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On graph-restrictive permutation groups

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A B S T R A C T

Let Γ be a connected G-vertex-transitive graph, let v be a vertex of Γ and let $L = G_v^{\Gamma(v)}$ be the permutation group induced by the action of the vertex-stabiliser G_v on the neighbourhood $\Gamma(v)$. Then (Γ, G) is said to be *locally-L*. A transitive permutation group L is *graph-restrictive* if there exists a constant $c(L)$ such that, for every locally-L pair (Γ, G) and an arc (u, v) of Γ, the inequality $|G_{uv}| \leqslant c(L)$ holds.

Using this terminology, the Weiss Conjecture says that primitive groups are graph-restrictive. We propose a very strong generalisation of this conjecture: a group is graph-restrictive if and only if it is semiprimitive. (A transitive permutation group is said to be *semiprimitive* if each of its normal subgroups is either transitive or semiregular.) Our main result is a proof of one of the two implications of this conjecture, namely that graph-restrictive groups are semiprimitive. We also collect the known results and prove some new ones regarding the other implication.

1. Introduction

Unless explicitly stated, all graphs considered in this paper are finite and simple. A graph Γ is said to be *G-vertex-transitive* if G is a subgroup of $\mathrm{Aut}(\Gamma)$ acting transitively on the vertex-set $\mathrm{V}\Gamma$ of Γ. Similarly, Γ is said to be *G-arc-transitive* if G acts transitively on the arcs of Γ (that is, on the ordered pairs of adjacent vertices of Γ). When $G = \mathrm{Aut}(\Gamma)$, the prefix G in the above notation is sometimes omitted.

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One of the most important early results concerning arc-transitive graphs is the beautiful theorem of Tutte [15,16], saying that, in a connected 3-valent arc-transitive graph, the order of an arc-stabiliser divides 16. The immediate generalisation of Tutte's result to 4-valent graphs is false; there exist connected 4-valent arc-transitive graphs with arbitrarily large arc-stabilisers (for example, see [10]).

On the other hand, it can easily be deduced from the work of Gardiner [6] that, if Γ is a connected 4-valent G-arc-transitive graph and (u, v) is an arc of Γ, then $|G_{uv}| \leqslant 2^2 3^6$ unless $G_v^{\Gamma(v)} \cong \mathrm{D}_4$, where $G_v^{\Gamma(v)}$ denotes the permutation group induced by the action of G_v on the neighbourhood $\Gamma(v)$. In view of Gardiner's results, bounds on the order of the arc-stabiliser are now usually considered in terms of the *local action* $G_v^{\Gamma(v)}$ rather than simply the valency. This leads us to the following definitions.

Definition 1. Let Γ be a connected G-vertex-transitive graph, let v be a vertex of Γ and let L be a permutation group which is permutation isomorphic to $G_v^{\Gamma(v)}$. Then (Γ, G) is said to be *locally-L*. More generally, if $\mathcal{P}$ is a permutation group property, then (Γ, G) will be called *locally-$\mathcal{P}$* provided that $G_v^{\Gamma(v)}$ possesses the property $\mathcal{P}$.

Note that, if Γ has valency d, then the permutation group $G_v^{\Gamma(v)}$ has degree d and, up to permutation isomorphism, does not depend on the choice of v. In [18], the third author introduced the following notion.

Definition 2. A transitive permutation group L is *graph-restrictive* if there exists a constant $c(L)$ such that, for every locally-L pair (Γ, G) and an arc (u, v) of Γ, the inequality $|G_{uv}| \leqslant c(L)$ holds.

This definition makes it possible to give succinct formulations of many results and questions. For example, Tutte's theorem can be restated as follows: the symmetric group Sym(3) in its natural action on 3 points is graph-restrictive and the constant $c(\mathrm{Sym}(3))$ can be chosen to be 16. Several authors generalised Tutte's result from Sym(3) to other primitive groups, which eventually led Weiss to pose the following conjecture.

Weiss Conjecture. *(See [19, Conjecture 3.12].) Primitive groups are graph-restrictive.*

While much effort has been deployed in the attempts to prove the Weiss Conjecture, very few authors considered graph-restrictiveness of imprimitive groups. We will show that the main results which have been used to attack the Weiss Conjecture can be generalised from primitive groups to semiprimitive groups, which we conjecture are graph-restrictive. (A transitive permutation group is said to be *semiprimitive* if each of its normal subgroups is either transitive or semiregular. Contrary to [2], we consider regular groups to be semiprimitive.) In fact, encouraged by some results proved in this paper, we conjecture the following characterisation of graph-restrictive groups.

Conjecture 3. *A permutation group is graph-restrictive if and only if it is semiprimitive.*

Our goal in this paper is to collect the known results regarding this conjecture and to establish new ones. One of our main results is the following theorem (proved in Section 4), which proves one of the two implications of the above conjecture.

Theorem 4. *Every graph-restrictive group is semiprimitive.*

The structure of the paper is very simple. Section 2 is a summary of all important results, with each new theorem getting its own section later for details and proofs.

2. Summary

We start with a review of previously known results about graph-restrictiveness of permutation groups. It is a rather trivial observation that regular permutation groups L are graph-restrictive, with $c(L) = 1$. One of the earliest non-trivial results with important applications towards the Weiss Conjecture is Theorem 5, a variant of the Thompson–Wielandt theorem (see [17]). Given a G-arc-transitive graph Γ and (u, v) an arc of Γ, denote by $G_{uv}^{[1]}$ the subgroup of G fixing $\Gamma(u)$ and $\Gamma(v)$ point-wise.

Theorem 5. *(See [6, Corollary 2.3].) Let (Γ, G) be a locally-primitive pair and let (u, v) be an arc of Γ. Then $G_{uv}^{[1]}$ is a p-group for some prime p or $G_{uv}^{[1]} = 1$.*

Theorem 5 imposes a very strong restriction on the structure of G_{uv}. For example, if Γ has valency d, then G_{uv} contains a normal p-subgroup of index at most $(d-1)!^2$. As we will see, Theorem 5 was the first step in the proof of many results regarding the Weiss Conjecture.

Despite considerable effort by many authors, the Weiss Conjecture is still open. Some important subcases have however been dealt with. For example, the case of primitive groups of *affine type* is almost complete (see [20, Theorem]). (A primitive group L is said to be of affine type if it contains a regular abelian normal subgroup Q. In particular, as L is primitive, Q is an elementary abelian q-group, for some prime q, and L has degree a power of q.)

Note that [20, Theorem] was misquoted in [4, Theorem 3.1] where the authors concluded that all primitive groups of affine type are graph-restrictive, which does not follow from [20, Theorem]. This has unfortunate consequences. For example, they claim that primitive groups of degree at most 20 are graph-restrictive [4, Proposition 4.1], but their proof relies on [4, Theorem 3.1] in an essential way. In fact, to the best of our knowledge, it is still unknown whether the primitive group of affine type $\mathrm{Sym}(3) \operatorname{wr} \mathrm{Sym}(2)$ with its product action of degree 9 is graph-restrictive. We will return to the question of graph-restrictiveness of groups of small degree in Section 3.

The proof of [20, Theorem] depends heavily on Theorem 5. Another important case of the Weiss Conjecture where Theorem 5 plays a crucial role is the 2-transitive case, which has been settled as the culmination of work by Weiss and Trofimov.

Theorem 6. 2*-transitive groups are graph-restrictive.*

An explanation of the work involved in the proof of Theorem 6 can be found in the introduction of [21].

Note that, by Burnside's theorem [5, Theorem 3.5B], transitive groups of prime degree are either 2-transitive or of affine type. Together with Tutte's result, [20, Theorem] and Theorem 6 imply that transitive groups of prime degree are graph-restrictive.

Little was previously known about graph-restrictiveness of imprimitive groups. Recently, Sami [13] has shown that dihedral groups of odd degree are graph-restrictive, generalising Tutte's theorem (as $\mathrm{Sym}(3) \cong \mathrm{D}_3$), and the third author generalised this result to the wider class of so-called p-sub-regular groups [18, Theorem 1.2]. (Of course, in view of Theorem 4, all these groups are semiprimitive.) To the best of our knowledge, this concludes the list of permutation groups that were known to be graph-restrictive prior to this paper.

Praeger has proved [11] that a quasiprimitive group L is graph-restrictive if and only if there exists a constant $c'(L)$ such that, for every locally-L pair (Γ, G) with G quasiprimitive or biquasiprimitive and an arc (u, v) of Γ, the inequality $|G_{uv}| \leqslant c'(L)$ holds. (A transitive permutation group is called *quasiprimitive* if each of its non-trivial normal subgroups is transitive. It is called *biquasiprimitive* if it is not quasiprimitive and each of its non-trivial normal subgroups has at most two orbits.) This leads her to conjecture that quasiprimitive groups are graph-restrictive, a conjecture stronger than the Weiss Conjecture but weaker than Conjecture 3.

Let us now discuss our new results supporting the conjecture that semiprimitive groups are graph-restrictive. It turns out that many of the important tools that are available for primitive groups are also available for semiprimitive groups. For example, the starting point for most of the results in the primitive case is Theorem 5, which was recently generalised to the semiprimitive case by the second author.

Theorem 7. *(See [14, Corollary 3].) Let (Γ, G) be a locally-semiprimitive pair and let (u, v) be an arc of Γ. Then $G_{uv}^{[1]}$ is a p-group for some prime p or $G_{uv}^{[1]} = 1$.*

Theorem 7 is very useful to prove that certain semiprimitive permutation groups are graph-restrictive. We will give some examples of this, but first we must introduce the following more general version of Definition 2.

Definition 8. Let p be a prime. A transitive permutation group L is *p-graph-restrictive* if there exists a constant $c(p, L)$ such that, for every locally-L pair (Γ, G), the largest pth power dividing the order of an arc-stabiliser is bounded above by $c(p, L)$.

Note that a transitive permutation group L is graph-restrictive if and only if it is p-graph-restrictive for every prime p dividing the order of a point-stabiliser L_x. The proof of the next lemma can be extracted from page 44 of [20]. (Recall that, for a prime p, $\mathrm{O}_p(G)$ denotes the largest normal p-subgroup of G.)

Lemma 9. *Let (Γ, G) be a locally-transitive pair and let (u, v) be an arc of Γ. If $G_{uv}^{[1]}$ is a non-trivial p-group, then $\mathrm{O}_p(G_{uv}^{\Gamma(u)}) \neq 1$.*

Theorem 7 together with Lemma 9 easily yield the following corollary.

Corollary 10. *Let L be a semiprimitive group acting on Ω, let $x \in \Omega$ and let p be a prime. If $\mathrm{O}_p(L_x) = 1$, then L is p-graph-restrictive.*

In Section 5, we will use a result of Glauberman about normalisers of p-groups to prove Corollary 11, which is a generalisation of [18, Theorem 1.2].

Corollary 11. *Let p be a prime and let L be a transitive permutation group on Ω. Let $x \in \Omega$ and let P be a Sylow p-subgroup of L_x. Suppose that*

(1) *$|P| = p$, and*
(2) *there exists $l \in L$ such that $\langle P, P^l \rangle$ is transitive on Ω.*

Then, L is p-graph-restrictive. In fact, we can take $c(p, L) = p^6$ if p is odd and $c(p, L) = 16$ if $p = 2$.

3. Examples and groups of small degrees

In this section, we give a few examples of how Corollary 10 and Corollary 11 can be combined to show that certain semiprimitive permutation groups are graph-restrictive and examine the status of Conjecture 3 for groups of small degree.

Example 12. Let $n \geqslant 3$ be odd and let $L = \mathrm{D}_n$ be the dihedral group of order $2n$ in its natural action on n points. Then $|L_x| = 2$ and, if l is a generator of the cyclic subgroup $\mathrm{C}_n \leqslant L$, then $\langle L_x, L_x^l \rangle = L$. By Corollary 11, it follows that L is graph-restrictive (in fact, we can take $c(L) = 16$).

Example 13. Let p be a prime. Denote by Z the centre of $\mathrm{GL}(2, p)$, that is, the subgroup of $\mathrm{GL}(2, p)$ consisting of the diagonal matrices. Let G be a subgroup of $\mathrm{GL}(2, p)$ with $\mathrm{SL}(2, p) \leqslant G$, and let K be a subgroup of $G \cap Z$. Clearly, $\mathrm{GL}(2, p)$ acts as a group of automorphisms on the 2-dimensional vector space of row vectors $\mathbb{F}_p^2$. We let Ω denote the set of orbits of K on $\mathbb{F}_p^2 \setminus \{0\}$. Since $K \trianglelefteq G$, the group

G acts on Ω and, by a direct computation, we see that K is the kernel of the action of G on Ω. Write $L = G/K$ for the permutation group induced by G on Ω.

Claim. *The group L is graph-restrictive.*

We prove this claim as follows. Write $r = |G : \mathrm{SL}(2, p)|$. We let $e_1 = (1, 0)$, $e_2 = (0, 1)$, $\alpha = e_1^K$ and $\beta = e_2^K$. Since $\mathrm{SL}(2, p) \leqslant G$, the group G is transitive on $\mathbb{F}_p^2 \setminus \{0\}$ and hence

$$L_\alpha = \frac{G_{e_1}K}{K} \cong \frac{G_{e_1}}{K \cap G_{e_1}} \cong G_{e_1} = \left\{ \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} \,\middle|\, a, b \in \mathbb{F}_p,\ b^r = 1 \right\} \cong C_p \rtimes C_r.$$

It follows that L_α is a Frobenius group with Frobenius kernel of size p and with Frobenius complement of size r. In particular, $\mathrm{O}_q(L_\alpha) \neq 1$ if and only if $q = p$. Thus, by Corollary 10, L is q-graph-restrictive for each prime $q \neq p$.

Finally, let

$$U_{e_1} = \left\{ \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \,\middle|\, a \in \mathbb{F}_p \right\} \quad \text{and} \quad U_{e_2} = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \,\middle|\, a \in \mathbb{F}_p \right\}$$

be the unipotent radicals of G_{e_1} and G_{e_2}, and let $P_\alpha = U_{e_1}K/K$ and $P_\beta = U_{e_2}K/K$. We have $|P_\alpha| = |P_\beta| = p$ and P_α is a Sylow p-subgroup of L_α. Furthermore, since $\mathrm{SL}(2, p)$ is generated by the root subgroups U_{e_1} and U_{e_2}, we have that $\langle P_\alpha, P_\beta \rangle$ is transitive on Ω. Thus, by Corollary 11, L is p-graph-restrictive. Therefore L is graph-restrictive. $\square$

Using a computer algebra system containing a list of transitive permutation groups of small degree (for example `Magma` [3]), it is rather straightforward to generate an exhaustive list of semiprimitive groups of small degree. By going through such a list and applying a combination of [20, Theorem], Theorem 6, Corollary 10 and Corollary 11, we obtain the following result:

Proposition 14. *A semiprimitive permutation group of degree at most* 13 *is graph-restrictive unless possibly it is one of the following*:

(1) $\mathrm{Sym}(3) \,\mathrm{wr}\, \mathrm{Sym}(2)$ *with its product action of degree* 9,
(2) $\mathbb{Z}_3^2 \rtimes \mathbb{Z}_2$ *of degree* 9 ($\mathbb{Z}_3^2$ *acts regularly on itself by multiplication and* $\mathbb{Z}_2$ *acts on* $\mathbb{Z}_3^2$ *by inversion*),
(3) $\mathrm{Sym}(5)$ *acting on the* 10 *unordered pairs of a* 5*-set, or*
(4) $\mathrm{Sym}(4)$ *acting on the* 12 *ordered pairs of a* 4*-set.*

In particular, Conjecture 3 is true for permutation groups of degree at most 8. As we already noted, it is unknown whether the primitive group $\mathrm{Sym}(3) \,\mathrm{wr}\, \mathrm{Sym}(2)$ of degree 9 is graph-restrictive, hence even the Weiss Conjecture is not known to hold for groups of degree 9 or more.

4. Proof of Theorem 4

The proof of Theorem 4 relies on a construction inspired by the *wreath extension* (see [9, Section 8.1]), which has been used to solve some instances of the Embedding Galois Problem (see [8, Chapter 3]). We start by establishing some notation and some preliminary lemmas that will be necessary in the proof of Theorem 4.

Let L be a finite permutation group on a set Λ and let K be an intransitive normal subgroup of L. Denote by Δ the set of orbits of K on its action on Λ. Replacing K by $\bigcap_{\lambda \in \Lambda}(KL_\lambda)$, we may assume that K is the kernel of the action of L on Δ. We let S denote the permutation group induced by L on Δ and hence $S \cong L/K$. In particular, we have the short exact sequence

$$1 \to K \to L \xrightarrow{\pi} S \to 1.$$

Fix $\delta \in \Delta$ and $\lambda \in \delta$. Since K is transitive on δ, we have $L_\delta = KL_\lambda$, $S_\delta \cong L_\lambda/K_\lambda$ and the short exact sequence

$$1 \to K_\lambda \to L_\lambda \xrightarrow{\pi} S_\delta \to 1$$

(where we denote by π the restriction $\pi_{|L_\lambda} : L_\lambda \to S_\delta$).

Fix T a transversal for the set of right cosets of S_δ in S. We assume that $1 \in T$. Given $s \in S$, there exists a unique element of T, which we denote by s^τ, such that $S_\delta s = S_\delta s^\tau$. The correspondence $s \mapsto s^\tau$ defines a map $\tau : S \to T$ with $1^\tau = 1$.

Lemma 15. *If $x, s \in S$, then $(xs^{-1})^\tau s(x^\tau)^{-1} \in S_\delta$.*

Proof. We have $S_\delta xs^{-1} = S_\delta(xs^{-1})^\tau$ and hence $S_\delta x = S_\delta(xs^{-1})^\tau s$. Furthermore, as $S_\delta x = S_\delta x^\tau$, we obtain $S_\delta = S_\delta(xs^{-1})^\tau s(x^\tau)^{-1}$. □

Consider the set

$$\Omega = \{ f : S \to L_\lambda \mid f(yx) = f(x) \text{ for every } y \in S_\delta,\ x \in S \}.$$

The elements of Ω are functions $f : S \to L_\lambda$ which (for each $x \in S$) are constant on the right coset $S_\delta x$ of S_δ in S, and hence they can be thought of as functions from Δ to L_λ. The set Ω is a group isomorphic to $L_\lambda^{|\Delta|}$ under point-wise multiplication. Given $f \in \Omega$ and $g \in L$, let f^g be the element of Ω defined by

$$f^g(x) = f\bigl(x\bigl(g^\pi\bigr)^{-1}\bigr).$$

This defines a group action of L on Ω and the semidirect product $\Omega \rtimes L$ is isomorphic to the standard wreath product $L_\lambda \operatorname{wr}_\Delta L$. Moreover, given an arbitrary positive integer m, by extending this action of L on Ω to the (component-wise) action of L on Ω^m, we obtain a semidirect product $\Omega^m \rtimes L$ where the multiplication is given by

$$(g, f_1, \dots, f_m)\bigl(g', h_1, \dots, h_m\bigr) = \bigl(gg', f_1^{g'} h_1, \dots, f_m^{g'} h_m\bigr). \tag{1}$$

Consider the subset

$$\begin{aligned} A = \{ &(g, f_1, \dots, f_m) \mid g \in L,\ \text{for each } i \in \{1, \dots, m\},\ f_i \in \Omega, \text{and} \\ &\text{for every } x \in S,\ \bigl(f_i(x)\bigr)^\pi = \bigl(x\bigl(g^\pi\bigr)^{-1}\bigr)^\tau g^\pi \bigl(x^\tau\bigr)^{-1} \} \end{aligned} \tag{2}$$

of $\Omega^m \rtimes L$. Note that, by Lemma 15, the element $(x(g^\pi)^{-1})^\tau g^\pi (x^\tau)^{-1}$ in the definition of A lies in S_δ.

Lemma 16. *The set A is a subgroup of $\Omega^m \rtimes L$.*

Proof. Let $(g, f_1, \dots, f_m), (g', h_1, \dots, h_m) \in A$. For each $i \in \{1, \dots, m\}$, we have $f_i^{g'} h_i \in \Omega$. Fix $x \in S$. Then, for each $i \in \{1, \dots, m\}$, using the definition of A, we obtain

$$\begin{aligned} \bigl(\bigl(f_i^{g'} h_i\bigr)(x)\bigr)^\pi &= \bigl(f_i^{g'}(x)\bigr)^\pi \bigl(h_i(x)\bigr)^\pi = \bigl(f_i\bigl(x\bigl(g'^\pi\bigr)^{-1}\bigr)\bigr)^\pi \bigl(h_i(x)\bigr)^\pi \\ &= \bigl(\bigl(\bigl(x\bigl(g'^\pi\bigr)^{-1}\bigr)\bigl(g^\pi\bigr)^{-1}\bigr)^\tau g^\pi \bigl(\bigl(x\bigl(g'^\pi\bigr)^{-1}\bigr)^\tau\bigr)^{-1}\bigr)\bigl(\bigl(x\bigl(g'^\pi\bigr)^{-1}\bigr)^\tau g'^\pi \bigl(x^\tau\bigr)^{-1}\bigr) \\ &= \bigl(x\bigl(\bigl(gg'\bigr)^\pi\bigr)^{-1}\bigr)^\tau \bigl(gg'\bigr)^\pi \bigl(x^\tau\bigr)^{-1}. \end{aligned}$$

From (1) and (2), this shows that the product $(g, f_1, \dots, f_m)(g', h_1, \dots, h_m)$ lies in A. Denote by $e : S \to L_\lambda$ the function with $e(x) = 1$ for every $x \in S$. Clearly $(1, e, \dots, e)$ is the identity of $\Omega^m \rtimes S$ and lies in A. Finally, if $(g, f_1, \dots, f_m)$ is in A, then $(g^{-1}, (f_1^{-1})^{g^{-1}}, \dots, (f_m^{-1})^{g^{-1}})$ is the inverse of $(g, f_1, \dots, f_m)$. Fix $x \in S$. For each $i \in \{1, \dots, m\}$, using the definition of A, we obtain

$$\left(\left(\left(f_i^{-1}\right)^{g^{-1}}\right)(x)\right)^{\pi} = \left(\left(f_i\left(xg^{\pi}\right)\right)^{\pi}\right)^{-1} = \left(x^{\tau} g^{\pi}\left(\left(xg^{\pi}\right)^{\tau}\right)^{-1}\right)^{-1} = \left(xg^{\pi}\right)^{\tau}\left(g^{\pi}\right)^{-1}\left(x^{\tau}\right)^{-1}.$$

Thus $(g^{-1}, (f_1^{-1})^{g^{-1}}, \dots, (f_m^{-1})^{g^{-1}})$ lies in A. $\square$

Consider the map

$$\varphi : A \to L \quad \text{defined by } (g, f_1, \dots, f_m)^{\varphi} = g.$$

Lemma 17. *φ is a surjective homomorphism.*

Proof. From (1), φ is a homomorphism. For each $s \in S_\delta$, fix an element of L_λ, which we denote by s^{ε}, with $(s^{\varepsilon})^{\pi} = s$. Since $\pi : L_\lambda \to S_\delta$ is surjective, $\varepsilon : S_\delta \to L_\lambda$ is a well-defined mapping with $s^{\varepsilon\pi} = s$ for every $s \in S_\delta$.

Fix $g \in L$. Given $x \in S$, we see from Lemma 15 that $(x(g^{\pi})^{-1})^{\tau} g^{\pi}(x^{\tau})^{-1} \in S_\delta$. Therefore, the function $f_g : S \to L_\lambda$ with $f_g(x) = ((x(g^{\pi})^{-1})^{\tau} g^{\pi}(x^{\tau})^{-1})^{\varepsilon}$ is well defined. If $y \in S_\delta$ and $x \in S$, then $(yx)^{\tau} = x^{\tau}$ and $(yx(g^{\pi})^{-1})^{\tau} = (x(g^{\pi})^{-1})^{\tau}$. Therefore

$$f_g(yx) = \left(\left(yx\left(g^{\pi}\right)^{-1}\right)^{\tau} g^{\pi}\left((yx)^{\tau}\right)^{-1}\right)^{\varepsilon} = \left(\left(x\left(g^{\pi}\right)^{-1}\right)^{\tau} g^{\pi}\left(x^{\tau}\right)^{-1}\right)^{\varepsilon} = f_g(x).$$

Since this holds for arbitrary $y \in S_\delta$ and $x \in S$, it follows that $f_g \in \Omega$. Now

$$f_g(x)^{\pi} = \left(\left(\left(x\left(g^{\pi}\right)^{-1}\right)^{\tau} g^{\pi}\left(x^{\tau}\right)^{-1}\right)^{\varepsilon}\right)^{\pi} = \left(\left(x\left(g^{\pi}\right)^{-1}\right)^{\tau} g^{\pi}\left(x^{\tau}\right)^{-1}\right)^{\varepsilon\pi} = \left(x\left(g^{\pi}\right)^{-1}\right)^{\tau} g^{\pi}\left(x^{\tau}\right)^{-1}.$$

Finally, this shows that $(g, f_g, \dots, f_g) \in A$ and $g = (g, f_g, \dots, f_g)^{\varphi}$, which proves that φ is surjective. $\square$

Let M be the kernel of φ. It can be shown that if L_λ splits over K_λ, then A splits over M. If $(g, f_1, \dots, f_m) \in M$, then $g = (g, f_1, \dots, f_m)^{\varphi} = 1$ and, for each $i \in \{1, \dots, m\}$, we have $(f_i(x))^{\pi} = (x(g^{\pi})^{-1})^{\tau} g^{\pi}(x^{\tau})^{-1} = 1$. Hence $f_i(x) \in K_\lambda$ for every $x \in S$. Therefore

$$M = \left\{(1, f_1, \dots, f_m) \;\middle|\; f_i : S \to K_\lambda,\ f_i \in \Omega \text{ for each } i \in \{1, \dots, m\}\right\} \cong K_\lambda^{|\Delta|m}.$$

Consider the subset C of A defined by

$$C = \left\{(g, f_1, \dots, f_m) \in A \;\middle|\; g \in L_\lambda\right\}. \tag{3}$$

Since φ is a homomorphism, C is a subgroup of A.

Lemma 18. *M is the core of C in A and $M \cong K_\lambda^{|\Delta|m}$. The action of A on the right cosets of C is permutation isomorphic to the action of L on Λ.*

Proof. From Lemma 17, the map φ is a surjective homomorphism and hence $A/M \cong L$. Furthermore, $C^{\varphi} = L_\lambda$. As $M \leqslant C \leqslant A$, $M \trianglelefteq A$ and L_λ is core-free in L, we see that M is the core of C in A. Finally, the action of A on the right cosets of C is permutation isomorphic to the action of $L = A^{\varphi}$ on the right cosets of $L_\lambda = C^{\varphi}$, that is, permutation isomorphic to the action of L on Λ. $\square$

Assume that m is odd. Let $c = (g, f_1, \dots, f_m) \in C$ and set

$$c^{\iota} = (g, f_1, \dots, f_m)^{\iota} = \left(f_1(1), f_1^{\iota_{1,c}}, \dots, f_m^{\iota_{m,c}}\right)$$

where

$$f_i^{\iota_{i,c}}(x) = \begin{cases} g & \text{if } x \in S_\delta \text{ and } i = 1, \\ f_{i+1}(x) & \text{if } x \in S_\delta,\ 2 \leqslant i \leqslant m-1 \text{ and } i \text{ is even}, \\ f_{i-1}(x) & \text{if } x \in S_\delta,\ 3 \leqslant i \leqslant m \text{ and } i \text{ is odd}, \\ f_{i-1}(x) & \text{if } x \notin S_\delta,\ 2 \leqslant i \leqslant m-1 \text{ and } i \text{ is even}, \\ f_{i+1}(x) & \text{if } x \notin S_\delta,\ 1 \leqslant i \leqslant m-2 \text{ and } i \text{ is odd}, \\ f_m(x) & \text{if } x \notin S_\delta \text{ and } i = m. \end{cases} \tag{4}$$

Note that, for each $i \in \{1, \dots, m\}$, the definition of $\iota_{i,c}$ depends on i and on the element c. In particular, to compute $f_i^{\iota_{i,c}}$ one needs to know i and the coordinates of c.

Lemma 19. *The map ι is an automorphism of C with $\iota^2 = 1$.*

Proof. Let $c = (g, f_1, \dots, f_m) \in C$. We have $g^\pi \in L_\lambda^\pi = S_\delta$ and hence $(g^\pi)^\tau = ((g^\pi)^{-1})^\tau = 1$. Therefore, for each $i \in \{1, \dots, m\}$, we obtain

$$(\dagger) \qquad f_i(1)^\pi = \left(\left(g^\pi\right)^{-1}\right)^\tau g^\pi \left(1^\tau\right)^{-1} = g^\pi.$$

In particular, $f_1(1) \in L_\lambda$ and $f_1(1)^\pi = g^\pi$.

We first show that $c^\iota \in C$. From (4), we see that, for each $i \in \{1, \dots, m\}$ and for each $x \in S$, the function $f_i^{\iota_{i,c}}$ is constant on the right coset $S_\delta x$ of S_δ in S and therefore $f_i^{\iota_{i,c}} \in \Omega$. Let $x \in S$. We have to show that, for each $i \in \{1, \dots, m\}$, we have $(f_i^{\iota_{i,c}}(x))^\pi = (x(f_1(1)^\pi)^{-1})^\tau f_1(1)^\pi (x^\tau)^{-1}$. In particular, as $f_1(1)^\pi = g^\pi$, we have to show that $(f_i^{\iota_{i,c}}(x))^\pi = (x(g^\pi)^{-1})^\tau g^\pi (x^\tau)^{-1}$. Since $g^\pi \in S_\delta$ and $(g, f_1, \dots, f_m) \in A$, this is clear from the definition of $f_i^{\iota_{i,c}}$, except possibly when $i = 1$ and $x \in S_\delta$. Hence, we assume that $i = 1$ and $x \in S_\delta$. In particular, $(x(g^\pi)^{-1})^\tau = x^\tau = 1$. Thus, we have

$$\left(f_1^{\iota_{1,c}}(x)\right)^\pi = g^\pi = \left(x\left(g^\pi\right)^{-1}\right)^\tau g^\pi \left(x^\tau\right)^{-1}.$$

Therefore, $c^\iota \in C$.

Now we show that ι^2 is the identity permutation of C. We have

$$c^{\iota^2} = \left(f_1(1), f_1^{\iota_{1,c}}, \dots, f_m^{\iota_{m,c}}\right)^\iota = \left(f_1^{\iota_{1,c}}(1), \left(f_1^{\iota_{1,c}}\right)^{\iota_{1,c^\iota}}, \dots, \left(f_m^{\iota_{m,c}}\right)^{\iota_{m,c^\iota}}\right).$$

Now, from (4) we have $f_1^{\iota_{1,c}}(1) = g$ and hence the first coordinate of c^{ι^2} equals the first coordinate of c. Furthermore,

$$\left(f_1^{\iota_{1,c}}\right)^{\iota_{1,c^\iota}} = \begin{cases} f_1(x) & \text{if } x \in S_\delta, \\ f_2^{\iota_{1,c}}(x) = f_1(x) & \text{if } x \notin S_\delta, \end{cases}$$

and hence the second coordinate of c^{ι^2} equals the second coordinate of c. Assume that $2 \leqslant i \leqslant m-1$ is even. We have

$$\left(f_i^{\iota_{1,c}}\right)^{\iota_{1,c^\iota}} = \begin{cases} f_{i+1}^{\iota_{1,c}}(x) = f_i(x) & \text{if } x \in S_\delta, \\ f_{i-1}^{\iota_{1,c}}(x) = f_i(x) & \text{if } x \notin S_\delta, \end{cases}$$

and hence the $(i+1)$th coordinate of c^{ι^2} equals the $(i+1)$th coordinate of c. The proof that $(f_i^{\iota_{i,c}})^{\iota_{i,c^\iota}} = f_i$ when $3 \leqslant i \leqslant m$ and i is odd is very similar to the previous case and is left to the reader.

Finally, we show that ι is an automorphism of C. Let $e : S \to L_\lambda$ be such that $e(x) = 1$ for every $x \in S$. Now, let $c = (g, f_1, \dots, f_m)$ and $c' = (g', h_1, \dots, h_m)$ be in C. We have

$$\begin{aligned} \left(cc'\right)^\iota &= \left(gg', f_1^{g'} h_1, \dots, f_m^{g'} h_m\right)^\iota \\ &= \left(\left(f_1^{g'} h_1\right)(1), \left(f_1^{g'} h_1\right)^{\iota_{1,cc'}}, \dots, \left(f_m^{g'} h_m\right)^{\iota_{m,cc'}}\right) \end{aligned}$$

and

$$c^\iota c'^\iota = \big(f_1(1), f_1^{\iota_{1,c}}, \ldots, f_m^{\iota_{m,c}}\big)\big(h_1(1), h_1^{\iota_{1,c'}}, \ldots, h_m^{\iota_{m,c'}}\big)$$
$$= \big(f_1(1)h_1(1), \big(f_1^{\iota_{1,c}}\big)^{h_1(1)} h_1^{\iota_{1,c'}}, \ldots, \big(f_m^{\iota_{m,c}}\big)^{h_1(1)} h_m^{\iota_{m,c'}}\big).$$

Recall that from (†), we have $h_1(1)^\pi = g'^\pi$. As $g'^\pi \in S_\delta$ and f_1 is constant on S_δ, we obtain $(f_1^{g'} h_1)(1) = f_1((g'^\pi)^{-1})h_1(1) = f_1(1)h_1(1)$ and hence the first coordinate of $(cc')^\iota$ equals the first coordinate of $c^\iota c'^\iota$. We have

$$\big(f_1^{g'} h_1\big)^{\iota_{1,cc'}}(x) = \begin{cases} gg' & \text{if } x \in S_\delta, \\ (f_2^{g'} h_2)(x) & \text{if } x \notin S_\delta, \end{cases}$$

and

$$\big(\big(f_1^{\iota_{1,c}}\big)^{h_1(1)} h_1^{\iota_{1,c'}}\big)(x)$$
$$= \begin{cases} f_1^{\iota_{1,c}}(x(h_1(1)^\pi)^{-1})h_1^{\iota_{1,c'}}(x) = gg' & \text{if } x \in S_\delta, \\ f_1^{\iota_{1,c}}(x(h_1(1)^\pi)^{-1})h_2(x) = f_2(x(g'^\pi)^{-1})h_2(x) = (f_2^{g'} h_2)(x) & \text{if } x \notin S_\delta. \end{cases}$$

Therefore $(f_1^{g'} h_1)^{\iota_{1,cc'}} = (f_1^{\iota_{1,c}})^{h_1(1)} h_1^{\iota_{1,c'}}$ and hence the second coordinate of $(cc')^\iota$ equals the second coordinate of $c^\iota c'^\iota$. Assume that $2 \leqslant i \leqslant m-1$ is even. We have

$$\big(f_i^{g'} h_i\big)^{\iota_{i,cc'}}(x) = \begin{cases} (f_{i+1}^{g'} h_{i+1})(x) & \text{if } x \in S_\delta, \\ (f_{i-1}^{g'} h_{i-1})(x) & \text{if } x \notin S_\delta, \end{cases}$$

and $((f_i^{\iota_{i,c}})^{h_1(1)} h_i^{\iota_{i,c'}})(x)$ equals

$$\begin{cases} f_i^{\iota_{i,c}}\big(x\big(h_1(1)^\pi\big)^{-1}\big)h_{i+1}(x) = f_{i+1}\big(x\big(g'^\pi\big)^{-1}\big)h_{i+1}(x) = \big(f_{i+1}^{g'} h_{i+1}\big)(x) & \text{if } x \in S_\delta, \\ f_i^{\iota_{i,c}}\big(x\big(h_1(1)^\pi\big)^{-1}\big)h_{i-1}(x) = f_{i-1}\big(x\big(g'^\pi\big)^{-1}\big)h_{i-1}(x) = \big(f_{i-1}^{g'} h_{i-1}\big)(x) & \text{if } x \notin S_\delta. \end{cases}$$

Therefore $(f_i^{g'} h_i)^{\iota_{i,cc'}} = (f_i^{\iota_{i,c}})^{h_1(1)} h_i^{\iota_{i,c'}}$. The proof that $(f_i^{g'} h_i)^{\iota_{i,cc'}} = (f_i^{\iota_{i,c}})^{h_1(1)} h_i^{\iota_{i,c'}}$ when $3 \leqslant i \leqslant m$ and i is odd is very similar to the previous case and is left to the reader. □

Define

$$B = C \rtimes \langle \iota \rangle. \tag{5}$$

Lemma 20. *If N is a subgroup of C with $N \trianglelefteq A$ and $N \trianglelefteq B$, then $N = 1$.*

Proof. From Lemma 18, the group M is the core of C in A. Since $N \trianglelefteq A$ and $N \leqslant C$, we get $N \leqslant M$ and hence every element of N is of the form $(1, f_1, \ldots, f_m)$ where $f_1, \ldots, f_m \in \Omega$.

We first prove the following preliminary claim.

Claim. *Let $i \in \{1, \ldots, m\}$ and let $s' \in S$. Assume that, for some element $(1, f_1, \ldots, f_m)$ of N, we have $f_i(s') = 1$. Then $f_i(s) = 1$ for every $s \in S$.*

Let $(1, f_1, \ldots, f_m) \in N$. Fix g an arbitrary element of L. From Lemma 17, there exist $h_1, \ldots, h_m \in \Omega$ with $(g, h_1, \ldots, h_m) \in A$. As $N \trianglelefteq A$,

$$(1, f_1, \ldots, f_m)^{(g, h_1, \ldots, h_m)} = \big(1, h_1^{-1} f_1^g h_1, \ldots, h_m^{-1} f_m^g h_m\big) \in N.$$

By hypothesis, $1 = (h_i^{-1} f_i^g h_i)(s') = h_i(s')^{-1} f_i(s'(g^\pi)^{-1})h_i(s')$ and hence $f_i(s'(g^\pi)^{-1}) = 1$. Since g is an arbitrary element of L and $\pi : L \to S$ is surjective, we obtain $f_i = 1$. □

We argue by contradiction and we assume that $N \neq 1$. Let j be the minimal element of $\{1,\dots,m\}$ such that N contains an element $(1, f_1,\dots, f_m)$ with $f_j \neq 1$. Let $x = (1, f_1,\dots, f_m)$ be an arbitrary element of N. Since $N \trianglelefteq B$ and $\iota \in B$, we have that

$$(\dagger) \qquad (1, f_1,\dots, f_m)^\iota = \bigl(f_1(1), f_1^{\iota_{1,x}},\dots, f_m^{\iota_{m,x}}\bigr)$$

lies in N and hence $f_1(1) = 1$. Since x is an arbitrary element of N, by applying the claim with $i = 1$ and $s' = 1$, we get $f_1 = 1$. This proves that $j > 1$. Assume that j is even. Let $s' \in S \setminus S_\delta$. From (4) and from the minimality of j, we see that $f_j^{\iota_{j,x}}(s') = f_{j-1}(s') = 1$. Since $N^\iota = N$ and x is an arbitrary element of N, by applying the claim with $i = j$, we get $f_j = 1$ for every element $(1, f_1,\dots, f_m)$ of N, contradicting the choice of j. Assume that j is odd. From (4) and from the minimality of j, we see that $f_j^{\iota_{j,x}}(1) = f_{j-1}(1) = 1$. Since x^ι is an arbitrary element of $N^\iota = N$, by applying the claim with $i = j$ and $s' = 1$, we have $f_j = 1$ for every element $(1, f_1,\dots, f_m)$ of N, again contradicting the minimality of j. This last contradiction concludes the proof. □

Remark 21. The definition of ι in (4) depends on the fact that we have chosen m odd. Nevertheless, when m is even, it is possible to define (in a similar fashion) an involutory automorphism of C satisfying Lemma 20.

Now we recall the definition of *coset graph*. For a group G, a subgroup A and an element $b \in G$, the coset graph $\mathrm{Cos}(G, A, b)$ is the graph with vertex set the set of right cosets $G/A = \{Ag \mid g \in G\}$ and edge set $\{\{Ag, Abg\} \mid g \in G\}$. The following proposition is due to Sabidussi [12].

Proposition 22. *Let A be a core-free subgroup of G and let $b \in G$ with $G = \langle A, b\rangle$ and $b^{-1} \in AbA$. Then $\Gamma = \mathrm{Cos}(G, A, b)$ is a connected G-arc-transitive graph and the action of the stabiliser G_v of the vertex v of Γ on $\Gamma(v)$ is permutation isomorphic to the action of A on the right cosets of $A \cap A^b$ in A.*

The following lemma is well known but we include a proof for sake of completeness.

Lemma 23. *Let A, B and C be finite groups with $C = A \cap B$ and $|B : C| = 2$. Assume that 1 is the only subgroup of C normal both in A and in B. Then there exists a finite group $\bar{G}$ and a connected $\bar{G}$-arc-transitive graph Γ such that the stabiliser $\bar{G}_v$ of the vertex v of Γ is isomorphic to A and the action of $\bar{G}_v$ on $\Gamma(v)$ is permutation isomorphic to the action of A on the right cosets of C in A.*

Proof. Let $G = A *_C B$ be the free product of A with B amalgamated over C. We identify A, B and C with their corresponding isomorphic copies in G. Fix $b \in B \setminus C$. Since G is the free product of A and B and since b normalises the subgroup C of B, we have $C = A \cap A^b$. It is shown in [1, Theorem 2] that G is a residually finite group. As B is a finite group and $AA^b = \{xy^b \mid x, y \in A\}$ is a finite set, G contains a normal subgroup of finite index N with $B \cap N = 1$ and $AA^b \cap N = 1$. Let $\bar{G} = G/N$ and denote by $- : G \to \bar{G}$ the natural projection (in the rest of the proof we use the bar convention, that is, we denote by $\bar{X}$ the image of X under $-$).

Since $G = \langle A, B\rangle = \langle A, b\rangle$, we have $\bar{G} = \langle \bar{A}, \bar{b}\rangle$. Clearly, $\bar{C} = \overline{A \cap A^b} \leqslant \bar{A} \cap \bar{A}^{\bar{b}}$. Let $\bar{x} \in \bar{A} \cap \bar{A}^{\bar{b}}$. We have $x \in AN \cap A^bN$ and hence $x = a_1n_1 = a_2^b n_2$ for some $a_1, a_2 \in A$ and $n_1, n_2 \in N$. Thus $a_1^{-1}a_2^b = n_1n_2^{-1} \in AA^b \cap N = 1$, $a_1 = a_2^b$ and $x \in (A \cap A^b)N = CN$. It follows that $\bar{x} \in \bar{C}$. This shows that $\bar{A} \cap \bar{A}^{\bar{b}} = \bar{C}$. Since there is no non-trivial normal subgroup of $A \cap A^b$ normal in both A and B, there is no non-trivial normal subgroup of $\bar{A} \cap \bar{A}^{\bar{b}}$ normal in both $\bar{A}$ and $\bar{B}$. Since $|B : C| = 2$, we have $b^{-1} \in Cb \subseteq Ab$ and $\bar{b}^{-1} \in \bar{A}\bar{b} \subseteq \bar{A}\bar{b}\bar{A}$. Since $A \cap N = 1$, the restriction of $-$ to A is an isomorphism from A to $\bar{A}$ mapping C to $\bar{C}$. Therefore the action of A on the right cosets of $A \cap A^b = C$ in A is permutation isomorphic to the action of $\bar{A}$ on the right cosets of $\bar{A} \cap \bar{A}^{\bar{b}}$ in $\bar{A}$. The conclusion then follows from Proposition 22 applied to $\bar{G}, \bar{A}$ and $\bar{b}$. □

Finally, we are ready to prove the main theorem of this section.

Proof of Theorem 4. Let L be graph-restrictive. Suppose, by contradiction, that L is not semiprimitive and let K be an intransitive normal subgroup of L which is not semiregular. Let m be an arbitrary odd positive integer, let A be as in (2), let C be as in (3) and let B be as in (5). From Lemma 20, we see that we may apply Lemma 23 to A, B and C. Therefore, there exists a G-arc-transitive graph Γ with the stabiliser G_v of the vertex v isomorphic to A and with the action of G_v on $\Gamma(v)$ permutation isomorphic to the action of A on the right cosets of C, which, by Lemma 18, is equivalent to the action of L on Λ. Thus (Γ, G) is locally-L. Now, using Lemma 18 again, we obtain that the kernel of the action of G_v on $\Gamma(v)$ is isomorphic to $K_\lambda^{|\Delta|m}$. Since K is not semiregular, we have $|K_\lambda| \neq 1$. Furthermore, as m is an arbitrary odd integer, we have that the size of $G_v^{[1]}$ cannot be bounded above by a function of L. Thus L is not graph-restrictive. $\square$

5. *p*-graph-restrictive groups

In this section using the following theorem of Glauberman we prove Corollary 11.

Theorem 24. *(See [7, Theorem 1].) Suppose P is a subgroup of a finite group G, $g \in G$, and $P \cap P^g$ is a normal subgroup of prime index p in P^g. Let n be a positive integer, and let $\tilde{G} = \langle P, P^g, \dots, P^{g^n} \rangle$. Assume that:*

(1) *g normalises no non-identity normal subgroup of P, and*
(2) *$P \cap Z(\tilde{G}) = 1$.*

Then $|P| = p^t$ for some positive integer t for which $t \leqslant 3n$ and $t \neq 3n - 1$. Moreover, if $n = 2$, $p = 2$ and $t = 6$, then P contains a non-identity normal subgroup of $\tilde{G}$.

Proof of Corollary 11. Let p be a prime and let L be a transitive permutation group on Ω. Let $x \in \Omega$ and let $\bar{P}$ be a Sylow p-subgroup of L_x such that $|\bar{P}| = p$, and there exists $\bar{l} \in L$ such that $\langle \bar{P}, \bar{P}^{\bar{l}} \rangle$ is transitive on Ω. We must show that L is p-graph-restrictive.

Let (Γ, G) be a locally-L pair, let (u, v) be an arc of Γ and let $K = G_v^{[1]}$ be the kernel of the action of G_v on the neighbourhood of v. Let P be a Sylow p-subgroup of G_{uv}. By hypothesis, we have $|P : K \cap P| = p$ and, moreover, there exists $l \in G_v$ with $\langle P, P^l \rangle$ transitive on $\Gamma(v)$.

Let R be a Sylow p-subgroup of KP^l containing $Q = K \cap P$. Note that $\langle P, R \rangle$ is transitive on $\Gamma(v)$ and hence $P \neq R$. Let $w = u^l \in \Gamma(v)$ and note that $KP^l \leqslant (G_{uv})^l = G_{wv}$ and hence R is a Sylow p-subgroup of G_{wv}. It follows that Q has index p in both P and R and, in particular, is normal in both of them. Since $P \neq R$, we have that $Q = P \cap R$.

Since Γ is G-arc-transitive, there exists $\sigma \in G$ such that $(u, v)^\sigma = (v, w)$. Since both P^σ and R are Sylow p-subgroups of G_{wv}, there exists $h \in G_{wv}$ such that $R = P^{\sigma h}$. Writing $g = \sigma h$, we get $R = P^g$ and $(u, v)^g = (v, w)$. As $\langle P, P^g \rangle = \langle P, R \rangle$ is transitive on $\Gamma(v)$, it follows that $\langle P^g, P^{g^2} \rangle$ is transitive on $\Gamma(w)$. Hence $\tilde{G} = \langle P, P^g, P^{g^2} \rangle$ is transitive on the edges of Γ and $\langle g, P \rangle$ is transitive on the arcs of Γ. Since $P \leqslant G_{uv}$, the element g normalises no non-identity normal subgroup of P, P contains no non-trivial normal subgroup of $\tilde{G}$ and $P \cap \mathrm{Z}(\tilde{G}) = 1$. We can then use Theorem 24 with $n = 2$ to conclude. $\square$

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