CORE
Metadata, citation and similar papers at core.ac.uk
Provided by Elsevier - Publisher Connector

Journal of Number Theory 129 (2009) 1976–1989

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Diophantine exponents of affine subspaces: The simultaneous approximation case

Yuqing Zhang

Brandeis University, Waltham, MA, United States

ARTICLE INFO

Article history:
Received 5 September 2008
Revised 6 February 2009
Available online 28 April 2009
Communicated by David Goss

ABSTRACT

We apply nondivergence estimates for flows on homogeneous spaces to compute Diophantine exponents of affine subspaces of $\mathbb{R}^n$ and their nondegenerate submanifolds.

© 2009 Elsevier Inc. All rights reserved.

1. Introduction

Given any $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$ we set its norm as follows:

$$\|\mathbf{y}\| = \max\{|y_1|, |y_2|, \ldots, |y_n|\}. \tag{1.1}$$

We may view $\mathbf{y}$ as a linear form and define its Diophantine exponent as

$$\omega(\mathbf{y}) = \sup\{v \mid \exists\infty \text{ many } \mathbf{q} \in \mathbb{Z}^n \text{ with } |\mathbf{qy} + p| < \|\mathbf{q}\|^{-v} \text{ for some } p \in Z\} \tag{1.2}$$

where $\mathbf{qy} = q_1y_1 + q_2y_2 + \cdots + q_ny_n$.

Alternatively we may define Diophantine exponent of $\mathbf{y}$ in the context of simultaneous approximation:

$$\sigma(\mathbf{y}) = \sup\{v \mid \exists\infty \text{ many } q \in \mathbb{Z} \text{ with } \|q\mathbf{y} + \mathbf{p}\| < |q|^{-v} \text{ for some } \mathbf{p} \in \mathbb{Z}^n\}. \tag{1.3}$$

It can be deduced from Dirichlet's Theorem [C] that

$$\sigma(\mathbf{y}) \geqslant \frac{1}{n}, \qquad \omega(\mathbf{y}) \geqslant n \quad \forall \mathbf{y} \in \mathbb{R}^n. \tag{1.4}$$

E-mail address: yuqingzh@gmail.com.

0022-314X/$ – see front matter © 2009 Elsevier Inc. All rights reserved.
doi:10.1016/j.jnt.2009.02.013

Khintchine's Transference Theorem (see Chapter V of [C] for instance) tells

$$\frac{\omega(\mathbf{y})-n+1}{n} \geqslant \sigma(\mathbf{y}) \geqslant \frac{1}{n-1+n/\omega(\mathbf{y})}, \quad \forall \mathbf{y} \in \mathbb{R}^n. \tag{1.5}$$

In particular $\sigma(\mathbf{y}) = \frac{1}{n}$ if and only if $\omega(\mathbf{y}) = n$. We call $\mathbf{y}$ not very well approximable if $\sigma(\mathbf{y}) = \frac{1}{n}$ and call $\mathbf{y}$ very well approximable otherwise. It is known that $\sigma(\mathbf{y}) = \frac{1}{n}$ and $\omega(\mathbf{y}) = n$ for a.e. $\mathbf{y}$, hence the set of not very well approximable vectors has full Lebesgue measure.

Following [K2] the Diophantine exponent $\omega(\mu)$ of a Borel measure μ is set to be the μ-essential supremum of the ω function, that is,

$$\omega(\mu) = \sup\bigl\{v \bigm| \mu\bigl\{\mathbf{y} \bigm| \omega(\mathbf{y}) > v\bigr\} > 0\bigr\}. \tag{1.6}$$

If M is a smooth submanifold of $\mathbb{R}^n$ and μ is the measure class of the Riemannian volume on M (more precisely put, μ is the pushforward $\mathbf{f}_*\lambda$ of λ by any smooth map $\mathbf{f}$ parameterizing M), then the Diophantine exponent of M, $\omega(M)$, is set to be equal to $\omega(\mu)$. In the spirit of (1.6) let us define

$$\sigma(M) = \sigma(\mu) \stackrel{\text{def}}{=} \sup\bigl\{v \bigm| \mu\bigl\{\mathbf{y} \bigm| \sigma(\mathbf{y}) > v\bigr\} > 0\bigr\}. \tag{1.7}$$

$\omega(M) \geqslant n$ and $\sigma(M) \geqslant \frac{1}{n}$ by Dirichlet's Theorem combined with (1.6) and (1.7). M is called extremal if $\sigma(M) = \frac{1}{n}$ or $\omega(M) = n$. A trivial example of an extremal submanifold of $\mathbb{R}^n$ is $\mathbb{R}^n$ itself.

K. Mahler [M] conjectured in 1932 that

$$M = \bigl\{\bigl(x, x^2, \ldots, x^n\bigr) \bigm| x \in \mathbb{R}\bigr\} \tag{1.8}$$

is an extremal submanifold. This was proved by Sprindžuk [Sp1] in 1964. The curve of (1.8) has a notable property that it does not lie in any affine subspace of $\mathbb{R}^n$. We might describe and formalize this property in terms of nondegeneracy condition as follows. Let $\mathbf{f} = (f_1, \ldots, f_n) : U \to \mathbb{R}^n$ be a differentiable map where U is an open subset of $\mathbb{R}^d$. $\mathbf{f}$ is called nondegenerate in an affine subspace L of $\mathbb{R}^n$ at $\mathbf{x} \in U$ if $\mathbf{f}(U) \subset L$ and the span of all the partial derivatives of $\mathbf{f}$ at $\mathbf{x}$ up to some order coincides with the linear part of L. If M is a d-dimensional submanifold of L we will say that M is nondegenerate in L at $\mathbf{y} \in M$ if any diffeomorphism of $\mathbf{f}$ between an open subset U of $\mathbb{R}^d$ and a neighborhood of $\mathbf{y}$ in M is nondegenerate in L at $\mathbf{f}^{-1}(\mathbf{y})$. We will say M is nondegenerate in L if it is nondegenerate in L at almost all points of M.

It was conjectured by Sprindžuk [Sp2] in 1980 that almost all points on a nondegenerate analytic submanifold of $\mathbb{R}^n$ are not very well approximable. In 1998 D. Kleinbock and G.A. Margulis proved that

Theorem 1. *(See [KM1].) Let M be a smooth nondegenerate submanifold of R^n, then M is extremal, i.e. almost all points of M are not very well approximable.*

[K1] studies the conditions under which an affine subspace is extremal and showed that an affine space is extremal if and only if its nondegenerate submanifolds are extremal. [K2] derives formulas for computing $\omega(L)$ and $\omega(M)$ when L is not extremal and M is an arbitrary nondegenerate submanifold in it. This breakthrough is achieved through sharpening of some nondivergence estimates in the space of unimodular lattices (see Lemmas 8 and 9 for review). [K2] proves that

Theorem 2. *(See Theorem 0.3 of [K2].) If L is an affine subspace of $\mathbb{R}^n$ and M is a nondegenerate submanifold in L, then*

$$\omega(M) = \omega(L) = \inf\bigl\{\omega(\mathbf{x}) \bigm| \mathbf{x} \in L\bigr\} = \inf\bigl\{\omega(\mathbf{x}) \bigm| \mathbf{x} \in M\bigr\}. \tag{1.9}$$

This paper goes on to compute Diophantine exponents of nonextremal subspaces in the σ context. We follow the strategy of associating Diophantine property of vectors with behavior of certain trajectories in the space of lattices. Combined with dynamics we use nondivergence estimates in its strengthened format (Lemma 8) to prove the following:

Theorem 3. *If L is an affine subspace of $\mathbb{R}^n$ and M is a nondegenerate submanifold in L, then*

$$\sigma(M) = \sigma(L) = \inf\{\sigma(\mathbf{x}) \mid \mathbf{x} \in L\} = \inf\{\sigma(\mathbf{x}) \mid \mathbf{x} \in M\}. \tag{1.10}$$

Theorem 3 shows that simultaneous Diophantine exponents of affine subspaces are inherited by their nondegenerate submanifolds. Though Theorems 2 and 3 look much alike, the latter cannot be deduced directly from the former. A simplified account for this can be found in (1.5). When $\omega(\mathbf{y}) > n$, $\frac{\omega(\mathbf{y})-n+1}{n} > \frac{1}{n-1+n/\omega(\mathbf{y})}$ and $\omega(\mathbf{y})$ might take on any value between the two fractions (we refer readers to [J] for such examples).

We will also compute explicitly Diophantine exponents of affine subspaces in terms of the coefficients of their parameterizing maps. One instance of our accomplishment is the derivation of $\sigma(L)$ where L is a hyperplane: Consider $L \subset \mathbb{R}^n$ parameterized by

$$(x_1, x_2, \dots, x_{n-1}) \to (a_1x_1 + \cdots + a_{n-1}x_{n-1} + a_n, x_1, \dots, x_{n-1}). \tag{1.11}$$

If we denote the vector $(a_1, \dots, a_n)$ by $\mathbf{a}$, then in Section 4 we will establish

Theorem 4. *For L as described in* (1.11)

$$\sigma(L) = \max\left\{1/n, \frac{\omega(\mathbf{a})}{n + (n-1)\omega(\mathbf{a})}\right\}. \tag{1.12}$$

The main result of this paper is actually much more general than Theorem 3. We will be considering maps from Besicovitch metric spaces endowed with Federer measures (we postpone definitions of terminology till Section 2). We will be able to include in our results measures of the form $\mathbf{f}_*\mu$ where μ satisfies certain decay conditions as discussed in [KLW].

In Section 4 we will also study examples where $\sigma(L)$ is determined by the coefficients of its parameterizing map in a more intricate manner. In Section 5 we will give an illustration as to how the process of ascertaining $\sigma(L)$ differs from that of ascertaining $\omega(L)$.

2. Quantitative nondivergence

We will study homogeneous dynamics and how these relate to Diophantine approximation of vectors. First we define the space of unimodular lattices as follows:

$$\Omega_{n+1} \stackrel{\text{def}}{=} \mathrm{SL}(n+1,\mathbb{R}) / \mathrm{SL}(n+1,\mathbb{Z}). \tag{2.1}$$

Ω_{n+1} is noncompact, and can be decomposed as

$$\Omega_{n+1} = \bigcup_{\epsilon>0} K_\epsilon \tag{2.2}$$

where

$$K_\epsilon = \{\Lambda \in \Omega_{n+1} \mid \|v\| \geqslant \epsilon \text{ for all nonzero } v \in \Lambda\}. \tag{2.3}$$

Each K_ϵ is compact by Mahler's compactness criterion (see [M]).

Remark 5. $\|\ \|$ can be either the maximum or Euclidean norm on $\mathbb{R}^{n+1}$ and both can be used for decomposing Ω_{n+1} into union of compact subspaces because for each $v=(v_1,\ldots,v_{n+1})$ there exist $C_1>0$ and $C_2>0$ such that $C_1\max\{|v_1|,\ldots,|v_{n+1}|\}\leqslant\sqrt{v_1^2+\cdots+v_{n+1}^2}\leqslant C_2\max\{|v_1|,\ldots,|v_{n+1}|\}$. We assume it to be the maximum here and extend to the space of discrete subgroups of $\mathbb{R}^{n+1}$. For nonzero Γ we let $\|\Gamma\|$ be the volume of the quotient space $\Gamma_{\mathbb{R}}/\Gamma$, where $\Gamma_{\mathbb{R}}$ is the $\mathbb{R}$ linear span of Γ. If $\Gamma=\{0\}$, we set $\|\Gamma\|=1$.

Next we set

$$\Sigma_v \stackrel{\text{def}}{=} \bigl\{\mathbf{y}\in\mathbb{R}^n \bigm| \exists\infty \text{ many } q\in\mathbb{Z} \text{ such that } \|q\mathbf{y}-\mathbf{p}\|<|q|^{-v}\bigr\}.$$

Obviously $\sigma(\mathbf{y})=\sup\{v \mid \mathbf{y}\in\Sigma_v\}$.

Set $g_t=\operatorname{diag}\{\underbrace{e^{t/n},e^{t/n},\ldots,e^{t/n}}_{n},e^{-t}\}\in\mathrm{SL}(n+1,\mathbb{R})$ with $t\geqslant 0$ and associate $\mathbf{y}\in\mathbb{R}^n$ with matrix

$$u_{\mathbf{y}}=\begin{pmatrix} I_n & \mathbf{y}\\ 0 & 1\end{pmatrix}. \tag{2.4}$$

Consider lattice

$$\left\{\begin{pmatrix} q\mathbf{y}+\mathbf{p}\\ q\end{pmatrix} \Bigm| q\in\mathbb{Z},\ \mathbf{p}\in\mathbb{Z}^n\right\}=u_{\mathbf{y}}\mathbb{Z}^{n+1}. \tag{2.5}$$

When we have g_t act on vectors in $u_{\mathbf{y}}\mathbb{Z}^{n+1}$ as defined by (2.5), the first n components will be expanded and the last one (q) will be contracted. A definitive correlation between $\sigma(\mathbf{y})$ and trajectory of certain lattices in Ω_{n+1} was proposed and proved in [K1]. This is a special case of Theorem 8.5 of [KM2] on logarithm laws.

Lemma 6. *Suppose we are given a set $E\in\mathbb{R}^2$ which is discrete and homogeneous with respect to positive integers, and take $a,b>0$, $v>a/b$. Define c by $c=\frac{bv-a}{v+1}$, then the following are equivalent*:

1. *$\exists(x,z)\in E$ with arbitrarily large $|z|$ such that $|x|\leqslant|z|^{-v}$;*
2. *$\exists$ arbitrarily large $t>0$ such that for some $(x,z)\in E$ one has $\max(e^{at}|x|,e^{-bt}|z|)\leqslant e^{-ct}$.*

In the light of Lemma 6, if we set $v>1/n$, $\mathbf{y}\in\mathbb{R}^n$ and $E=\{(\|q\mathbf{y}+\mathbf{p}\|,|q|)\mid q\in\mathbb{Z},\ \mathbf{p}\in\mathbb{Z}^n\}$, (1) of Lemma 6 is equivalent to

$$\sigma(\mathbf{y})\mathbf{y}\in\Sigma_v.$$

By setting $a=1/n$, $b=1$ and $\mathbb{R}_+=\{x\in\mathbb{R}\mid x\geqslant 0\}$ one sees (2) of Lemma 6 is equivalent to

$$g_tu_y\mathbb{Z}^{n+1}\notin K_{e^{-ct}} \quad \text{for an unbounded set of } t\in\mathbb{R}_+, \tag{2.6}$$

where $\|\ \|$ is the maximum norm and

$$c=\frac{v-1/n}{v+1} \quad\Leftrightarrow\quad v=\frac{1/n+c}{1-c}=\frac{1+nc}{n(1-c)}. \tag{2.7}$$

If, in compliance with the definition of $\sigma(\mathbf{y})$, we set

$$\gamma(\mathbf{y})=\sup\bigl\{c \bigm| g_tu_{\mathbf{y}}\mathbb{Z}^{n+1}\notin K_{e^{-ct}} \text{ for an unbounded set of } t\in\mathbb{R}_+\bigr\} \tag{2.8}$$

then by (2.7) we have

$$\sigma(\mathbf{y}) = \frac{1 + n\gamma(\mathbf{y})}{n(1 - \gamma(\mathbf{y}))}. \tag{2.9}$$

Suppose ν is a measure on $\mathbb{R}^n$, and $v \geqslant 1/n$, by (1.6) and what ensues $\sigma(\nu) \leqslant v$ if and only if

$$\nu(\Sigma_u) = 0 \quad \forall u > v. \tag{2.10}$$

(2.10) is equivalent to

$$\nu\big(\big\{\mathbf{y} \mid g_t u_{\mathbf{y}} \mathbb{Z}^{n+1} \notin K_{e^{-dt}} \text{ for an unbounded set of } t \in \mathbb{R}_+\big\}\big) = 0, \quad \forall d > c, \tag{2.11}$$

where c is related to v via fractions of (2.7).

(2.11) can be further simplified into

$$\nu\big(\big\{\mathbf{y} \mid g_t u_{\mathbf{y}} \mathbb{Z}^{n+1} \notin K_{e^{-dt}} \text{ for an unbounded set of } t \in \mathbb{N}\big\}\big) = 0, \quad \forall d > c. \tag{2.12}$$

By the Borel–Cantelli Lemma, a sufficient condition for $\sigma(\nu) \leqslant v$, or (2.12) is

$$\sum_{t=1}^{\infty} \nu\big(\big\{\mathbf{y} \mid g_t u_{\mathbf{y}} \mathbb{Z}^{n+1} \notin K_{e^{-dt}}\big\}\big) < \infty, \quad \forall d > c. \tag{2.13}$$

The following lemma, established in [K2], serves as a sharpening of quantitative nondivergence. First an assembly of relevant concepts from the same resource (to trace their historical development see also [KM1,KLW]).

A metric space X is called *N-Besicovitch* if for any bounded subset A and any family β of nonempty open balls of X such that each $x \in A$ is a center of some ball of β, there is a finite or countable subfamily $\{\beta_i\}$ of β covering A with multiplicity at most N. X is *Besicovitch* if it is *N-Besicovitch* for some N.

Let μ be a locally finite Borel measure on X, U an open subset of X with $\mu(U) > 0$. Following [KLW] we call μ *D-Federer* on U if

$$\sup_{\substack{x \in \operatorname{supp}\mu,\, r>0 \\ B(x,3r) \subset U}} \frac{\mu(B(x,3r))}{\mu(B(x,r))} < D.$$

μ is said to be *Federer* if for μ-a.e. $x \in X$ there exists a neighborhood U of x and $D > 0$ such that μ is *D-Federer* on U.

An important illustration of the above notions is that $\mathbb{R}^d$ is Besicovitch and λ, the Lebesgue measure is *Federer*. Many natural measures supported on fractals are also known to be *Federer* (see [K2] for technical details).

For a subset B of X and a function f from B to a normed space with norm $\|\ \|$, we define $\|f\|_B = \sup_{x \in B} \|f(x)\|$. If μ is a locally finite Borel measure on X and B a subset of X with $\mu(B) > 0$ $\|f\|_{\mu,B}$ is set to be $\|f\|_{B \cap \operatorname{supp}\mu}$.

A function $f : X \to \mathbb{R}$ is called (C,α)-good on $U \subset X$ with respect to μ if for any open ball B centered in $\operatorname{supp}\mu$ one has

$$\forall \varepsilon > 0 \quad \mu\big(\big\{x \in B \mid |f(x)| < \varepsilon\big\}\big) \leqslant C\left(\frac{\varepsilon}{\|f\|_{\mu,B}}\right)^{\alpha} \mu(B).$$

Roughly speaking a function is (C,α)-good if the set of points where it takes small value has small measure. In Lemma 8 we will see that functions of the form $\mathbf{x} \to \|h(\mathbf{x})\Gamma\|$, where Γ runs through subgroups of $\mathbb{Z}^{n+1}$, are (C,α)-good with uniform C and α.

Let $\mathbf{f} = (f_1, \dots, f_n)$ be a map from X to $\mathbb{R}^n$. Following [K2] we say that $(\mathbf{f}, \mu)$ is good at $x \in X$ if there exists a neighborhood V of x such that any linear combination of $1, f_1, \dots, f_n$ is (C,α)-good on V with respect to μ and $(\mathbf{f}, \mu)$ is good if $(\mathbf{f}, \mu)$ is good at μ-almost every point. Reference to measure will be omitted if $\mu = \lambda$, and we will simply say that $\mathbf{f}$ is good or good at x. For example polynomial maps are good. [K1] proved the following result:

Lemma 7. *Let L be an affine subspace of $\mathbb{R}^n$ and let $\mathbf{f}$ be a smooth map from U, an open subset of $\mathbb{R}^d$ to L which is nondegenerate at $\mathbf{x} \in U$, then $\mathbf{f}$ is good at $\mathbf{x}$.*

Furthermore if L is an affine subspace of $\mathbb{R}^n$ and $\mathbf{f}$ a map from X into L, following [K2] we say $(\mathbf{f}, \mu)$ is nonplanar in L at $x \in \operatorname{supp}\mu$ if L is equal to the intersection of all affine subspaces containing $\mathbf{f}(B \cap \operatorname{supp}\mu)$ for some open neighborhood B of x. $(\mathbf{f}, \mu)$ is nonplanar in L if $(\mathbf{f}, \mu)$ is nonplanar in L at μ-a.e. x. We skip saying μ when $\mu = \lambda$ and skip L if $L = \mathbb{R}^n$. From definition $(\mathbf{f}, \mu)$ is nonplanar if and only if for any open B of positive measure, the restrictions of $1, f_1, \dots, f_n$ to $B \cap \operatorname{supp}\mu$ are linearly independent over $\mathbb{R}$. Clearly nondegeneracy in L implies nonplanarity in L. Nondegenerate smooth maps from $\mathbb{R}^d$ to $\mathbb{R}^n$ as in Lemma 7 give typical examples of nonplanarity.

Let Γ be any discrete subgroup of $\mathbb{R}^k$, we denote by $rk(\Gamma)$ the rank of Γ when viewed as a $\mathbb{Z}$-module. We denote by $S_{n+1,j}$ the set of subgroups of order j in $\mathbb{Z}^{n+1}$ for $1 \leqslant j \leqslant n+1$.

Lemma 8. *Let $k, N \in \mathbb{N}$ and $C, D, \alpha, \rho > 0$ and suppose we are given an N-Besicovitch metric space X, a ball $B = B(x_0, r_0) \subset X$, a measure μ which is D-Federer on $\tilde{B} = B(x_0, 3^k r_0)$ and a map $h : \tilde{B} \to \mathrm{GL}_k(\mathbb{R})$. Assume the following two conditions hold*:

1. *$\forall \Gamma \subset \mathbb{Z}^k$, the function $x \to \|h(x)\Gamma\|$ is (C,α)-good on $\tilde{B}$ with respect to μ;*
2. *$\forall \Gamma \subset \mathbb{Z}^k$, $\|h(\cdot)\Gamma\|_{\mu,B} \geqslant \rho^{rk(\Gamma)}$.*

Then for any positive $\epsilon \leqslant \rho$ one has

$$\mu\big(\big\{x \in B \mid h(x)\mathbb{Z}^k \notin K_\epsilon\big\}\big) \leqslant kC\big(ND^2\big)^k \left(\frac{\epsilon}{\rho}\right)^\alpha \mu(B). \tag{2.14}$$

Historically one theorem of [KM1] established the above lemma in its weaker form:

Lemma 9. *Let $k, N \in \mathbb{N}$ and $C, D, \alpha, \rho > 0$ and suppose we are given an N-Besicovitch metric space X, a ball $B = B(x_0, r_0) \subset X$, a measure μ which is D-Federer on $\tilde{B} = B(x_0, 3^k r_0)$ and a map $h : \tilde{B} \to \mathrm{GL}_k(\mathbb{R})$. Assume the following two conditions hold*:

1. *$\forall \Gamma \subset \mathbb{Z}^k$, the function $x \to \|h(x)\Gamma\|$ is (C,α)-good on $\tilde{B}$ with respect to μ;*
2. *$\forall \Gamma \subset \mathbb{Z}^k$, $\|h(\cdot)\Gamma\|_{\mu,B} \geqslant \rho$.*

Then for any positive $\epsilon \leqslant \rho$ one has

$$\mu\big(\big\{x \in B \mid h(x)\mathbb{Z}^k \notin K_\epsilon\big\}\big) \leqslant kC\big(ND^2\big)^k \left(\frac{\epsilon}{\rho}\right)^\alpha \mu(B). \tag{2.15}$$

For the wide number-theoretic applications of Lemma 9 we refer readers to papers like [KM1, KLW], to name a few. [K2] proves with an inductive process that one can replace the second condition $\|h(\cdot)\Gamma\|_{\mu,B} \geqslant \rho$ of Lemma 9 with $\|h(\cdot)\Gamma\|_{\mu,B} \geqslant \rho^{rk(\Gamma)}$ and thus obtains a strengthening of nondivergence estimates as recorded in Lemma 8. Both [K2] and the present paper exploit Lemma 8 to get Diophantine exponents of nonextremal spaces.

Proposition 10. *Let X be a Besicovitch metric space, $B = B(x, r) \subset X$, μ a measure which is D-Federer on $\tilde{B} = B(x, 3^{n+1}r)$ for some $D > 0$ and $\mathbf{f}$ a continuous map from $\tilde{B}$ to $\mathbb{R}^n$. Take $c \geqslant 0$ and assume that*

1. *$\exists C, \alpha > 0$ such that all the functions $x \to \|g_t u_{\mathbf{f}(x)} \Gamma\|$, $\Gamma \subset \mathbb{Z}^{n+1}$ are (C, α)-good on $\tilde{B}$ with respect to μ;*
2. *for any $d > c$, $\exists T = T(d) > 0$ such that for any $t \geqslant T$ and any $\Gamma \subset \mathbb{Z}^{n+1}$ one has*

$$\|g_t u_{\mathbf{f}(\cdot)} \Gamma\|_{\mu, B} \geqslant e^{-rk(\Gamma)\, dt}. \tag{2.16}$$

Then $\sigma(\mathbf{f}_*(\mu|_B)) \leqslant v$, where $v = \frac{1/n + c}{1-c}$.

Proof. Apply Lemma 8 with $k = n + 1$, $\mu = \mathbf{f}_*(\mu|_B)$, $h(x) = g_t u_{\mathbf{f}(x)}$, $\rho = e^{-ct}$ and $\epsilon = e^{-dt}$. $d \geqslant c \Leftrightarrow \epsilon \leqslant \rho$. It follows that

$$\mu\left(\left\{x \in B \;\middle|\; h(x)\mathbb{Z}^{n+1} \notin K_{e^{-dt}}\right\}\right) \leqslant \mathrm{const} \cdot e^{-\alpha \frac{d-c}{2} t} \mu(B) \quad \forall t \geqslant T. \tag{2.17}$$

Hence

$$\sum_{t=1}^{\infty} \mu\left(\left\{x \in B \;\middle|\; h(x)\mathbb{Z}^{n+1} \notin K_{e^{-dt}}\right\}\right) < \infty \quad \forall d > c.$$

By previous discussion concerning (2.13), we conclude that $\sigma(\mathbf{f}_*(\mu|_B)) \leqslant v$ for $v = \frac{1/n + c}{1-c}$, as desired. $\square$

To get an appreciation of the purport of the proposition, let us turn to the consequences of one of the conditions failing to be met.

Lemma 11. *Let μ be a measure on a set $B \subset \mathbb{R}^n$, take $c > 0$, $v > 1/n$ and $c = \frac{v - 1/n}{v+1}$. Let $\mathbf{f}$ be a map from B to $\mathbb{R}^n$ such that* (2.16) *does not hold, then*

$$\mathbf{f}(B \cap \operatorname{supp} \mu) \subset \Sigma_u \quad \text{for some } u > v. \tag{2.18}$$

Proof. If (2.16) does not hold, $\exists j$ with $1 \leqslant j \leqslant n + 1$, a sequence $t_i \to \infty$ and a sequence of discrete subgroups $\Gamma_i \in S_{n+1, j}$ such that for some $d > c$

$$\forall x \in B \cap \operatorname{supp} \mu \quad \|g_t u_{\mathbf{f}(x)} \Gamma_i\| < e^{-j d t_i}. \tag{2.19}$$

By Minkowski's lemma, we have $\forall i$, $\forall x \in B \cap \operatorname{supp} \mu$ there exists nonzero vector $v \in g_{t_i} u_{\mathbf{f}(x)} \Gamma_i$ with $\|v\| \leqslant 2^j e^{-dt_i}$ therefore

$$g_{t_i} u_{\mathbf{f}(x)} \mathbb{Z}^{n+1} \notin K_{2^j e^{-dt_i}} \quad \text{for an unbounded set of } t. \tag{2.20}$$

Hence $\gamma(\mathbf{f}(x)) \geqslant d$ by (2.8) and $\sigma(\mathbf{f}(x)) \geqslant u$ for some $u > v$ by (2.9). $\square$

3. Applications and calculations

In this part we will utilize the theories established in Section 2 to get some tangible applications. Let L be an s-dimensional affine subspace of $\mathbb{R}^n$. Throughout we will parameterize it as

$$\mathbf{x} \to (\tilde{\mathbf{x}}A, \mathbf{x}) \tag{3.1}$$

where $\tilde{\mathbf{x}}$ stands for $(\mathbf{x}, 1)$, $\mathbf{x} \in \mathbb{R}^s$ and $A \in M_{s+1,n-s}$ where $M_{s+1,n-s}$ denotes the set of matrices of dimension $(s+1) \times (n-s)$.

We record the following observation:

Proposition 12. *Let L be an s-dimensional affine subspace of $\mathbb{R}^n$ described by (3.1), then*

$$\frac{1}{n} \leqslant \sigma(L) \leqslant \frac{1}{s}. \tag{3.2}$$

Proof. Note that $\sigma(\mathbf{y}) \geqslant \frac{1}{n}$ for all $\mathbf{y} \in L$ hence $\sigma(L) \geqslant \frac{1}{n}$.

Also by (3.1) for all $\mathbf{y} \in L$, $\sigma(\mathbf{y}) \leqslant \sigma(x_1, \ldots, x_s)$, hence $\sigma(L) \leqslant \sigma(\mathbb{R}^s) = \frac{1}{s}$. □

Although for any particular $\mathbf{y} \in L$, $\sigma(\mathbf{y})$ is determined by how L is parameterized, later development will show that $\sigma(L)$ is independent of parameterization. In brief, we are merely interested in whether a set is null or not, and that is unaltered under invertible linear transformations.

For any matrix $A \in M_{s+1,n-s}$ we define

$$\omega(A) = \sup\bigl\{v \bigm| \exists\infty \text{ many } \mathbf{q} \in \mathbb{Z}^{n-s} \text{ with } \|A\mathbf{q} + \mathbf{p}\| < \|\mathbf{q}\|^{-v} \text{ for some } \mathbf{p} \in \mathbb{Z}^{s+1}\bigr\}. \tag{3.3}$$

Comparing (3.3) with (1.3) and (1.2), we see that given vector $\mathbf{y} = (y_1, \ldots, y_n)$

$$\omega(A) = \omega(\mathbf{y}) \quad \text{if } A = \mathbf{y}, \qquad \omega(A) = \sigma(\mathbf{y}) \quad \text{if } A = \mathbf{y}^T. \tag{3.4}$$

Suppose $\mathbb{R}^{n+1}$ has standard basis $\mathbf{e}_1, \ldots, \mathbf{e}_{n+1}$, and if we extend the Euclidean structure of $\mathbb{R}^{n+1}$ to $\bigwedge^j(\mathbb{R}^{n+1}) = \bigotimes^j(\mathbb{R}^{n+1}) \backslash W_j$ where W_j is the subspace of j-tensors generated by transposition, then for all

$$I = \{i_1, i_2, \ldots, i_j\} \subset \{1, 2, \ldots, n+1\}, \quad i_1 < i_2 < \cdots < i_j$$

$\{\mathbf{e}_I \mid \mathbf{e}_I = \mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2} \wedge \cdots \wedge \mathbf{e}_{i_j},\ \#I = j\}$ form an orthogonal basis of $\bigwedge^j(\mathbb{R}^{n+1})$.

If a discrete subgroup $\Gamma \subset \mathbb{R}^{n+1}$ of rank j is viewed as a $\mathbb{Z}$-module with basis $\mathbf{v}_1, \ldots, \mathbf{v}_j$ then we may represent it by exterior product $\mathbf{w} = \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_j$. Observing $\|\Gamma\| = \|\mathbf{w}\|$, we will be able to compute $\|g_t u_{\mathbf{f}} \Gamma\|_{\mu,B}$ as in (2.16) directly.

Further computation shows (up to $\pm$ signs of permutations)

$$u_y \mathbf{e}_i = \begin{cases} \mathbf{e}_i & \text{if } i \neq n+1, \\ \sum_{i=1}^n y_j \mathbf{e}_j + \mathbf{e}_{n+1} & \text{if } i = n+1. \end{cases} \tag{3.5}$$

Hence according to properties of exterior algebra,

$$u_y \mathbf{e}_I = \begin{cases} \mathbf{e}_I & \text{if } n+1 \notin I, \\ \sum_{i=1}^n y_j \mathbf{e}_{I \backslash \{n+1\} \cup i} + \mathbf{e}_I & \text{if } n+1 \in I. \end{cases} \tag{3.6}$$

Therefore $\mathbf{w} \in \bigwedge^j(\mathbb{R}^{n+1})$ under left multiplication of $u_{\mathbf{y}}$ results in

$$u_{\mathbf{y}} \mathbf{w} = \pi(\mathbf{w}) + \sum_{i=1}^{n+1} C_i(\mathbf{w}) y_i \tag{3.7}$$

where

$$\pi(\mathbf{w}) = \sum_{\substack{\#I = j \\ n+1 \in I}} \langle \mathbf{e}_I, \mathbf{w} \rangle \mathbf{e}_I, \tag{3.8}$$

$$C_i(\mathbf{w}) = \sum_{\substack{i\in I\\ I\subset\{1,2,\ldots,n\}}} \langle \mathbf{e}_{I\setminus\{i\}\cup\{n+1\}}, \mathbf{w}\rangle \mathbf{e}_I, \quad 1 \leqslant i \leqslant n,$$

$$C_{n+1}(\mathbf{w}) = \sum_{I\subset\{1,2,\ldots,n\}} \langle \mathbf{e}_I, \mathbf{w}\rangle \mathbf{e}_I, \quad y_{n+1} = 1. \tag{3.9}$$

Note that $\sum_{i=1}^{n+1} C_i(\mathbf{w}) y_i$ denotes the image of $u_{\mathbf{y}}\mathbf{w}$ under the projection from $\bigwedge^j(\mathbb{R}^{n+1})$ to $\bigwedge^j(V)$, where V is the space spanned by $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$. Apparently $\bigwedge^j(V)$ is orthogonal to $\pi(\mathbf{w})$.

$$g_t u_{\mathbf{y}} \mathbf{w} = e^{-\frac{n+1-j}{n}} \pi(\mathbf{w}) + e^{\frac{jt}{n}} \sum_{i=1}^{n+1} C_i(\mathbf{w}) y_i. \tag{3.10}$$

(3.10) shows that g_t action tends to contract the $\pi(\mathbf{w})$ part while extracting its orthogonal complement. As for the norm, up to some constant,

$$\|g_t u_{\tilde{f}} \mathbf{w}\| = \max\bigl(e^{-\frac{n+1-j}{n}} \|\pi(\mathbf{w})\|, e^{\frac{jt}{n}} \|\tilde{f}(\ )C(\mathbf{w})\|\bigr) \tag{3.11}$$

where $\tilde{f} = (f_1, \ldots, f_n, 1)$, and

$$C(\mathbf{w}) = \begin{pmatrix} C_1(\mathbf{w}) \\ C_2(\mathbf{w}) \\ \vdots \\ C_{n+1}(\mathbf{w}) \end{pmatrix}.$$

Denote by $\Theta_{\mu,B}$ the $\mathbb{R}$-linear span of the restriction of $(f_1, \ldots, f_n, 1)$ to $B \cap \operatorname{supp}\mu$. Suppose $\Theta_{\mu,B}$ has dimension $s+1$. Let $\mathbf{g} = (g_1, \ldots, g_s, 1)$ be a basis of the above space, then $\exists R \in M_{s+1,n+1}$ such that $\tilde{f} = \mathbf{g}R$. $\|\tilde{f}C(\mathbf{w})\| = \|\mathbf{g}RC(\mathbf{w})\|$. As the elements of $\mathbf{g}$ are independent, up to some constant

$$\|\tilde{f}C(\mathbf{w})\| = \|RC(\mathbf{w})\|.$$

(2.16) is equivalent to $\forall d > c$, $\exists T$ such that $\forall t \geqslant T$, $\forall j = 1, \ldots, n+1$ and $\forall \mathbf{w} \in S_{n+1,j}$ one has

$$\max\bigl(e^{-\frac{n+1-j}{n}} \|\pi(\mathbf{w})\|, e^{\frac{jt}{n}} \|RC(\mathbf{w})\|\bigr) \geqslant e^{-jdt}. \tag{3.12}$$

We may restate (3.12) in the language of Lemma 6 in the following manner.

Set $E = \{(\|RC(\mathbf{w})\|, \|\pi(\mathbf{w})\|) \mid \mathbf{w} \in S_{n+1,j}\}$ which is discrete and homogeneous with respect to positive integers.

Set $a = \frac{j}{n}$, $b = \frac{n+1-j}{n}$, then (3.12) means $\forall c > c_0 = j\frac{v-n}{v+1}$ the second assumption of Lemma 6 does not hold for large enough $\|\pi(\mathbf{w})\|$.

This, by the same lemma, is equivalent to the first assumption not being met with v replaced by any number greater than

$$\frac{a+c_0}{b-c_0} = \frac{jv}{v+1-jv}. \tag{3.13}$$

Therefore (3.12) becomes equivalent to $\forall j = 1, 2, \ldots, n$, $\forall u > \frac{jv}{v+1-jv}$ and $\forall \mathbf{w} \in S_{n+1,j}$ with large enough $\|\pi(\mathbf{w})\|$ one has

$$\|RC(\mathbf{w})\| > \|\pi(\mathbf{w})\|^{-u}. \tag{3.14}$$

Now we will adopt strategies devised in [K2] to prove Theorem 3. Let $L = \tilde{f}(B \cap \operatorname{supp}\mu)$. Suppose L has dimension s and $\mathbf{h}: \mathbb{R}^s \to L$ is an affine isomorphism then $\exists R \in M_{s+1,n+1}$ such that

$$(h_1, h_2, \dots, h_n, 1)(x) = (x_1, x_2, \dots, x_s, 1)R, \quad \forall \mathbf{x} \in \mathbb{R}^s. \tag{3.15}$$

Theorem 13. *Let μ be a Federer measure on a Besicovitch metric space X, L an affine subspace of $\mathbb{R}^n$, and let $f: X \to L$ be a continuous map which is (f, μ)-good and (f, μ)-nonplanar in L. Then the following statements are equivalent for $v \geqslant 1/n$:*

1. $\{x \in \operatorname{supp}\mu \mid f(x) \notin \Sigma_u\}$ *is nonempty for any $u > v$;*
2. $\sigma(f_*\mu) \leqslant v$;
3. (3.14) *holds for any R satisfying* (3.15).

Proof. Suppose the second statement holds, then the set in the first statement has full measure hence is nonempty.

If the third statement holds previous discussion shows that (3.14) $\Leftrightarrow$ (3.12) $\Leftrightarrow$ (2.16). We may apply Proposition 10 to get the second statement.

If the third statement fails to hold, then no ball B intersecting $\operatorname{supp}\mu$ satisfies (2.16). By Lemma 11 $f(B \cap \operatorname{supp}\mu) \subset \Sigma_u$ for some $u > v$. This would undermine the first statement. □

From Theorem 13 we see that $\sigma(L) \leqslant \inf\{\sigma(\mathbf{y}) \mid \mathbf{y} \in L\}$ because the first statement implies the second one. $\sigma(L) \geqslant \inf\{\sigma(\mathbf{y}) \mid \mathbf{y} \in L\}$ is apparent by definition.

$\sigma(L)$ is inherited by its nondegenerate submanifolds because nondegeneracy implies (f, μ)-goodness and (f, μ)-nonplanarity by previous conceptual discussions. Therefore $\sigma(L) = \sigma(M) = \inf\{\sigma(\mathbf{y}) \mid \mathbf{y} \in L\} = \inf\{\sigma(\mathbf{y}) \mid \mathbf{y} \in M\}$ and Theorem 3 is established.

Besides, Theorem 13 establishes that

$$\sigma(L) = \sup\bigl\{v \bigm| (2.16) \text{ does not hold}\bigr\}. \tag{3.16}$$

More significantly, it yields a more general result than Theorem 3.

Theorem 14. *Let μ be a Federer measure on a Besicovitch metric space X, L an affine subspace of $\mathbb{R}^n$, and let $f: X \to L$ be a continuous map such that (f, μ) is good and nonplanar in L then*

$$\sigma(f_*\mu) = \sigma(L) = \inf\bigl\{\sigma(\mathbf{y}) \bigm| \mathbf{y} \in L\bigr\} = \inf\bigl\{\sigma\bigl(f(x)\bigr) \bigm| x \in \operatorname{supp}\mu\bigr\}. \tag{3.17}$$

[KLW], for instance, studied 'absolutely decaying and Federer' measures and proved that if μ is absolutely decaying and Federer, and f is nondegenerate at μ-a.e. points of $\mathbb{R}^d$, then (f, μ) is good and nonplanar. Theorem 14 is applicable to such generalized situations.

To make all this more explicit, first note that for an affine subspace L of dimension s matrix A as described in (3.1) can be read from matrix R as in (3.15) and vice versa, since

$$R = (\,A \quad I_{s+1}\,). \tag{3.18}$$

Set

$$\begin{aligned} &\sigma_j(A) \ (1 \leqslant j \leqslant n+1) \\ &\quad = \sup\Bigl\{v \Bigm| \exists \mathbf{w} \in S_{n+1,j} \text{ with arbitrarily large } \bigl\|\pi(\mathbf{w})\bigr\| \text{ and } \bigl\|RC(\mathbf{w})\bigr\| < \bigl\|\pi(\mathbf{w})\bigr\|^{-\frac{jv}{v+1-jv}}\Bigr\} \end{aligned} \tag{3.19}$$

and we derive

Corollary 15. *If L is an s-dimensional affine subspace of $\mathbb{R}^n$ parameterized by* (3.1)*, then*

$$\sigma(L) = \max\{1/n, \sigma_1(A), \sigma_2(A), \ldots, \sigma_n(A)\}. \tag{3.20}$$

Proof. Note (3.14) ⇔ (3.12) ⇔ (2.16) then apply (3.16).

It remains to elucidate $\sigma_{n+1}(A)$. $\bigwedge^{n+1}(\mathbb{R}^{n+1})$ is spanned by a single element $\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{n+1}$, hence $\|g_t u_{\tilde{f}} \mathbf{w}\|_{B,\mu}$ has a positive lower bound. (2.16) is always met as long as $v > 1/n$ for $j = n+1$, and we replace $\sigma_{n+1}(A)$ with $1/n$ in (3.20). □

4. Several examples

Corollary 15 will prove to be effective for deriving explicit formulas of $\sigma(L)$. First we have

Theorem 16. $\sigma_n(A) = \frac{\omega(A)}{n+(n-1)\omega(A)}$.

Proof. For $\mathbf{w} \in S_{n+1,n}$,

$$\mathbf{w} = \sum_{j=1}^{n+1} x_j \mathbf{e}_{T \smallsetminus \{j\}} \tag{4.1}$$

where $T = \{1, 2, \ldots, n, n+1\}$ and $x_j \in \mathbb{Z}$.

Therefore

$$C_j(\mathbf{w}) = x_j \mathbf{e}_{T \smallsetminus \{n+1\}}, \quad 1 \leqslant j \leqslant n+1, \tag{4.2}$$

$\pi(\mathbf{w}) = \sum_{j=1}^n x_j \mathbf{e}_{T \smallsetminus \{j\}}$, $\|\pi(\mathbf{w})\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$,

$$\|RC(\mathbf{w})\| = \left\| \begin{pmatrix} A & I_{s+1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n+1} \end{pmatrix} \right\| = \left\| \begin{pmatrix} x_{n-s+1} \\ \vdots \\ x_n \\ x_{n+1} \end{pmatrix} + A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-s} \end{pmatrix} \right\|.$$

$\sigma_n(A)$ is consequently equal to the supremum of v such that there exists $\mathbf{x} = (x_1, \ldots, x_{n+1}) \in Z^{n+1}$ with arbitrarily large $\sqrt{x_1^2 + \cdots + x_n^2}$ and

$$\left\| \begin{pmatrix} x_{n-s+1} \\ \vdots \\ x_n \\ x_{n+1} \end{pmatrix} + A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-s} \end{pmatrix} \right\| < \left(\sqrt{x_1^2 + \cdots + x_n^2}\right)^{-\frac{nv}{v+1-nv}}. \tag{4.3}$$

In order for $\|RC(\mathbf{w})\|$ to be sufficiently small, left side of the inequality of (4.3) has to be less than 1, hence $(x_{n-s+1}, \ldots, x_n, x_{n+1}) \in \mathbb{Z}^{s+1}$ are determined by $(x_1, x_2, \ldots, x_{n-s}) \in \mathbb{Z}^{n-s}$. Up to some constant

$$\sqrt{x_1^2 + \cdots + x_n^2} \asymp \sqrt{x_1^2 + \cdots + x_{n-s}^2}. \tag{4.4}$$

Now that the definitions of $\sigma_n(A)$ and $\omega(A)$ differ only by the exponents, we conclude that

$$\frac{n\sigma_n(A)}{1 + (n-1)\sigma_n(A)} = \omega(A) \quad \Rightarrow \quad \sigma_n(A) = \frac{\omega(A)}{n + (n-1)\omega(A)}. \qquad \square \tag{4.5}$$

Next we set out to prove Theorem 4 noting that we will be able to eliminate all $\sigma_j(A)$ for $j<n$ and only $\sigma_n(A)$ matters.

Proof of Theorem 4. The aforementioned matrix R satisfies $(a_1x_1+\cdots+a_{n-1}x_{n-1}+a_n, x_1, \dots, x_{n-1}, 1) = (x_1, \dots, x_{n-1}, 1)R$. Therefore

$$R = \begin{pmatrix} A & I_n \end{pmatrix},$$

$$A = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix},$$

$$\bigl\|RC(\mathbf{w})\bigr\| = \max\bigl\{\bigl\|C_2(\mathbf{w}) + a_1C_1(\mathbf{w})\bigr\|, \dots, \bigl\|C_{n+1}(\mathbf{w}) + a_nC_1(\mathbf{w})\bigr\|\bigr\}. \tag{4.6}$$

We claim that $\sigma_j(A)$ $(1 \leqslant j \leqslant n-1)$ remain zero. To see this, consider $\mathbf{w} \in S_{n+1,j}$: as the norm of $\pi(\mathbf{w})$ is tending to ∞ it suffices to show that $\|RC(\mathbf{w})\| > \epsilon$ for some $\epsilon > 0$ for large $\|\pi(\mathbf{w})\|$.

Suppose not and assume first that $a_1, a_2, \dots, a_{n-1}$ are all nonzero. Recall

$$\pi(\mathbf{w}) = \sum_{\substack{\#I=j \\ n+1\in I}} \langle \mathbf{e}_I, \mathbf{w}\rangle \mathbf{e}_I,$$

$$C_j(\mathbf{w}) = \sum_{\substack{j\in I \\ I\subset\{1,2,\dots,n\}}} \langle \mathbf{e}_{I\setminus\{j\}\cup\{n+1\}}, \mathbf{w}\rangle \mathbf{e}_I, \quad 1 \leqslant j \leqslant n,$$

$$C_{n+1}(\mathbf{w}) = \sum_{I\subset\{1,2,\dots,n\}} \langle \mathbf{e}_I, \mathbf{w}\rangle \mathbf{e}_I.$$

Each term in $C_1(\mathbf{w})$ is of the form $\langle \mathbf{e}_{I\setminus\{1\}\cup\{n+1\}}, \mathbf{w}\rangle \mathbf{e}_I$, $1 \in I$. For an arbitrary one, since the index set I has $i<n$ elements, $\exists k>1$ such that $k \notin I$.

Consider $\|C_k(\mathbf{w}) + a_{k-1}C_1(\mathbf{w})\|$ (it cannot have a positive lower bound by assumption), $\|C_k(\mathbf{w}) + a_{k-1}C_1(\mathbf{w})\| \geqslant \|a_{k-1}\langle \mathbf{e}_{I\setminus\{1\}\cup\{n+1\}}, \mathbf{w}\rangle \mathbf{e}_I\|$, therefore $\langle \mathbf{e}_{I\setminus\{1\}\cup\{n+1\}} \mathbf{w}\rangle \mathbf{e}_I$ must be equal to zero. Consequently $C_1(\mathbf{w}) = 0$. $C_i(\mathbf{w}) = 0$ $(2 \leqslant i \leqslant n)$ are forced to be zero by (4.6). This contradicts the fact that $\pi(\mathbf{w})$ is nonzero.

For arbitrary $A^t = (a_1, \dots, a_t, 0, \dots, 0, a_n)$ first note that according to (4.6) we have $C_i(\mathbf{w}) = 0$ $(t+1 \leqslant i \leqslant n)$ or $\|RC(\mathbf{w})\|$ cannot be arbitrarily small.

$$\bigl\|RC(\mathbf{w})\bigr\| \geqslant \max\bigl\{\bigl\|C_2(\mathbf{w}) + a_1C_1(\mathbf{w})\bigr\|, \dots, \bigl\|C_{t+1}(\mathbf{w}) + a_tC_1(\mathbf{w})\bigr\|\bigr\}. \tag{4.7}$$

Employing previous analysis on (4.7) shows that $C_i(\mathbf{w}) = 0$ $(1 \leqslant i \leqslant t)$ hence $\pi(\mathbf{w})$ has to be zero.

Therefore $\sigma_j(A) = 0$ for $j<n$. Combining Corollary 15 and Theorem 16, Theorem 4 is established. □

Remark 17. When A^t is of a special form $(0, 0, \dots, 0, a)$ our conclusion coincides with Satz 3 of [J]. The latter proved this special case with elementary method. This method, however, is not easily adjustable to more general situations.

In the next theorem we will study other subspaces than hyperplanes which highlight $\sigma_j(A)$ with $j<n$.

Theorem 18. *Consider a line $(L) \subset \mathbb{R}^3$ that passes through the origin parameterized by $x \to (ax, bx, x)$. Set $\mathbf{y} = (a, b) \in \mathbb{R}^2$, then*

$$\sigma(L) = \max\left\{1/3, \frac{\sigma(\mathbf{y})}{2+\sigma(\mathbf{y})}, \frac{\omega(\mathbf{y})}{3+2\omega(\mathbf{y})}\right\}. \tag{4.8}$$

Proof. Because $A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ in this case, by Corollary 15 and Theorem 16 we only need to prove $\sigma_1(A) = 0$ and $\sigma_2(A) = \frac{\sigma(\mathbf{y})}{2+\sigma(\mathbf{y})}$.

For $\mathbf{w} \in S_{4,1}$, $\mathbf{w}$ can be expressed as $x_1\mathbf{e}_1 + \cdots + x_4\mathbf{e}_4$ with $x_i \in \mathbb{Z}$.

$$\pi(\mathbf{w}) = x_4\mathbf{e}_4,$$

$$C_j(\mathbf{w}) = x_4\mathbf{e}_j, \quad 1 \leqslant j \leqslant 3, \qquad C_4(\mathbf{w}) = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3,$$

$$\left\|RC(\mathbf{w})\right\| = \left\|(A \quad I_2)\, C(\mathbf{w})\right\| \geqslant |x_4| = \left\|\pi(\mathbf{w})\right\|.$$

By (3.19), $\sigma_1(A)$ is zero as $\|RC(\mathbf{w})\|$ becomes unbounded. Moreover

$$\mathbf{w} \in S_{4,2} = \sum_{1 \leqslant i < j \leqslant 4} x_{i,j}\mathbf{e}_i \wedge \mathbf{e}_j, \quad x_{i,j} \in \mathbb{Z}. \tag{4.9}$$

We have by (3.19) $\sigma_2(A)$ equal to the supremum of v such that there exists $(p_1, p_2, q) \in Z^3$ with arbitrarily large $|q|$ and

$$\left\| \begin{matrix} qa + p_1 \\ qb + p_2 \end{matrix} \right\| < |q|^{-\frac{2v}{v+1-2v}}. \tag{4.10}$$

Hence $\sigma_2(A) = \frac{\sigma(\mathbf{y})}{2+\sigma(\mathbf{y})}$ as desired. $\square$

5. Further remarks

We study one low dimension example to see some distinction between $\sigma(L)$ and $\omega(L)$. Let $L = \{(x, a) \mid x \in \mathbb{R}\}$ with $\sigma(a) > 2$. From definition we know at once that $\omega(x, a) \geqslant \sigma(a)$ $\forall x$, hence

$$\omega(L) \geqslant \sigma(a). \tag{5.1}$$

[K2] by using dynamics showed that the lower bound was actually attained

$$\omega(L) = \sigma(a). \tag{5.2}$$

From (1.5) and (5.2) we derive that $\sigma(L) \geqslant \frac{1}{1+2/\sigma(a)}$. However there seems to be no way to know the exact value of $\sigma(L)$ simply from results of $\omega(L)$.

On the other hand from Theorem 4 we derive that

$$\sigma(L) = \frac{1}{1+2/\sigma(a)}. \tag{5.3}$$

It turns out (5.3) suffices to generate the exact value of $\omega(L)$ if we consider the following argument: by (1.5)

$$\sigma(L) \geqslant \frac{1}{1+2/\omega(L)}. \tag{5.4}$$

By (5.3) and (5.1) and the fact that $f(x) = \frac{1}{1+2/x}$ is increasing we have

$$\sigma(L) \leqslant \frac{1}{1 + 2/\omega(L)}. \tag{5.5}$$

(5.4) and (5.5) show that $\sigma(L) = \frac{1}{1+2/\omega(L)}$. Comparing this with (5.3) we see that $\omega(L) = \sigma(a)$ as desired.

Acknowledgment

The author is grateful to Professor Kleinbock for helpful discussions.

References

[C] J. Cassels, An Introduction to Diophantine Approximation, Cambridge Tracts in Math., vol. 45, Cambridge Univ. Press, Cambridge, 1957.
[J] V. Jarnik, Eine Bemerkung zum Übertragimgssatz, Büalgar. Akad. Nauk Izv. Mat. Inst. (3) (1959) 169–175.
[K1] D. Kleinbock, Extremal subspaces and their submanifolds, Geom. Funct. Anal. 13 (2003) 437–466.
[K2] D. Kleinbock, An extension of quantitative nondivergence and application to Diophantine exponents, Trans. Amer. Math. Soc. 360 (2008) 6497–6523.
[KLW] D. Kleinbock, E. Lindenstrauss, B. Weiss, On fractal measures and Diophantine approximation, Selecta Math. 10 (2004) 479–523.
[KM1] D. Kleinbock, G.A. Margulis, Flows on homogeneous spaces and Diophantine approximation on manifolds, Ann. of Math. 148 (1998) 339–360.
[KM2] D. Kleinbock, G.A. Margulis, Logarithm laws for flows on homogeneous spaces, Invent. Math. 138 (1999) 451–494.
[M] K. Mahler, Über das Mass der Menge aller S-ZAHLEN, Math. Ann. 106 (1932) 131–139.
[Sp1] V. Sprindžuk, More on Mahler's conjecture, Dokl. Akad. Nauk SSSR 155 (1964) 54–56 (in Russian).
[Sp2] V. Sprindžuk, Achievements and problems in Diophantine approximation theory, Russian Math. Surveys 35 (1980) 1–80.