Journal of Algebra 321 (2009) 2068–2084

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Algebraic structures on Grothendieck groups of a tower of algebras ☆

Nantel Bergeron, Huilan Li *

Department of Mathematics and Statistics, York University, Toronto, Ontario M3J 1P3, Canada

ARTICLE INFO

Article history:
Received 24 February 2007
Available online 4 February 2009
Communicated by Jean-Yves Thibon

Keywords:
Graded algebra
Hopf algebra
Grothendieck group
Representation

ABSTRACT

The Grothendieck group of the tower of symmetric group algebras has a self-dual graded Hopf algebra structure. Inspired by this, we introduce by way of axioms, a general notion of a tower of algebras and study two Grothendieck groups on this tower linked by a natural paring. Using representation theory, we show that our axioms give a structure of graded Hopf algebras on each Grothendieck groups and these structures are dual to each other. We give some examples to indicate why these axioms are necessary. We also give auxiliary results that are helpful to verify the axioms. We conclude with some remarks on generalized towers of algebras leading to a structure of generalized bialgebras (in the sense of Loday) on their Grothendieck groups.

1. Introduction

In 1977, L. Geissinger realized *Sym* (symmetric functions in countably many variables) as a self-dual graded Hopf algebra [6]. Using the work of Frobenius and Schur [21], this can be interpreted as the self-dual Grothendieck Hopf algebra of the tower of symmetric group algebras $\bigoplus_{n\geqslant 0}\mathbb{C}\mathfrak{S}_n$. Since then, we have encountered many instances of combinatorial Hopf algebras. In each instance, we study a pair of dual Hopf algebras, and find that this duality can be interpreted as the duality of the Grothendieck groups of an appropriate tower of algebras. For example, C. Malvenuto and C. Reutenauer established the duality between the Hopf algebra of *NSym* (noncommutative symmetric functions) and the Hopf algebra of *QSym* (quasi-symmetric functions) [14]. Later, D. Krob and J.-Y. Thibon showed that this

☆ This work is supported in part by CRC and NSERC.
* Corresponding author.
E-mail addresses: bergeron@mathstat.yorku.ca (N. Bergeron), lihuilan@mathstat.yorku.ca (H. Li).
URL: http://www.math.yorku.ca/bergeron (N. Bergeron).

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2008.12.005

duality can be interpreted as the duality of the Grothendieck groups associated with $\bigoplus_{n\geqslant 0} H_n(0)$ the tower of Hecke algebras at $q=0$ [10]. More recently, it was shown that if one uses $\bigoplus_{n\geqslant 0} \mathrm{HCl}_n(0)$ the tower of Hecke–Clifford algebras at $q=0$, then one gets a similar interpretation for the duality between the *Peak* algebra and its dual [2]. In [20] Sergeev constructed semi-simple super algebras Se_n ($n\geqslant 0$) and a characteristic map from the super modules of $\mathrm{Se}=\bigoplus_n \mathrm{Se}_n$ to Schur's Q-functions $\Gamma=\mathbb{C}[p_1,p_3,\ldots]\subseteq \mathit{Sym}$. The space Γ is a self-dual graded Hopf subalgebra of *Sym*. In [7] the tower of 0-Ariki–Koike–Shoji algebras $\bigoplus_{n\geqslant 0} H_{n,r}(0)$ is shown to be related to the Mantaci–Reutenauer descent algebras [16], and their duals, a generalization of quasi-symmetric functions, are introduced by Poirier [19]. More examples are found in [9].

Our present goal is to describe a general setting that includes all the examples above. We study the relationship between some graded algebras A and the algebraic structure on their Grothendieck groups $G_0(A)$ and $K_0(A)$. More precisely, $A=(\bigoplus_{n\geqslant 0} A_n,\rho)$ is a graded algebra where each homogeneous component A_n is itself an algebra (with a different product). We will call A a tower of algebras if it satisfies some axioms given in Section 3.1. This list of axioms implies that their Grothendieck groups are graded Hopf algebras. Moreover, our axioms allow us to define a paring and to show that the corresponding Grothendieck groups are graded dual to each other. We also discuss how to weaken our axioms and still get similar results. This is core of our paper and is found in Section 3.

In Section 5 we discuss how our axioms may be adapted to verify that the Grothendieck groups $G_0(A)$ and $K_0(A)$ have a structure of generalized bialgebra in the sense of Loday [12]. This leads to the notion of generalized towers of algebras. In Section 2 we recall some definitions and propositions about bialgebras and Grothendieck groups. In Section 4 we give some examples. We also give some general results that are helpful to check the axioms.

2. Notations and propositions

We give brief review of the theory of bialgebras [6] and Grothendieck groups [4] which will be useful for later discussion.

Definition 2.1. Let K be a commutative ring. A *K-algebra B* is a K-module with *multiplication* $\pi: B\otimes_K B\to B$ and *unit map* $\mu: K\to B$ satisfying the *associativity* and the *unitary property*. We denote this algebra by the triple (B,π,μ).

A *K-coalgebra C* is a K-module with *comultiplication* $\Delta: C\to C\otimes C$ and *counit map* $\epsilon: C\to R$ satisfying *coassociativity* and *counitary property*. We denote this coalgebra by the triple (C,Δ,ϵ).

If a K-module B is simultaneously an algebra and a coalgebra, it is called a *bialgebra* provided these structures are compatible in the sense that the comultiplication and counit are algebra homomorphisms. We denote this bialgebra by the 5-tuple $(B,\pi,\mu,\Delta,\epsilon)$.

A K-linear map $\gamma: H\to H$ on a bialgebra H is an *antipode* if for all h in H, $\sum h_i\gamma(h_i')=\epsilon(h)1_H=\sum\gamma(h_i)h_i'$ when $\Delta h=\sum h_i\otimes h_i'$. A *Hopf algebra* is a bialgebra with an antipode.

Definition 2.2. An algebra B is a *graded algebra* if there is a direct sum decomposition $B=\bigoplus_{i\geqslant 0} B_i$ such that $\pi(B_p\otimes B_q)\subseteq B_{p+q}$, and $\mu(K)\subseteq B_0$.

A coalgebra C is a *graded coalgebra* if there is a direct sum decomposition $C=\bigoplus_{i\geqslant 0} C_i$ such that $\Delta(C_n)\subseteq\bigoplus(C_k\otimes C_{n-k})$ and $\epsilon(C_n)=0$ if $n\geqslant 1$.

A bialgebra $H=\bigoplus_{n\geqslant 0} H_n$ over K is called *graded connected* if $H_0=K1_H$. It is well known that a graded connected bialgebra is a Hopf algebra [15].

For $H=\bigoplus_{n\geqslant 0} H_n$ a graded bialgebra, its *graded dual* $H^{*\mathrm{gr}}=\bigoplus_{n\geqslant 0} H_n^*$ is also a graded bialgebra if all H_n are finitely generated and $H_i^*\otimes H_k^*\cong(H_i\otimes H_k)^*$ for all i and k.

We now recall the definition of Grothendieck groups. Let B be an arbitrary algebra. Denote

$$_B\mathrm{mod} = \text{the category of all finitely generated left } B\text{-modules},$$

$$\mathcal{P}(B) = \text{the category of all finitely generated projective left } B\text{-modules}.$$

Definition 2.3. Let $\mathcal{C}$ be one of the above categories. Let $\mathbf{F}$ be the free abelian group generated by the symbol (M), one for each isomorphism class of modules M in $\mathcal{C}$. Let $\mathbf{F_0}$ be the subgroup of $\mathbf{F}$ generated by all expressions $(M)–(L)–(N)$ arising from all short exact sequences

$$0 \to L \to M \to N \to 0$$

in $\mathcal{C}$. The *Grothendieck group* $\mathcal{K}_0(\mathcal{C})$ of the category $\mathcal{C}$ is defined by the quotient $\mathbf{F}/\mathbf{F_0}$. For $M \in \mathcal{C}$, we denote by $[M]$ its image in $\mathcal{K}_0(\mathcal{C})$. We then set

$$G_0(B) = \mathcal{K}_0({}_B\text{mod}) \quad \text{and} \quad K_0(B) = \mathcal{K}_0\big(\mathcal{P}(B)\big).$$

Now let B be a finite-dimensional algebra over a field K. Let $\{V_1, \ldots, V_s\}$ be a complete list of nonisomorphic simple B-modules. Then their projective covers $\{P_1, \ldots, P_s\}$ are a complete list of nonisomorphic indecomposable projective B-modules [1]. With these lists, we have

Proposition 2.4.

$$G_0(B) = \bigoplus_{i=1}^{s} \mathbb{Z}[V_i] \quad \textit{and} \quad K_0(B) = \bigoplus_{i=1}^{s} \mathbb{Z}[P_i].$$

Let A be an algebra and $B \subseteq A$ a subalgebra, or more generally let $\varphi : B \to A$ be an injection of algebra preserving unities. Let M be a (left) A-module and N a (left) B-module. Then the *induction* of N from B to A is $\text{Ind}_B^A N = A \otimes_B N$ a (left) A-module and the *restriction* of M from A to B is $\text{Res}_B^A M = \text{Hom}_A(A, M)$ a (left) B-module. In the case of $\varphi : B \to A$, the expression $A \otimes_B N$ is the tensor $A \otimes N$ modulo the relations $a \otimes bn \equiv a\varphi(b) \otimes n$, and the left B-action on $\text{Hom}_A(A, M)$ is defined by $bf(a) = f(a\varphi(b))$, for $f \in \text{Hom}_A(A, M)$ and $b \in B$.

3. Grothendieck groups of a tower of algebras

We now present our axiomatic definition of a *tower* of algebras. The starting ingredient is a graded algebra $A = (\bigoplus_{n \geqslant 0} A_n, \rho)$, such that each homogeneous component is itself a finite-dimensional algebra. For all $n, m \geqslant 0$, we require the maps $\rho_{n,m}$ obtained from the products ρ restricted to $A_n \otimes A_m$ to be injective homomorphisms of algebras (preserving unities). Our axioms will allow us to define a notion of induction and restriction on the Grothendieck groups $G_0(A) = \bigoplus_{n \geqslant 0} G_0(A_n)$ and $K_0(A) = \bigoplus_{n \geqslant 0} K_0(A_n)$. This will be the basic construction to put a structure of graded dual Hopf algebras on $G_0(A)$ and $K_0(A)$.

3.1. Tower of algebras (preserving unities)

Let $A = (\bigoplus_{n \geqslant 0} A_n, \rho)$ be a graded algebra. We call it *a tower of algebras* over field $K = \mathbb{C}$ if the following conditions are satisfied:

(1) A_n is a finite-dimensional algebra with unit 1_n, for each n. $A_0 \cong K$.
(2) The (external) multiplication $\rho_{m,n} : A_m \otimes A_n \to A_{m+n}$ is an injective homomorphism of algebras, for all m and n (sending $1_m \otimes 1_n$ to 1_{m+n}).
(3) A_{m+n} is a two-sided projective $A_m \otimes A_n$-module with the action defined by $a \cdot (b \otimes c) = a\rho_{m,n}(b \otimes c)$ and $(b \otimes c) \cdot a = \rho_{m,n}(b \otimes c)a$, for all $m, n \geqslant 0$, $a \in A_{m+n}$, $b \in A_m$, $c \in A_n$ and $m, n \geqslant 0$.
(4) For every primitive idempotent g in A_{m+n}, $A_{m+n}g \cong \bigoplus (A_m \otimes A_n)(e \otimes f)$ as (left) $A_m \otimes A_n$-modules if and only if $gA_{m+n} \cong \bigoplus (e \otimes f)(A_m \otimes A_n)$ as (right) $A_m \otimes A_n$-modules for the same index of idempotents $(e \otimes f)$'s in $A_m \otimes A_n$.

(5) The following equality holds for $G_0(A)=\bigoplus_{n\geqslant 0} G_0(A_n)$ (or equivalently for $K_0(A)=\bigoplus_{n\geqslant 0} K_0(A_n)$)

$$\big[\operatorname{Res}^{A_{m+n}}_{A_k\otimes A_{m+n-k}}\operatorname{Ind}^{A_{m+n}}_{A_m\otimes A_n}(M\otimes N)\big]=\sum_{t+s=k}\big[\widetilde{\operatorname{Ind}}^{A_k\otimes A_{m+n-k}}_{A_t\otimes A_{m-t}\otimes A_s\otimes A_{n-s}}\big(\operatorname{Res}^{A_m}_{A_t\otimes A_{m-t}}M\otimes \operatorname{Res}^{A_n}_{A_s\otimes A_{n-s}}N\big)\big]$$

for all $0<k<m+n$, M an A_m-module and N an A_n-module, or M a projective A_m-module and N a projective A_n-module. Here the twisted induction

$$\begin{aligned}&\widetilde{\operatorname{Ind}}^{A_k\otimes A_{m+n-k}}_{A_t\otimes A_{m-t}\otimes A_s\otimes A_{n-s}}(M_1\otimes M_2)\otimes(N_1\otimes N_2)\\&\quad=(A_k\otimes A_{m+n-k})\,\widetilde{\otimes}_{A_t\otimes A_{m-t}\otimes A_s\otimes A_{n-s}}\big((M_1\otimes M_2)\otimes(N_1\otimes N_2)\big).\end{aligned}$$

This is the usual tensor quotient by the (twisted) relations

$$\begin{aligned}&(a\otimes b)\otimes\big[(c_1\otimes c_2)\cdot(w_1\otimes w_2)\otimes(d_1\otimes d_2)\cdot(u_1\otimes u_2)\big]\\&\quad\equiv\big[a\rho_{t,s}(c_1\otimes d_1)\otimes b\rho_{m-t,n-s}(c_2\otimes d_2)\big]\otimes(w_1\otimes u_1\otimes w_2\otimes u_2).\end{aligned}$$

Condition (1) guarantees that their Grothendieck groups are graded connected. Conditions (2) and (3) insure that the induction and restriction are well defined on $G_0(A)$ and $K_0(A)$. The duality follows from (4). Finally (5) gives an analogue of Mackey's formula. This gives us the compatibility relation between the multiplication and comultiplication that we will define on $G_0(A)$ and $K_0(A)$.

3.2. *Induction and restriction on $G_0(A)$*

For M a left A_m-module and N a left A_n-module, let $M\otimes N$ be the left $A_m\otimes A_n$-module defined by $(a\otimes b)\cdot(w\otimes u)=aw\otimes bu$, for $a\in A_m$, $b\in A_n$, $w\in M$ and $u\in N$. We define induction on $G_0(A)$ as follows:

$$\begin{aligned}i_{m,n}:G_0(A_m)\otimes G_0(A_n)&\to G_0(A_{m+n})\\ [M]\otimes[N]&\mapsto\big[\operatorname{Ind}^{A_{m+n}}_{A_m\otimes A_n}M\otimes N\big].\end{aligned}$$

For the restriction, we define

$$\begin{aligned}r_{k,l}:G_0(A_{k+l})&\to G_0(A_k)\otimes G_0(A_l)\\ [N]&\mapsto\big[\operatorname{Res}^{A_{k+l}}_{A_k\otimes A_l}N\big].\end{aligned}$$

Proposition 3.1. *i and r are well defined on $G_0(A)$.*

Proof. If $d:M_1\to M_2$ and $\delta:N_1\to N_2$ are isomorphisms, then

$$A_{m+n}\otimes_{A_m\otimes A_n}(M_1\otimes N_1)\cong A_{m+n}\otimes_{A_m\otimes A_n}(M_2\otimes N_2)$$

with the map

$$\big(1\otimes_{A_{m+n}}(d\otimes\delta)\big)(a\otimes w\otimes u)\overset{\text{def}}{=}a\otimes\big(d(w)\otimes\delta(u)\big).$$

This is well defined since

$$\begin{aligned}\big(1\otimes_{A_{m+n}}(d\otimes\delta)\big)\big(a\otimes(bw\otimes cu)\big) &= a\otimes\big(d(bw)\otimes\delta(cu)\big)=a\otimes\big(bd(w)\otimes c\delta(u)\big)\\ &= a\rho(b\otimes c)\otimes\big(d(w)\otimes\delta(u)\big)\\ &= \big(1\otimes_{A_{m+n}}(d\otimes\delta)\big)\big(a\rho(b\otimes c)\otimes(w\otimes u)\big).\end{aligned}$$

Hence $[\mathrm{Ind}_{A_m\otimes A_n}^{A_{m+n}} M_1\otimes N_1]=[\mathrm{Ind}_{A_m\otimes A_n}^{A_{m+n}} M_2\otimes N_2]$. Without loss of generality, assume $[M]=[M']+[M'']$. So there is a short exact sequence

$$0\to M'\to M\to M''\to 0.$$

Since N is a finitely generated left A_n-module, it is a projective K-module. We have

$$0\to M'\otimes N\to M\otimes N\to M''\otimes N\to 0$$

exact as K-modules (also exact as $A_m\otimes A_n$-modules). Since A_{m+n} is a (right) projective $A_m\otimes A_n$-module, we have

$$0\to A_{m+n}\otimes_{A_m\otimes A_n}(M'\otimes N)\to A_{m+n}\otimes_{A_m\otimes A_n}(M\otimes N)\to A_{m+n}\otimes_{A_m\otimes A_n}(M''\otimes N)\to 0$$

exact. Hence

$$\big[\mathrm{Ind}_{A_m\otimes A_n}^{A_{m+n}} M\otimes N\big]=\big[\mathrm{Ind}_{A_m\otimes A_n}^{A_{m+n}} M'\otimes N\big]+\big[\mathrm{Ind}_{A_m\otimes A_n}^{A_{m+n}} M''\otimes N\big].$$

Similarly,

$$\big[\mathrm{Ind}_{A_m\otimes A_n}^{A_{m+n}} M\otimes N\big]=\big[\mathrm{Ind}_{A_m\otimes A_n}^{A_{m+n}} M\otimes N'\big]+\big[\mathrm{Ind}_{A_m\otimes A_n}^{A_{m+n}} M\otimes N''\big]$$

for $[N]=[N']+[N'']$. Hence i is well defined on $G_0(A)$ by induction.

Now we show that r is well defined. Given that $\mathrm{Hom}_{A_n}(A_n,M)\cong M$ for any A_n-module M we have that if $N_1\cong N_2$ then $\mathrm{Hom}_{A_n}(A_n,N_1)\cong N_1\cong N_2\cong \mathrm{Hom}_{A_n}(A_n,N_2)$. That is $[\mathrm{Res}_{A_k\otimes A_l}^{A_n} N_1]=[\mathrm{Res}_{A_k\otimes A_l}^{A_n} N_2]$. Without loss of generality, assume $[N]=[N']+[N'']$. So there is a short exact sequence

$$0\to N'\to N\to N''\to 0$$

of A_n-modules. Then we have

$$0\to \mathrm{Hom}_{A_n}(A_n,N')\to \mathrm{Hom}_{A_n}(A_n,N)\to \mathrm{Hom}_{A_n}(A_n,N'')\to 0$$

exact as $A_k\otimes A_l$-modules. Hence

$$\big[\mathrm{Res}_{A_k\otimes A_l}^{A_n} N\big]=\big[\mathrm{Res}_{A_k\otimes A_l}^{A_n} N'\big]+\big[\mathrm{Res}_{A_k\otimes A_l}^{A_n} N''\big]$$

and again r is well defined by induction on $G_0(A)$. □

We can now define a multiplication and a comultiplication using i and r and define a unit and a counit on $G_0(A)$ as follows:

$$\pi : G_0(A) \otimes G_0(A) \to G_0(A) \quad \text{where } \pi|_{G_0(A_k)\otimes G_0(A_l)} = i_{k,l},$$
$$\Delta : G_0(A) \to G_0(A) \otimes G_0(A) \quad \text{where } \Delta|_{G_0(A_n)} = \sum_{k+l=n} r_{k,l},$$
$$\mu : \mathbb{Z} \to G_0(A) \quad \text{where } \mu(a) = a[K] \in G_0(A_0), \text{ for } a \in \mathbb{Z},$$
$$\epsilon : G_0(A) \to \mathbb{Z} \quad \text{where } \epsilon\big([M]\big) = \begin{cases} a & \text{if } [M] = a[K], \text{ where } a \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

In Section 3.5 we will prove the associativity of π, the unity of μ, the coassociativity of Δ and the counity of ϵ.

3.3. *Induction and restriction on $K_0(A)$*

As before, we define induction and restriction on $K_0(A)$:

$$i'_{m,n} : K_0(A_m) \otimes K_0(A_n) \to K_0(A_{m+n})$$
$$[P] \otimes [Q] \mapsto \big[\mathrm{Ind}_{A_m\otimes A_n}^{A_{m+n}} P \otimes Q\big],$$

and

$$r'_{k,l} : K_0(A_{k+l}) \to K_0(A_k) \otimes K_0(A_l)$$
$$[R] \mapsto \big[\mathrm{Res}_{A_k\otimes A_l}^{A_{k+l}} R\big].$$

Proposition 3.2. *i' and r' are well defined on $K_0(A)$.*

Proof. The proof is similar to Proposition 3.1 we only need to show here that $\mathrm{Ind}_{A_m\otimes A_n}^{A_{m+n}} P \otimes Q = A_{m+n} \otimes_{A_m\otimes A_n} (P \otimes Q)$ is a projective A_{m+n}-module. Assume that $P \oplus P' \cong (A_m)^s$ and $Q \oplus Q' \cong (A_n)^t$ for some s and t. Since

$$\begin{aligned}
&A_{m+n} \otimes_{A_m\otimes A_n} (P \otimes Q) \oplus A_{m+n} \otimes_{A_m\otimes A_n} (P' \otimes Q) \oplus A_{m+n} \otimes_{A_m\otimes A_n} \big((A_m)^s \otimes Q'\big) \\
&\quad\cong A_{m+n} \otimes_{A_m\otimes A_n} \big((A_m)^s \otimes (A_n)^t\big) \cong A_{m+n} \otimes_{A_m\otimes A_n} (A_m \otimes A_n)^{st} \\
&\quad\cong \big(A_{m+n} \otimes_{A_m\otimes A_n} (A_m \otimes A_n)\big)^{st} \cong (A_{m+n})^{st},
\end{aligned}$$

we have that $\mathrm{Ind}_{A_m\otimes A_n}^{A_{m+n}} P \otimes Q$ is a projective A_{m+n}-module.

Assume $R \oplus R' \cong (A_n)^s$ for some s. Then there is a split short exact sequence

$$0 \to R \to (A_n)^s \to R' \to 0.$$

Since $\mathrm{Hom}_{A_n}(A_n, M) \cong M$ as $A_k \otimes A_l$-modules for any $k + l = n$ and any A_n-module M, the short sequence

$$0 \to \mathrm{Hom}_{A_n}(A_n, R) \to \mathrm{Hom}_{A_n}\big(A_n, (A_n)^s\big) \to \mathrm{Hom}_{A_n}(A_n, R') \to 0$$

is exact and split. That means

$$\mathrm{Hom}_{A_n}\big(A_n, (A_n)^s\big) \cong \mathrm{Hom}_{A_n}(A_n, R) \oplus \mathrm{Hom}_{A_n}(A_n, R')$$

as $A_k \otimes A_l$-modules. Since $\mathrm{Hom}_{A_n}(A_n, A_n^s) \cong A_n^s$ and A_n is a projective (left) $A_k \otimes A_l$-module, so is $(A_n)^s$. From above, it follows that

$$\mathrm{Hom}_{A_n}(A_n, R) \oplus \mathrm{Hom}_{A_n}(A_n, R') \cong (A_n)^s$$

as $A_k \otimes A_l$-modules, i.e., $\mathrm{Hom}_{A_n}(A_n, R)$ is a summand of $(A_n)^s$. Therefore, $\mathrm{Hom}_{A_n}(A_n, R)$ is a projective $A_k \otimes A_l$-module. □

Using i' and r' we also define a multiplication, a comultiplication, a unit and a counit on $K_0(A)$.

$$\pi' : K_0(A) \otimes K_0(A) \to K_0(A) \quad \text{where } \pi'|_{K_0(A_k)\otimes K_0(A_l)} = i'_{k,l},$$

$$\Delta' : K_0(A) \to K_0(A) \otimes K_0(A) \quad \text{where } \Delta'|_{K_0(A_n)} = \sum_{k+l=n} r'_{k,l},$$

$$\mu' : \mathbb{Z} \to K_0(A) \quad \text{where } \mu'(a) = a[K] \in K_0(A_0), \text{ for } a \in \mathbb{Z},$$

$$\epsilon' : K_0(A) \to \mathbb{Z} \quad \text{where } \epsilon'\big([M]\big) = \begin{cases} a & \text{if } [M] = a[K], \text{ where } a \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

In Section 3.5, we will see that the operations above have the desired properties.

3.4. *Pairing on $K_0(A) \times G_0(A)$*

To show the duality between $K_0(A)$ and $G_0(A)$ we define a pairing $\langle\, , \rangle : K_0(A) \times G_0(A) \to \mathbb{Z}$ where

$$\big\langle [P], [M] \big\rangle = \begin{cases} \dim_K(\mathrm{Hom}_{A_n}(P, M)) & \text{if } [P] \in K_0(A_n) \text{ and } [M] \in G_0(A_n), \\ 0 & \text{otherwise.} \end{cases}$$

We also define $\langle\, , \rangle : (K_0(A) \otimes K_0(A)) \times (G_0(A) \otimes G_0(A)) \to \mathbb{Z}$ where

$$\big\langle [P] \otimes [Q], [M] \otimes [N] \big\rangle = \begin{cases} \dim_K(\mathrm{Hom}_{A_k \otimes A_l}(P \otimes Q, M \otimes N)) \\ \qquad \text{if } [P] \otimes [Q] \in K_0(A_k) \otimes K_0(A_l) \text{ and } [M] \otimes [N] \in G_0(A_k) \otimes G_0(A_l), \\ 0 \quad \text{otherwise.} \end{cases}$$

Proposition 3.3. *$\langle\, , \rangle$ is a well-defined bilinear pairing on $K_0(A) \times G_0(A)$ satisfying the following identities*:

$$\big\langle [P] \otimes [Q], [M] \otimes [N] \big\rangle = \big\langle [P], [M] \big\rangle \big\langle [Q], [N] \big\rangle,$$

$$\big\langle \pi'\big([P] \otimes [Q]\big), [M] \big\rangle = \big\langle [P] \otimes [Q], \Delta[M] \big\rangle,$$

$$\big\langle \Delta'[P], [M] \otimes [N] \big\rangle = \big\langle [P], \pi\big([M] \otimes [N]\big) \big\rangle,$$

$$\big\langle \mu'(1), [M] \big\rangle = \epsilon\big([M]\big),$$

$$\big\langle [P], \mu(1) \big\rangle = \epsilon'\big([P]\big).$$

Proof. Assume $[M] = [M'] + [M'']$. We have $0 \to M' \to M \to M'' \to 0$ a short exact sequence. Since P is a projective module, the short sequence

$$0 \to \mathrm{Hom}_{A_n}(P, M') \to \mathrm{Hom}_{A_n}(P, M) \to \mathrm{Hom}_{A_n}(P, M'') \to 0$$

is exact. Hence $\langle [P],[M]\rangle = \langle [P],[M']\rangle + \langle [P],[M'']\rangle$. If we assume $[P]=[P']+[P'']$, then $P\cong P'\oplus P''$ and we have $\mathrm{Hom}_{A_n}(P,M)\cong \mathrm{Hom}_{A_n}(P',M)\oplus \mathrm{Hom}_{A_n}(P'',M)$. Hence $\langle [P],[M]\rangle = \langle [P'],[M]\rangle + \langle [P''],[M]\rangle$. Therefore $\langle\,,\rangle$ is a well-defined bilinear pairing on $K_0(A)\times G_0(A)$.

The identity $\langle [P]\otimes[Q],[M]\otimes[N]\rangle = \langle [P],[M]\rangle\langle [Q],[N]\rangle$ is clear from the isomorphism $\mathrm{Hom}_{A_k\otimes A_l}(P\otimes Q, M\otimes N)\cong \mathrm{Hom}_{A_k}(P,M)\otimes_K \mathrm{Hom}_{A_l}(Q,N)$. For $\langle \pi'([P]\otimes[Q]),[M]\rangle = \langle [P]\otimes[Q],\Delta[M]\rangle$ we use the Adjointness Theorem [4]. We have

$$\begin{aligned}\mathrm{Hom}_{A_{k+l}}\big(\mathrm{Ind}_{A_k\otimes A_l}^{A_{k+l}} P\otimes Q, M\big) &\cong \mathrm{Hom}_{A_{k+l}}\big(A_{k+l}\otimes_{A_k\otimes A_l}(P\otimes Q), M\big)\\ &\cong \mathrm{Hom}_{A_k\otimes A_l}\big(P\otimes Q, \mathrm{Hom}_{A_{k+l}}(A_{k+l},M)\big)\\ &\cong \mathrm{Hom}_{A_k\otimes A_l}\big(P\otimes Q, \mathrm{Res}_{A_k\otimes A_l}^{A_{k+l}} M\big),\end{aligned}$$

which gives us

$$\dim_K\big(\mathrm{Hom}_{A_{k+l}}\big(\mathrm{Ind}_{A_k\otimes A_l}^{A_{k+l}} P\otimes Q, M\big)\big) = \dim_K\big(\mathrm{Hom}_{A_k\otimes A_l}\big(P\otimes Q, \mathrm{Res}_{A_k\otimes A_l}^{A_{k+l}} M\big)\big).$$

Thus $\langle i'_{k,l}([P]\otimes[Q]),[M]\rangle = \langle [P]\otimes[Q], r_{k,l}[M]\rangle$ and the desired identity follows.

To show $\langle \Delta'[P],[M]\otimes[N]\rangle = \langle [P],\pi([M]\otimes[N])\rangle$, we need to prove the identity $\langle r'_{k,l}[P],[M]\otimes[N]\rangle = \langle [P], i_{k,l}([M]\otimes[N])\rangle$, for all $[P]\in K_0(A_{k+l})$, $[M]\in K_0(A_k)$ and $[N]\in G_0(A_l)$. This is not as straightforward as before. Here we need the equality

$$\dim_K\big(\mathrm{Hom}_{A_{k+l}}\big(P, A_{k+l}\otimes_{A_k\otimes A_l}(M\otimes N)\big)\big) = \dim_K\big(\mathrm{Hom}_{A_k\otimes A_l}\big(\mathrm{Hom}_{A_{k+l}}(A_{k+l},P), M\otimes N\big)\big). \tag{3.1}$$

Clearly, without lost of generality we can restrict our attention to indecomposable projective modules P. For such a P, there is a primitive idempotent $g\in A_{k+l}$ such that $P\cong A_{k+l}g$. We know that for any finite-dimensional algebra B over K, M a left B-module and e a primitive idempotent, we have $\mathrm{Hom}_B(Be,M)\cong eM$ as vector spaces (see [17]). Hence

$$\dim_K\big(\mathrm{Hom}_{A_{k+l}}\big(A_{k+l}g, A_{k+l}\otimes_{A_k\otimes A_l}(M\otimes N)\big)\big) = \dim_K\big(gA_{k+l}\otimes_{A_k\otimes A_l}(M\otimes N)\big).$$

To prove (3.1), we expect that

$$\dim_K\big(gA_{k+l}\otimes_{A_k\otimes A_l}(M\otimes N)\big) = \dim_K\big(\mathrm{Hom}_{A_k\otimes A_l}(A_{k+l}g\downarrow_{A_k\otimes A_l}, M\otimes N)\big).$$

Since $gA_{k+l}\cong\bigoplus(e\otimes f)(A_k\otimes A_l)$ as a (right) $A_k\otimes A_l$-module for some idempotents $(e\otimes f)$'s in $A_k\otimes A_l$, we have

$$\begin{aligned}gA_{k+l}\otimes_{A_k\otimes A_l}(M\otimes N) &\cong \bigoplus(e\otimes f)(A_k\otimes A_l)\otimes_{A_k\otimes A_l}(M\otimes N)\\ &\cong \bigoplus(e\otimes f)(M\otimes N).\end{aligned}$$

At the same time from condition (4) $A_{k+l}g\cong\bigoplus(A_k\otimes A_l)(e\otimes f)$ as a (left) $A_k\otimes A_l$-module for the same idempotents $(e\otimes f)$'s in $A_k\otimes A_l$. Hence

$$\begin{aligned}\mathrm{Hom}_{A_k\otimes A_l}(A_{k+l}g\downarrow_{A_k\otimes A_l}, M\otimes N) &\cong \mathrm{Hom}_{A_k\otimes A_l}\Big(\bigoplus(A_k\otimes A_l)(e\otimes f), M\otimes N\Big)\\ &\cong \bigoplus(e\otimes f)(M\otimes N).\end{aligned}$$

Therefore (3.1) holds.

We know $\mu'(1) = [K]$ and

$$\dim_K\bigl(\mathrm{Hom}_K(K, M)\bigr) = \begin{cases} a & \text{if } [M] = a[K], \text{ where } a \in \mathbb{Z}, \\ 0 & \text{otherwise,} \end{cases}$$

therefore $\langle \mu'(1), [M]\rangle = \epsilon([M])$. Similarly, $\langle [P], \mu(1)\rangle = \epsilon([P])$. □

Let $\{V_1, \ldots, V_s\}$ be a complete list of nonisomorphic simple A_n-modules. Then the set of their projective covers $\{P_1, \ldots, P_s\}$ is a complete list of nonisomorphic indecomposable projective A_n-modules. The proposition below is well known (see [4]).

Proposition 3.4. *$\langle [P_i], [V_j]\rangle = \delta_{i,j}$ for $1 \leqslant i, j \leqslant s$.*

3.5. Main result 1

Theorem 3.5.

(1) *π and π' are associative. Hence $(G_0(A), \pi, \mu)$ and $(K_0(A), \pi', \mu')$ are algebras.*
(2) *Δ and Δ' are coassociative. Hence $(G_0(A), \Delta, \epsilon)$ and $(K_0(A), \Delta', \epsilon')$ are coalgebra.*
(3) *If $G_0(A)$ satisfies the condition* (5)*, then Δ and ϵ are algebra homomorphisms and $G_0(A)$ is a connected graded Hopf algebra, as is $K_0(A)$ by duality. Equivalently, the same results hold if instead $K_0(A)$ satisfies the condition* (5)*.*

Proof. (1) We only need to show the associativity of π. The associativity of π' follows from Propositions 3.3 and 3.4. From the associativity of ρ it is straightforward to verify that

$$\begin{aligned}
\mathrm{Ind}_{A_{l+m}\otimes A_n}^{A_{l+m+n}}\bigl(\mathrm{Ind}_{A_l\otimes A_m}^{A_{l+m}} L\otimes M\bigr)\otimes N &= A_{l+m+n}\otimes_{A_{l+m}\otimes A_n}\bigl(\bigl(A_{l+m}\otimes_{A_l\otimes A_m}(L\otimes M)\bigr)\otimes N\bigr)\\
&= A_{l+m+n}\otimes_{A_l\otimes A_m\otimes A_n}(L\otimes M\otimes N)\\
&= A_{l+m+n}\otimes_{A_l\otimes A_{l+n}}\bigl(L\otimes\bigl(A_{m+n}\otimes_{A_m\otimes A_n}(M\otimes N)\bigr)\bigr)\\
&= \mathrm{Ind}_{A_l\otimes A_{m+n}}^{A_{l+m+n}} L\otimes\bigl(\mathrm{Ind}_{A_m\otimes A_n}^{A_{m+n}} M\otimes N\bigr).
\end{aligned}$$

Hence $i_{l+m,n}\cdot(i_{l,m}\otimes 1_n) = i_{l,m+n}\cdot(1_l\otimes i_{m,n})$ and the associativity of π follows.

(2) Again we only need to show the coassociativity of Δ, that is, $(r_{l,m}\otimes 1)\cdot r_{l+m,n} = (1\otimes r_{m,n})\cdot r_{l,m+n}$. From the definition of r we have

$$\begin{aligned}
\mathrm{Res}_{A_l\otimes A_m\otimes A_n}^{A_{l+m}\otimes A_n}\mathrm{Res}_{A_{l+m}\otimes A_n}^{A_{l+m+n}} V &= \mathrm{Hom}_{A_{l+m}\otimes A_n}\bigl(A_{l+m}\otimes A_n, \mathrm{Hom}_{A_{l+m+n}}(A_{l+m+n}, V)\bigr)\\
&\cong \mathrm{Hom}_{A_{l+m+n}}\bigl(A_{l+m+n}\otimes_{A_{l+m}\otimes A_n}(A_{l+m}\otimes A_n), V\bigr)
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Res}_{A_l\otimes A_m\otimes A_n}^{A_l\otimes A_{m+n}}\mathrm{Res}_{A_l\otimes A_{m+n}}^{A_{l+m+n}} V &= \mathrm{Hom}_{A_l\otimes A_{m+n}}\bigl(A_l\otimes A_{m+n}, \mathrm{Hom}_{A_{l+m+n}}(A_{l+m+n}, V)\bigr)\\
&\cong \mathrm{Hom}_{A_{l+m+n}}\bigl(A_{l+m+n}\otimes_{A_l\otimes A_{m+n}}(A_l\otimes A_{m+n}), V\bigr).
\end{aligned}$$

Now we want to show that

$$\mathrm{Hom}_{A_{l+m+n}}\bigl(A_{l+m+n}\otimes_{A_{l+m}\otimes A_n}(A_{l+m}\otimes A_n), V\bigr)\cong \mathrm{Hom}_{A_{l+m+n}}\bigl(A_{l+m+n}\otimes_{A_l\otimes A_{m+n}}(A_l\otimes A_{m+n}), V\bigr) \quad (3.2)$$

as $A_l \otimes A_m \otimes A_n$-modules. In fact, $A_{l+m+n} \otimes_{A_{l+m} \otimes A_n} (A_{l+m} \otimes A_n) \cong A_{l+m+n}$ as A_{l+m+n}–$(A_{l+m} \otimes A_n)$-bimodules and $A_{l+m+n} \otimes_{A_l \otimes A_{m+n}} (A_l \otimes A_{m+n}) \cong A_{l+m+n}$ as A_{l+m+n}–$(A_l \otimes A_{m+n})$-bimodules. Hence (3.2) holds as $A_l \otimes A_m \otimes A_n$-modules with the action defined by $((a \otimes b \otimes c) \cdot f)(d) = f(d\rho_{l,m,n}(a \otimes b \otimes c))$ for $a \in A_l$, $b \in A_m$, $c \in A_n$, $d \in A_{l+m+n}$ and $f \in \mathrm{Hom}_{A_{l+m+n}}(A_{l+m+n}, V)$. This completes the proof.

(3) Without loss of generality, we suppose $G_0(A)$ satisfies the identity in condition (5). For $[M] \in G_0(A_m)$, $[N] \in G_0(A_n)$, we know that

$$\Delta\big(\pi\big([M] \otimes [N]\big)\big) = \sum_{k=0}^{m+n} \big[\mathrm{Hom}_{A_{m+n}}\big(A_{m+n}, A_{m+n} \otimes_{A_m \otimes A_n} (M \otimes N)\big) \downarrow_{A_k \otimes A_{m+n-k}}\big].$$

We use "$\downarrow_{A_k \otimes A_{m+n-k}}$" to remind us that the module should be viewed as an $A_k \otimes A_{m+n-k}$-module. On the other hand, we have in $A \otimes A$ the following product

$$\Delta[M]\Delta[N] = \sum_{k=0}^{m+n} \sum_{t+s=k} \big[(A_k \otimes A_{m+n-k}) \widetilde{\otimes}_{A_t \otimes A_{m-t} \otimes A_s \otimes A_{n-s}} \big(\mathrm{Hom}_{A_m}(A_m, M) \otimes \mathrm{Hom}_{A_n}(A_n, N)\big)\big].$$

To prove that Δ is an algebra homomorphism we need $\Delta(\pi([M] \otimes [N])) = \Delta[M]\Delta[N]$. For this it is enough to show the equality of the corresponding terms for all $0 \leqslant k \leqslant m+n$ in the expressions above. When $k = 0$, $A_0 \cong K$ we have

$$\begin{aligned}
&\big[\mathrm{Hom}_{A_{m+n}}\big(A_{m+n}, A_{m+n} \otimes_{A_m \otimes A_n} (M \otimes N)\big) \downarrow_{A_0 \otimes A_{m+n}}\big] \\
&\quad = \big[\mathrm{Hom}_{A_{m+n}}\big(A_{m+n}, A_{m+n} \otimes_{A_m \otimes A_n} (M \otimes N)\big)\big] \\
&\quad = \big[A_{m+n} \otimes_{A_m \otimes A_n} (M \otimes N)\big] \\
&\quad = \big[(A_0 \otimes A_{m+n}) \widetilde{\otimes}_{A_0 \otimes A_m \otimes A_0 \otimes A_n} (M \otimes N)\big] \\
&\quad = \big[(A_0 \otimes A_{m+n}) \widetilde{\otimes}_{A_0 \otimes A_m \otimes A_0 \otimes A_n} \big(\mathrm{Hom}_{A_m}(A_m, M) \otimes \mathrm{Hom}_{A_n}(A_n, N)\big)\big].
\end{aligned}$$

A similar computation holds for $k = m+n$. For $0 < k < m+n$, the equality follows from our condition (5):

$$\big[\mathrm{Res}^{A_{m+n}}_{A_k \otimes A_{m+n-k}} \mathrm{Ind}^{A_{m+n}}_{A_m \otimes A_n}(M \otimes N)\big] = \sum_{t+s=k} \big[\widetilde{\mathrm{Ind}}^{A_k \otimes A_{m+n-k}}_{A_t \otimes A_{m-t} \otimes A_s \otimes A_{n-s}}\big(\mathrm{Res}^{A_m}_{A_t \otimes A_{m-t}} M \otimes \mathrm{Res}^{A_n}_{A_s \otimes A_{n-s}} N\big)\big].$$

We have that $(G_0, \pi, \mu, \Delta, \epsilon)$ is a graded bialgebra, hence a graded Hopf algebra. By duality $K_0(A)$ is also a graded Hopf algebra. □

Now we are in the position to state our first main result:

Theorem 3.6. *If A is a tower of algebras satisfying conditions* (1)–(5)*, then we can construct on their Grothendieck groups $G_0(A)$ and $K_0(A)$ a bialgebra structure as above. Moreover, $(G_0, \pi, \mu, \Delta, \epsilon)$ and $(K_0, \pi', \mu', \Delta', \epsilon')$ are dual to each other as connected graded bialgebras.*

3.6. Tower of algebras (not preserving unities) and result 2

In [3], we consider a semi-tower of algebras with ρ not preserving unities. If we weaken the condition of ρ and modify the definitions of induction and restriction we can still get results similar as above. We include only a sketch of the ideas; the details can be found in [11].

The usual definitions of induction and restriction as in Section 2 may cause problems when ρ does not preserve the unities. For this we need to find a weaker definition. Let $\varphi : B \to A$ be an algebra injection not necessarily preserving unities. Let M be a left A-module. We let $\operatorname{Res}_B^A M = \{x \in M \mid \varphi(1)x = x\} \subseteq M$ be a submodule. For $x \in \operatorname{Res}_B^A M$ and $b \in B$ the action is defined by $\varphi(b)x$. When φ preserves the unities, clearly this definition agrees with the one in Section 2. For induction, we have to be careful only in the case of projective modules. Assume that P is an indecomposable left B-modules (we extend our definition linearly). Hence $P \cong Be$ for some primitive idempotent $e \in B$. We let $\operatorname{Ind}_B^A P = A\varphi(e)$. Again, when φ preserves the unities, it is straightforward to check that this agrees with the definition of induction in Section 2.

With this in hand, one can adapt all the steps in Sections 3.2, 3.3 and 3.4 to obtain

Theorem 3.7. *(See [11].) If A is a tower of algebras satisfying conditions* (1)–(5)*, then we can construct on their Grothendieck groups $G_0(A)$ and $K_0(A)$ a bialgebra structure as above. Moreover, $(G_0, \pi, \mu, \Delta, \epsilon)$ and $(K_0, \pi', \mu', \Delta', \epsilon')$ are dual to each other as graded bialgebras.*

4. Examples

In this section, we verify that $\bigoplus_{n\geqslant 0} \mathbb{C}\mathfrak{S}_n$ and $\bigoplus_{n\geqslant 0} H_n(0)$ satisfy all the axioms listed in Section 3.1. They are towers of algebras and we already know that their Grothendieck groups are dual Hopf algebras, respectively. We also give an example of graded algebra which do not satisfy all the axioms and consequently, its Grothendieck groups are not dual Hopf algebras.

4.1. Examples satisfying all the axioms

Example 4.1. Let $A = (\bigoplus_{n\geqslant 0} A_n, \rho)$ with $A_n = \mathbb{C}\mathfrak{S}_n$, where $\mathfrak{S}_n$ is the n-permutation group, and

$$\rho_{m,n} : \mathbb{C}\mathfrak{S}_m \otimes \mathbb{C}\mathfrak{S}_n \to \mathbb{C}\mathfrak{S}_{m+n},$$

where $\rho_{m,n}(\sigma \otimes \tau) = \sigma(1)\sigma(2)\cdots\sigma(m)(m+\tau(1))(m+\tau(2))\cdots(m+\tau(n))$. We use the one line notation of permutations. For example, $\rho_{2,3}(21 \otimes 312) = 21534$. It is clear that ρ preserves unities and satisfies associativity. It is also easy to check that ρ is injective and preserves multiplication.

Now since $\mathbb{C}\mathfrak{S}_n$ is a semi-simple algebra, we know that $\mathbb{C}\mathfrak{S}_{m+n}$ is a two-sided projective $\mathbb{C}\mathfrak{S}_m \otimes \mathbb{C}\mathfrak{S}_n$-module.

For finite group G, simple left modules are obtained from primitive idempotents $g \in \mathbb{C}G$. It is easy to show that the left module Gg is isomorphic to the right module gG (look at their characters). The condition (4) for $A = (\bigoplus_{n\geqslant 0} A_n, \rho)$ is thus satisfied. Condition (5) is just the Mackey Theorem [21].

Hence $A = \bigoplus_{n\geqslant 0} \mathbb{C}\mathfrak{S}_n$ is a tower of algebras and since $\mathbb{C}\mathfrak{S}_n$ is a semi-simple algebra we have that the Grothendieck group $G_0(A) = K_0(A)$ is a self-dual graded Hopf algebra. The characteristic map $\mathrm{ch} : G_0(A) \to \Lambda$, where $\mathrm{ch}([V]) = \sum_\mu z_\mu^{-1} \operatorname{tr} X_\mu^V p_\mu$, is then an isomorphism of graded Hopf algebras between $G_0(A)$ and Λ the Hopf algebra of symmetric functions (see [13]).

Remark 4.2. The Sergeev algebra Se_n is the cross product of symmetric group $\mathfrak{S}_n$ and the Clifford algebra Cliff_n [20], which is a semisimple superalgebra. Here consider the Grothendieck groups in categories of finitely generated supermodules and finitely generated projective supermodules over these superalgebras. One can modify our axioms to sit in the category of supermodules over superalgebras. Its Grothendieck groups G_0 and K_0 coincide and have the Hopf algebra structure which is self-dual. It is possible to check that this tower satisfies the modified conditions (1)–(5). And $\bigoplus_{n\geqslant 0} \mathrm{Se}_n$ is a tower of superalgebras.

Example 4.3. Let $A = (\bigoplus_{n\geqslant 0} H_n(0), \rho)$ be the direct sum of Hecke algebras [10] where ρ is defined by $\rho_{m,n}(T_i \otimes 1) = T_i$ and $\rho_{m,n}(1 \otimes T_j) = T_{j+m}$. The T_i for $1 \leqslant i \leqslant m-1$ are the generators of $H_m(0)$ satisfying

$$T_i^2 = -T_i,$$
$$T_i T_j = T_j T_i, \quad |i-j| > 1,$$
$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}.$$

It is easy to check that ρ preserves unities and satisfies associativity. Since the T_i's satisfy the braid relations, one can associate to each permutation $\sigma \in \mathfrak{S}_n$ the element T_σ in $H_n(0)$ defined by $T_\sigma = T_{i_1} \cdots T_{i_r}$, where $s_{i_1} \cdots s_{i_r}$ is an arbitrary reduced decomposition of σ and s_i is the simple transposition $(i, i+1)$. The set $\{T_\sigma \mid \sigma \in \mathfrak{S}_n\}$ forms a basis for $H_n(0)$ and the multiplication of basis elements is determined by

$$T_i T_\sigma = \begin{cases} T_{s_i \sigma} & \text{if } \ell(s_i\sigma) = \ell(\sigma) + 1, \\ -T_\sigma & \text{if } \ell(s_i\sigma) = \ell(\sigma) - 1. \end{cases}$$

Here $\ell(\sigma)$ is the length of a reduced expression for σ.

In $\mathfrak{S}_{m+n}$, we denote by $X_{(m,n)}$ the set of minimal length coset representatives of $\mathfrak{S}_{m+n}/\mathfrak{S}_m \times \mathfrak{S}_n$. We have $\mathfrak{S}_{m+n} = \bigoplus_{\tau \in X_{(n,m)}} \tau(\mathfrak{S}_m \times \mathfrak{S}_n)$. Moreover, our choice of representative implies that $\ell(\tau\sigma) = \ell(\tau) + \ell(\sigma)$, for all $\tau \in X_{(n,m)}$ and $\sigma \in \mathfrak{S}_m \times \mathfrak{S}_n$ [8]. This implies that

$$H_{m+n}(0) = \bigoplus_{\tau \in X_{(n,m)}} T_\tau \big(H_m(0) \otimes H_n(0)\big).$$

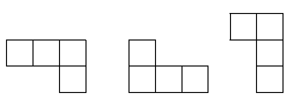
Therefore, when we consider H_{m+n} as a right $H_m(0) \otimes H_n(0)$-module it is a direct sum of $(m+n)!/m!n!$ copies of $H_m(0) \otimes H_n(0)$. Hence $H_{m+n}(0)$ is a right projective $H_m(0) \otimes H_n(0)$-module. Analogously, $H_{m+n}(0)$ is a left projective $H_m(0) \otimes H_n(0)$-module.

Now consider $H_N(0)$. To check the axiom (4) we need to better understand the simple modules and projective indecomposable modules of $H_N(0)$. For this we need to recall some results from [10, 18]. For $i \in [1, N-1]$, let $\square_i = 1 + T_i$. These elements satisfy the relations

$$\square_i^2 = \square_i,$$
$$\square_i \square_j = \square_j \square_i, \quad |i-j| > 1,$$
$$\square_i \square_{i+1} \square_i = \square_{i+1} \square_i \square_{i+1}.$$

In particular, the morphism defined by $T_i \to \square_i$ is an involution of $H_N(0)$. Since the $\square_i$'s also satisfy the braid relations, one can associate to each permutation $\sigma \in \mathfrak{S}_N$ the element $\square_\sigma$ of $H_N(0)$ defined by $\square_\sigma = \square_{i_1} \cdots \square_{i_r}$, where $s_{i_1} \cdots s_{i_r}$ is an arbitrary reduced decomposition of σ.

For a composition $I = (i_1, \ldots, i_r)$ of n, the corresponding *ribbon diagram* of I consists of n boxes with i_1 boxes in the first row, i_2 boxes in the second row, ..., i_r boxes in the rth row and the first box in the next row is under the last one in the previous row. We denote by $\bar{I} = (i_r, \ldots, i_1)$ its mirror image and by $I^\sim$ its conjugate composition, i.e., the composition obtained by writing from right to left the lengths of the columns of the ribbon diagram of I. For example, let $I = (3, 1)$. Then $\bar{I} = (1, 3)$ and $I^\sim = (2, 1, 1)$. The corresponding ribbon diagrams are

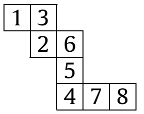


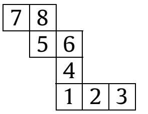
There are 2^{N-1} simple and 2^{N-1} indecomposable projective H_N-modules. They can be realized as minimal left ideals and indecomposable left ideals of $H_N(0)$, respectively. All the simple modules are of dimension 1.

To describe the generators of the simple and indecomposable projective H_N-modules, we associate with a composition I of N two permutations $\alpha(I)$ and $\omega(I)$ of $\mathfrak{S}_N$ defined by

- $\alpha(I)$ is the permutation obtained by filling the columns of the ribbon diagram of shape I from bottom to top and from left to right with the numbers $1, 2, \ldots, N$;
- $\omega(I)$ is the permutation obtained by filling the rows of the ribbon diagram of shape I from left to right and from bottom to top with the numbers $1, 2, \ldots, N$.

For example, consider the composition $I = (2, 2, 1, 3)$ of 8. The fillings of the ribbon diagram of shape I corresponding to $\alpha(I)$ and $\omega(I)$ are





$\alpha(2, 2, 1, 3) = 13265478$ $\omega(2, 2, 1, 3) = 78564123.$

Let $I = (i_1, \ldots, i_r)$ be a composition and $\sigma \in \mathfrak{S}_N$. The *descent set* of σ is $\mathrm{Des}(\sigma) = \{i\colon\ \sigma(i) > \sigma(i+1)\}$ and we also define $D(I) = \{i_1, i_1 + i_2, \ldots, i_1 + \cdots + i_{r-1}\}$. The descent class $D_I = \{\sigma \in \mathfrak{S}_N\colon\ \mathrm{Des}(\sigma) = D(I)\}$ is the interval $[\alpha(I), \omega(I)]$ in the weak order on $\mathfrak{S}_N$ (see [10, Lemma 5.2]).

The simple $H_N(0)$-modules are indexed by all compositions of N. The simple $H_N(0)$-module associated to a composition I is given by the minimal left ideal $C_I = H_N(0)\eta_I$, where $\eta_I = T_{\omega(\bar{I})}\Box_{\alpha(I^{\sim})}$. These modules form a complete system of simple $H_N(0)$-modules and

$$T_i \eta_I = \begin{cases} -\eta_I & \text{if } i \in D(I), \\ 0 & \text{if } i \notin D(I). \end{cases}$$

We associate to I an indecomposable projective $H_N(0)$-module M_I such that $M_I/\mathrm{rad}(M_I) \cong H_N(0)\eta_I$. This module is realized as the left ideal

$$M_I = H_N(0)\nu_I,$$

where $\nu_I = T_{\alpha(I)}\Box_{\alpha(\bar{I}^{\sim})}$. A basis of M_I is given by $\{T_\sigma \Box_{\alpha(\bar{I}^{\sim})}\colon\ \sigma \in [\alpha(I), \omega(I)]\}$. The family $(M_I)_{|I|=N}$ forms a complete system of projective indecomposable $H_N(0)$-modules, and $H_N(0) = \bigoplus_{|I|=N} M_I$.

Remark 4.4. The results above are remarkable, specially considering the fact that the ν_I are not the minimal idempotents of $H_N(0)$; this is an open problem in general. The ν_I are not even idempotent in general. For example, let $I = (2, 1)$. Then $\bar{I}^{\sim} = (1, 2)$, $\alpha(I) = 132 = s_2$ and $\alpha(\bar{I}^{\sim}) = 213 = s_1$. $\nu_I^2 = T_2\Box_1 T_2 \Box_1 = T_2(1 + T_1)T_2(1 + T_1) = (T_2 + T_2T_1)(T_2 + T_2T_1) = T_2^2 + T_2^2T_1 + T_2T_1T_2 + T_2T_1T_2T_1 = -T_2 - T_2T_1 + T_2T_1T_2 - T_2T_1T_2 = -T_2 - T_2T_1 = -T_2(1 + T_1) = -T_2\Box_1 = -\nu_I \neq \nu_I$.

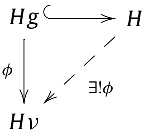
From [18], we know that $H_N(0)T_{\alpha(I)}\Box_{\alpha(\bar{I}^{\sim})} \cong H_N(0)\Box_{\alpha(\bar{I}^{\sim})}T_{\alpha(I)}$ as left ideals (also as left modules). Denote by "-1" the anti-morphism of $H_N(0)$ which reverses the order of the product of the generators in all monomials. For instance, $(T_{i_1} \cdots T_{i_r})^{-1} = T_{i_r} \cdots T_{i_1}$, i.e., $(T_\sigma)^{-1} = T_{\sigma^{-1}}$. This identity also holds when we replace T_i by $\Box_i$. Since $\alpha(I)^{-1} = \alpha(I)$ we have $H_N(0)\nu_I \cong H_N(0)\nu_I^{-1}$. Similarly, $\nu_I H_N(0) \cong \nu_I^{-1} H_N(0)$ as right modules.

Let g_I be a primitive idempotent such that $H_N(0)g_I \cong H_N(0)\nu_I$. Obviously g_I^{-1} is also a primitive idempotent in $H_N(0)$ with $g_I^{-1}H_N(0) \cong \nu_I^{-1}H_N(0) \cong \nu_I H_N(0)$. If $H_N(0)g_I \cong \bigoplus (H_k(0) \otimes H_l(0))(e_J \otimes f_L)$ where $k + l = N$, e_J and f_L are primitive idempotents in $H_k(0)$ and $H_l(0)$, respectively, then at the same time we have $g_I^{-1}H_N(0) \cong \bigoplus (e_J^{-1} \otimes f_L^{-1})(H_k(0) \otimes H_l(0))$. To show axiom (4) we need an auxiliary result:

Proposition 4.5. *Let H be a self-injective algebra and g be an element in H such that Hg is a projective H-module. Then $Hg \cong H\nu$ as H-modules for some $\nu \in H$ if and only if there exist $a, b, c, d \in H$ such that $a\nu = gb$, $cg = \nu b$, $acg = g$, $ca\nu = \nu$, $gbd = g$ and $\nu db = \nu$.*

Proof. Suppose that there exist $a, b, c, d \in H$ such that $a\nu = gb$, $cg = \nu b$, $acg = g$, $ca\nu = \nu$, $gbd = g$ and $\nu db = \nu$. Define $\phi : Hg \to H\nu$ as a (left) H-module homomorphism by $\phi(g) = a\nu$. Then $\phi(cg) = c\phi(g) = ca\nu = \nu$. Define $\psi : H\nu \to Hg$ as a (left) H-module homomorphism by $\psi(\nu) = cg$. Since $(\phi \circ \psi)(\nu) = \phi(\psi(\nu)) = \phi(cg) = c\phi(g) = ca\nu = \nu$ and $(\psi \circ \phi)(g) = \psi(\phi(g)) = \psi(a\nu) = a\psi(\nu) = acg = g$, $\psi = \phi^{-1}$ and ϕ is an isomorphism from Hg to $H\nu$.

Conversely, suppose that H is a self-injective algebra, g is an element in H such that Hg is a projective H-module. Let $\phi : Hg \to H\nu$ be a (left) H-module isomorphism. Then $\phi(g) = a\nu$ and $\phi^{-1}(\nu) = cg$ for some $a, c \in H$. Hence $\nu = \phi(cg) = c\phi(g) = ca\nu$ and $g = \phi^{-1}(a\nu) = a\phi^{-1}(\nu) = acg$. Since H is self-injective, i.e., an H-module is projective if and only if it is injective [1], $H\nu$ is an injective module and $\phi : Hg \to H\nu$ can be extended to a homomorphism from H to $H\nu$ such that the following diagram



is commutative. For convenience we also write the homomorphism $\phi : H \to H\nu$. Similarly, $\phi^{-1} : H\nu \to Hg$ can be extended to a homomorphism $\phi^{-1} : H \to Hg$. Let $\phi(1) = b$ and $\phi^{-1}(1) = d$ for some $b, d \in H$. Then $a\nu = \phi(g) = g\phi(1) = gb$, $cg = \phi^{-1}(\nu) = \nu\phi^{-1}(1) = \nu d$, $\nu = \phi(cg) = \phi(\nu d) = \nu d\phi(1) = \nu db$ and $g = \phi^{-1}(a\nu) = \phi^{-1}(gb) = gb\phi^{-1}(1) = gbd$. □

Since $H_N(0)$ is self-injective [5], we have

Corollary 4.6. *If $H_N(0)g_I \cong H_N(0)\nu_I$ for some primitive idempotent $g_I \in H_N(0)$, then $H_N(0)g_I^{-1} \cong H_N(0)\nu_I$, i.e., $H_N(0)g_I \cong H_N(0)g_I^{-1}$. Similarly $g_I H_N(0) \cong g_I^{-1} H_N(0)$.*

Proof. Since $H_N(0)g_I \cong H_N(0)\nu_I$ there exist $a, b, c, d \in H_N(0)$ such that $a\nu_I = g_I b$, $cg_I = \nu_I d$, $acg_I = g_I$, $ca\nu_I = \nu_I$, $g_I bd = g_I$ and $\nu_I db = \nu_I$. Applying "$^{-1}$" to these equations we get $d^{-1}\nu_I^{-1} = g_I^{-1}c^{-1}$, $b^{-1}g_I^{-1} = \nu_I^{-1}a^{-1}$, $d^{-1}b^{-1}g_I^{-1} = g_I^{-1}$, $b^{-1}d^{-1}\nu_I^{-1} = \nu_I^{-1}$, $g_I^{-1}c^{-1}a^{-1} = g_I^{-1}$ and $\nu_I^{-1}a^{-1}c^{-1} = \nu_I^{-1}$. Setting $a' = d^{-1}$, $b' = c^{-1}$, $c' = b^{-1}$ and $d' = a^{-1}$ we obtain the equations needed to show that $H_N(0)g_I^{-1} \cong H_N(0)\nu_I^{-1} \cong H_N(0)\nu_I$. □

Hence, condition (4) holds.

Next we prove the identity in condition (5) for $G_0(A)$. First we need to introduce the definition of shuffle. Let A be a totally ordered alphabet. A^* denotes the set of all finite-length words formed from the elements in A. The *shuffle* is the bilinear operation of $\mathbb{N}\langle A\rangle$ [10] denoted by $\sqcup\!\sqcup$ and recursively defined on words by the relations

$$1 \sqcup\!\sqcup u = u \sqcup\!\sqcup 1 = u,$$

$$(au) \sqcup\!\sqcup (bv) = a(u \sqcup\!\sqcup bv) + b(au \sqcup\!\sqcup v),$$

where 1 is the empty word, $u, v \in A^*$ and $a, b \in A$. One can show that $\sqcup\!\sqcup$ is associative. For convenience, we also denote $u \sqcup\!\sqcup v$ the set of all words occur in the sum of the shuffle. For instance,

$$21 \sqcup\!\sqcup 34 = 2134 + 2314 + 2341 + 3214 + 3241 + 3421.$$

It also means that $21 \sqcup\!\sqcup 34 = \{2134, 2314, 2341, 3214, 3241, 3421\}$.

From Proposition 5.7 in [10], let I and J be compositions of m and n. Let also $\sigma \in \mathfrak{S}_{[1,m]}$ and $\tau \in \mathfrak{S}_{[m+1,m+n]}$ such that $\mathrm{Des}(\sigma) = D(I)$ and $\mathrm{Des}(\tau) = D(J)$. Then

$$\big[\mathrm{Ind}_{H_m(0)\otimes H_n(0)}^{H_{m+n}(0)} C_I \otimes C_J\big] = \sum_{\omega\in\sigma \sqcup\!\sqcup \tau} [C_{C(\omega)}],$$

where $C(\omega)$ denotes the composition associated with the descent set of ω.

Proposition 4.7. *The following identity holds*

$$a_1\cdots a_m \sqcup\!\sqcup b_1\cdots b_n = \sum_{i=0}^{k} (a_1\cdots a_i \sqcup\!\sqcup b_1\cdots b_{k-i})(a_{i+1}\cdots a_m \sqcup\!\sqcup b_{k-i+1}\cdots b_n)$$

for any $0 \leqslant k \leqslant m$.

Proof. We proceed by induction. When $k = 0$, we have the trivial identity

$$a_1\cdots a_m \sqcup\!\sqcup b_1\cdots b_n = (1 \sqcup\!\sqcup 1)(a_1\cdots a_m \sqcup\!\sqcup b_1\cdots b_n).$$

For $k = 1$, we obtain the defining recursion of shuffle:

$$a_1\cdots a_m \sqcup\!\sqcup b_1\cdots b_n = (1 \sqcup\!\sqcup b_1)(a_1\cdots a_m \sqcup\!\sqcup b_2\cdots b_n) + (a_1 \sqcup\!\sqcup 1)(a_2\cdots a_m \sqcup\!\sqcup b_1\cdots b_n).$$

For $k > 1$ we start with

$$a_1\cdots a_m \sqcup\!\sqcup b_1\cdots b_n = a_1(a_2\cdots a_m \sqcup\!\sqcup b_1\cdots b_n) + b_1(a_1\cdots a_m \sqcup\!\sqcup b_2\cdots b_n)$$

and use the induction hypothesis to get

$$\begin{aligned}
&a_1(a_2\cdots a_m \sqcup\!\sqcup b_1\cdots b_n) + b_1(a_1\cdots a_m \sqcup\!\sqcup b_2\cdots b_n)\\
&\quad= a_1 \sum_{i=1}^{k+1} (a_2\cdots a_i \sqcup\!\sqcup b_1\cdots b_{k+1-i})(a_{i+1}\cdots a_m \sqcup\!\sqcup b_{k+1-i+1}\cdots b_n)\\
&\qquad+ b_1 \sum_{i=0}^{k} (a_1\cdots a_i \sqcup\!\sqcup b_2\cdots b_{k+1-i})(a_{i+1}\cdots a_m \sqcup\!\sqcup b_{k+1-i+1}\cdots b_n)\\
&\quad= b_1(1 \sqcup\!\sqcup b_2\cdots b_{k+1})(a_1\cdots a_m \sqcup\!\sqcup b_{k+2}\cdots b_n)\\
&\qquad+ \sum_{i=1}^{k} a_1(a_2\cdots a_i \sqcup\!\sqcup b_1\cdots b_{k+1-i})(a_{i+1}\cdots a_m \sqcup\!\sqcup b_{k+1-i+1}\cdots b_n)\\
&\qquad+ \sum_{i=1}^{k} b_1(a_1\cdots a_i \sqcup\!\sqcup b_2\cdots b_{k+1-i})(a_{i+1}\cdots a_m \sqcup\!\sqcup b_{k+1-i+1}\cdots b_n)\\
&\qquad+ a_1(a_2\cdots a_{k+1} \sqcup\!\sqcup 1)(a_{k+2}\cdots a_m \sqcup\!\sqcup b_1\cdots b_n)\\
&\quad= (1 \sqcup\!\sqcup b_1\cdots b_{k+1})(a_1\cdots a_m \sqcup\!\sqcup b_{k+2}\cdots b_n)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{k}(a_1\cdots a_i \sqcup\!\sqcup b_1\cdots b_{k+1-i})(a_{i+1}\cdots a_m \sqcup\!\sqcup b_{k+1-i+1}\cdots b_n)\\
&+(a_1\cdots a_{k+1}\sqcup\!\sqcup 1)(a_{k+2}\cdots a_m \sqcup\!\sqcup b_1\cdots b_n)\\
=&\sum_{i=0}^{k+1}(a_1\cdots a_i \sqcup\!\sqcup b_1\cdots b_{k+1-i})(a_{i+1}\cdots a_m \sqcup\!\sqcup b_{k+1-i+1}\cdots b_n). \qquad \square
\end{aligned}
$$

This implies that condition (5) holds for $G_0(A)$.

Remark 4.8. Consider the direct sum of 0-Hecke–Clifford algebras [2] $\mathrm{HCl}_n(0)$, $n \geqslant 0$, which are superalgebras. Here again it is possible to check that this tower satisfies the modified conditions (1)–(5) to show that $\bigoplus_{n\geqslant 0}\mathrm{HCl}_n(0)$ is also a tower of superalgebras.

4.2. An example not satisfying condition (5)

If one considers a direct sum of algebras that does not satisfy condition (3) then the induction and restriction may not be well defined. If it does not satisfy condition (4), then its Grothendieck groups are graded Hopf algebras respectively but not necessarily dual to each other. Hence we are mostly interested in finding structure that satisfies all our axioms but (5). We give some in [11] but the simplest one was given to us by F. Hivert:

Example 4.9. Let $A_n = \mathbb{C}[\mathbb{Z}/2\mathbb{Z}]^{\otimes n}$ and $\rho_{m,n} : \mathbb{C}[\mathbb{Z}/2\mathbb{Z}]^{\otimes m}\otimes\mathbb{C}[\mathbb{Z}/2\mathbb{Z}]^{\otimes n} \to \mathbb{C}[\mathbb{Z}/2\mathbb{Z}]^{\otimes(m+n)}$ be the identity map. It is clear that this tower satisfies all conditions (1)–(4). It does not satisfy condition (5). To see this, we know that there are two simple A_1-modules T, the trivial module and S, the sign module. They are also indecomposable projective A_1-modules. Any simple (or indecomposable projective) A_n-module is an n-tensor product of T's and S's. To see that (5) is not satisfied in general, consider the left-hand side of the formula

$$\big[\mathrm{Res}^{A_2}_{A_1\otimes A_1}\,\mathrm{Ind}^{A_2}_{A_1\otimes A_1}(T\otimes S)\big] = \big[\mathrm{Res}^{A_2}_{A_1\otimes A_1}(T\otimes S)\big] = [T\otimes S].$$

But the right-hand side is

$$
\begin{aligned}
&\big[\widetilde{\mathrm{Ind}}^{A_1\otimes A_1}_{A_0\otimes A_1\otimes A_1\otimes A_0}\big(\mathrm{Res}^{A_1}_{A_0\otimes A_1}T\otimes \mathrm{Res}^{A_1}_{A_1\otimes A_0}S\big)\big]+\big[\widetilde{\mathrm{Ind}}^{A_1\otimes A_1}_{A_1\otimes A_0\otimes A_0\otimes A_1}\big(\mathrm{Res}^{A_1}_{A_1\otimes A_0}T\otimes \mathrm{Res}^{A_1}_{A_0\otimes A_1}S\big)\big]\\
&\quad=[S\otimes T]+[T\otimes S].
\end{aligned}
$$

These are not equal.

Remark 4.10. The algebra $A = (\bigoplus A_n, \rho)$ above does not satisfy our condition (5) and its Grothendieck groups $G(A)$ and $K(A)$ are not Hopf algebra in the strict sense. Yet, in this case $G_0(A)$ and $K_0(A)$ are generalized bialgebras in the sense of Loday [12]. The multiplication π and the comultiplication Δ satisfy a very simple compatibility relation. Let $\hat{\Delta}(x) = \Delta(x) - 1\otimes x - x\otimes 1$. Then

$$\hat{\Delta}\circ\pi = Id\otimes Id + (\pi\otimes Id)\circ(Id\otimes\hat{\Delta}) + (Id\otimes\pi)\circ(\hat{\Delta}\otimes Id). \tag{4.1}$$

At the module level, this is equivalent to the following requirement:

(5)′ In $G_0(A)$ we have

$$\big[\mathrm{Res}^{A_{m+n}}_{A_k\otimes A_{m+n-k}}\,\mathrm{Ind}^{A_{m+n}}_{A_m\otimes A_n}(M\otimes N)\big] = \begin{cases} [(Id\otimes\mathrm{Ind}^{A_{m+n-k}}_{A_{m-k}\otimes A_n})((\mathrm{Res}^{A_m}_{A_k\otimes A_{m-k}}M)\otimes N)] & \text{if } k<m,\\ [M\otimes N] & \text{if } k=m,\\ [(\mathrm{Ind}^{A_k}_{A_m\otimes A_{k-m}}\otimes Id)(M\otimes(\mathrm{Res}^{A_n}_{A_{k-m}\otimes A_{m+n-k}}M))] & \text{if } k>m.\end{cases}$$

This is easy to check for $G_0(A)$. It is thus a self-dual bialgebra satisfying the compatibility relation (4.1).

5. Concluding remarks

In the last example of Section 4, we encountered a graded algebra that satisfies our conditions (1)–(4) but not (5). Yet, following Loday [12], we still have an interesting (generalized) bialgebra structure on its Grothendieck groups. We have given an alternative axiom, $(5)'$, that shows that we get the kind of algebra satisfying the compatibility relations (4.1).

This open the door to many avenues. The conditions (1)–(4) on a graded algebra A are essential to make sure that we can define a structure of graded algebra and of graded coalgebra on $G_0(A)$ and $K_0(A)$ with duality. Then one may ask what kind of compatibility one can get between the induction and the restriction. In this sense there are many alternatives to our condition (5). It would be interesting to find what is the required condition for each of the generalized bialgebras of [12] and to give examples for each cases. One can also define different kinds of inductions and restrictions to allow for different kind of operations on the Grothendieck groups of the tower. This is left to future work.

References

[1] M. Auslander, I. Reiten, S.O. Smalø, Representation Theory of Artinian Algebras, Cambridge Univ. Press, 1995.

[2] N. Bergeron, F. Hivert, J.Y. Thibon, The peak algebra and the Hecke–Clifford algebras at $q = 0$, J. Combin. Theory Ser. A 107 (1) (2004) 1–19.

[3] N. Bergeron, C. Hohweg, M. Rosas, M. Zabrocki, Grothedick bialgebra, partition lattices and symmetric functions in non-commutative variables, Electron. J. Combin. 13 (1) (2006), [R75], 19 pp., http://www.combinatorics.org/.

[4] C. Curtis, I. Reiner, Methods of Representation Theory, vol. I. With Applications to Finite Groups and Orders, John Wiley & Sons, Inc., New York, 1990.

[5] M. Fayers, 0-Hecke algebras of finite Coxeter groups, J. Pure Appl. Algebra 199 (2005) 27–41.

[6] L. Geissinger, Hopf algebras of symmetric functions and class functions, in: Combinatoire et Combinatoire et représentation du groupe symétrique, Actes Table Ronde C.N.R.S., Univ. Louis-Pasteur Strasbourg, Strasbourg, 1976, in: Lecture Notes in Math., vol. 579, Springer-Verlag, Berlin, 1977, pp. 168–181.

[7] F. Hivert, J.-C. Novelli, J.-Y. Thibon, Yang–Baxter bases of 0-Hecke algebras and representation theory of 0-Ariki–Koike–Shoji algebras, Adv. in Math. 205 (2006) 504–548.

[8] J.E. Humphreys, Reflection Groups and Coxeter Groups, Cambridge Univ. Press, Cambridge, 1990.

[9] M. Khovanov, NilCoxeter algebras categorify the Weyl algebra, Comm. Algebra 29 (11) (2001) 5033–5052.

[10] D. Krob, J.Y. Thibon, Noncommutative symmetric functions. IV. Quantum linear groups and Hecke algebras at $q = 0$, J. Algebraic Combin. 6 (4) (1997) 339–376.

[11] H. Li, Algebraic structures of Grothendieck groups of a tower of algebras, PhD thesis, York University, 2007.

[12] J.-L. Loday, Generalized bialgebras and triples of operads, Astérisque 320 (2008), vi+114 pp.

[13] I.G. Macdonald, Symmetric Functions and Hall Polynomials, Clarendon Press/Oxford Univ. Press, New York, 1995, 475 pp.

[14] C. Malvenuto, C. Reutenauer, Duality between quasi-symmetric functions and the Solomon descent algebra, J. Algebra 177 (3) (1995) 967–982.

[15] D. Manchon, Hopf Algebras in Renormalisation, Encyclopaedia Math., in press.

[16] R. Mantaci, C. Reutenauer, A generalization of Solomon's descent algebra for hyperoctahedral groups and wreath products, Comm. Algebra 23 (1995) 27–56.

[17] H. Nagao, Y. Tsushima, Representations of Finite Groups, Academic Press, Inc., 1989.

[18] P.N. Norton, 0-Hecke algebras, J. Austral. Math. Soc. Ser. A 27 (1979) 337–357.

[19] S. Poirier, Cycle type and descent set in wreath products, Discrete Math. 180 (1998) 315–343.

[20] A.N. Sergeev, Tensor algebra of the identity representation as a module over the Lie superalgebras $\mathrm{GL}(n;m)$ and $Q(n)$, Mat. USSR Sb. 51 (1985) 419–427.

[21] A.V. Zelevinsky, Representations of Finite Classical Groups. A Hopf Algebra Approach, Lecture Notes in Math., vol. 869, Springer-Verlag, Berlin, 1981.