leq 2k$, $x_i \in C_{2k+1} \cap C$, and $y_{2k+1} \in C_{2k+1} - C$. Then $G_i \{x_1, x_2, \ldots, x_{2k+1}, y_1, y_2, \ldots, y_{2k+1}\}$ is isomorphic to $A_1$ as $(v_1, v_2, \ldots, v_{2k+1}, u_1, u_2, \ldots, u_{2k+1}, u_{2k+1})$ is an isomorphism.

Case 3.2: There exists exactly one $G_i$ which dominates some other separated graphs.

Now by Lemma 2.8(i), $G_i$ is a strong separated graph. Wlg, $i = 1$. Let $x_i \in C_i - C$, $1 \leq i \leq 2k + 1$. Let $y_{i+1} \in C_i \cap C_{i+1}$, $2 \leq i \leq 2k$, $y_1 \in (C_{2k+1} \cap C) - C_1$, and $y_2 \in (C_2 \cap C) - C_1$. Since $\alpha$ is non-comparable, the existence of $y_i$, $1 \leq i \leq 2k + 1$, is assured. Again $y_i \in W(G_{i+1})$, $3 \leq i \leq 2k + 1$. As $C_2 \mid C_{2k+1}$, $y_1 \notin W(G_2)$ and $y_2 \notin W(G_1)$. Now $G_i \{x_1, x_2, \ldots, x_{2k+1}, y_1, y_2, \ldots, y_{2k+1}\}$ is isomorphic to $A_{13}$ as $(v_1, v_2, \ldots, v_{2k+1}, u_1, u_2, \ldots, u_{2k+1}, v_{2k+1})$ is an isomorphism.

Case 3.3: There exist more than one $G_i$ which dominates some other separated graphs.

Let $G_i$ and $G_j$ dominate some other separated graphs. Then by Lemma 2.8(ii), $G_i$ and $G_j$ are the only strong separated graphs and $G_i$ and $G_j$ occur consecutively in $\alpha$. Wlg, $i = 1$ and $j = 2k + 1$. Let $y_i \in C_i - C$, $1 \leq i \leq 2k + 1$. Let $x_i \in C_i \cap C_{i+1}$, $2 \leq i \leq 2k - 1$, $x_1 \in (C_2 \cap C) - C_1$, and $x_{2k} \in (C_{2k} \cap C) - C_{2k+1}$. Then $G_i \{x_1, x_2, \ldots, x_{2k}, y_1, y_2, \ldots, y_{2k+1}, y_{2k+1}\}$ is isomorphic to $A_{13}$ as $(v_1, v_2, \ldots, v_{2k+1}, u_1, u_2, \ldots, u_{2k+1}, v_{2k+1})$ is an isomorphism.
isomorphic to $A_{14}$ ($k > 1$) as $(x_1, x_2, \ldots, x_{2k}, y_1, y_2, \ldots, y_{2k+1}) \rightarrow (v_1, v_2, \ldots, v_{2k}, u_1, u_2, \ldots, u_{2k-1})$ is an isomorphism.  

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