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Representations versus numberings: on the relationship of two computability notions

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Abstract

This paper gives an answer to Weihrauch's (Computability, Springer, Berlin, 1987) question whether and, if not always, when an effective map between the computable elements of two represented sets can be extended to a (partial) computable map between the represented sets. Examples are known showing that this is not possible in general. A condition is introduced and for countably based topological T_0-spaces it is shown that exactly the (partial) effective maps meeting the requirement are extendable. For total effective maps the extra condition is satisfied in the standard cases of effectively given separable metric spaces and continuous directed-complete partial orders, in which the extendability is already known. In the first case a similar result holds also for partial effective maps, but not in the second. © 2001 Elsevier Science B.V. All rights reserved.

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0. Introduction

In a series of papers Weihrauch has developed a general approach to study constructivity in analytical mathematics (cf. e.g. [32]). The essential insight was that all sets studied there are equipped with some notion of approximation and that in important cases their elements can be *represented* as limits of certain sequences of points or (open) subsets approximating them. Moreover, in these cases the elements of the sequences can be coded by natural numbers in an easy way. Thus, every element of

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the space can be *named* by an arithmetic function, namely the one which generates the approximating sequence. The appropriate morphisms of such spaces are those maps which are defined by a computable transformation of the approximating sequences, i.e., by a recursive operator on the name space. They are called computable.

Elements that can be approximated by a computable sequence, i.e., which can be named by a total recursive function, are considered as computable. By coding the programs that compute the sequences (functions) one obtains a canonical indexing of the computable points. Indexed sets have been studied in great depth by Mal'cev [18] and Eršov [10–12]. The canonical morphisms of such sets are those maps which are defined by a recursive function on the indices. In our case this means that they are defined by operations which effectively transform the programs generating the approximating sequences. These morphisms are called effective.

As it is easy to see, the restriction of a computable map to the subspace of all computable points is effective. For certain special cases such as effectively given separable metric spaces and continuous directed-complete partial orders it is known that the converse of this implication is also true, i.e., every (total) effective map on the computable elements can be extended to a (partial) computable map. In [32] Weihrauch asks whether and, if not always, when such an extension result holds in general.

From the literature, examples are known which show that in general effective maps are not extendable to a computable map (cf. [13, 15, 17, 22, 35, 36]). Thus, the computability notion for mappings between represented spaces is stronger than the computability notion for maps between indexed spaces. The reason is that in the first case computability involves continuity, which is not true, in general, in the second case.

As a consequence of a general result in [28] it follows under a fairly general and natural assumption that exactly those *total* effective maps between the subspaces of all computable elements of a countably based T_0-space, which contains an effectively presented dense subset and satisfies some further condition, and another countably based T_0-space have a partial computable extension that possess a *witness for noninclusion*, which means: if some basic open set in the domain is *not* mapped into some basic open set in the codomain, then we must be able to effectively find a witness for this, that is, an element of the basic open set in the domain which is mapped outside the basic open set in the codomain. The additional condition on the domain space requires that the identity map on the subspace of computable points has a witness for noninclusion. We say in this case that the space has a realizer for noninclusion.

By slightly modifying the notion of having a witness for noninclusion given in [28], the present paper extends this result to the case of partial maps, thus improving results in the preliminary version [27]. As is shown in [30, 28], *total* effective maps always have a witness for noninclusion, if their domain of definition is domain-like, or their codomain is an effectively given separable metric space and the domain contains an effectively presented dense subset. By this way, the before mentioned extension theorems follow as special cases. In the case of effective maps with an effectively given separable metric space as codomain, the result remains true, if we allow the maps to be only partially defined. The domain space has to satisfy a suitable separability condition, then. Moreover, the recursive function realizing the effective map must not produce inconsistent information outside the domain of definition of the effective map. But, as follows from an example by Friedberg [13], it does not hold for partial effective maps the domain space of which is domain-like.

In [32] the theory of representations has been developed in great generality. Nevertheless, considering only countably based T_0-spaces is not a real restriction. Such spaces appear quite naturally in computer science. Scott [24] and Smyth [25] pointed out that data types can be thought of as T_0-spaces the basic open sets of which are the finitely describable properties of the data objects. Most structures considered in programming language semantics are equipped with a canonical topology. Prominent examples are metric spaces, Scott domains, and A- and f-spaces [7–9, 11]. As is shown by Stoltenberg–Hansen and Tucker [31], many algebraic structures, e.g., all term algebras over a finite signature, can be canonically embedded in complete ultrametric spaces as well as in Scott domains.

This paper is organized as follows: Section 1 contains basic definitions and facts from computability, numeration, and representation theory.

In Section 2, countably based T_0-spaces that satisfy some very general effectivity assumption and various representations with their derived numberings are studied. Moreover, some important, standard examples such as effectively given separable metric spaces, continuous directed-complete partial orders, A-, and f-spaces are considered. As a first step towards our final result, it is shown that every computable map between two countably based T_0-spaces is effectively continuous and, conversely, every effectively continuous map between the subspaces of their computable elements can be extended to a partial computable map between the spaces.

In Section 3, we present an example of Friedberg. This will provide us with the necessary counterexamples for the investigations in Sections 4 and 5.

In Section 4, the above noted modification of the notion of a witness for noninclusion is introduced and for maps between the subspaces of all computable elements of a countably based T_0-space, which contains an effectively presented dense subset and has a realizer for noninclusion, and another countably based T_0-space the statement is derived that exactly the (partial) effective maps having a witness for noninclusion are effectively continuous. From these two results the before mentioned answer to Weihrauch's question follows.

In Section 5, the results obtained in the preceding sections are applied to the standard examples introduced in Section 2 and the problem is studied when effective maps have a witness for noninclusion in these cases. Some final remarks appear in Section 6.

1. Basic definitions and properties

In what follows, let $\langle\, , \rangle : \omega^2 \to \omega$ be a recursive pairing function. Moreover, let $P^{(n)}$ ($R^{(n)}$) denote the set of all n-ary partial (total) recursive functions, and let W_i be the domain of the ith partial recursive function φ_i with respect to some Gödel numbering φ. We let $\varphi_i(a)\downarrow$ mean that the computation of $\varphi_i(a)$ stops, $\varphi_i(a)\downarrow \in A$ that it stops with value in A, and $\varphi_i(a)\downarrow_n$ that it stops within n steps. In the opposite case we write $\varphi_i(a)\uparrow$ and $\varphi_i(a)\uparrow_n$ respectively. If $A \subseteq \omega$ is not empty and (relatively) recursively enumerable (r.e.), A_s is the finite subset of A which can be enumerated in s steps with respect to some fixed (relatively) computable enumeration of A, i.e., $A_s = \{f(0), \ldots, f(s-1)\}$, where f is the fixed enumeration function. The cardinality of a set A is denoted by $|A|$. If F is a partial map from a set X into a set Y this is written as $F : X \rightharpoonup Y$.

Let S be a nonempty set. A (*partial*) *numbering* ν of S is a partial map $\nu : \omega \rightharpoonup S$ (onto) with domain $\mathrm{dom}(\nu)$. The value of ν at $n \in \mathrm{dom}(\nu)$ is denoted, interchangeably, by ν_n and $\nu(n)$.

Definition 1. Let ν and κ be numberings of the set S.
(1) $\nu \leqslant_m \kappa$, read ν is *many-one reducible* to κ, if there is a function $g \in P^{(1)}$ such that $\mathrm{dom}(\nu) \subseteq \mathrm{dom}(g)$, $g(\mathrm{dom}(\nu)) \subseteq \mathrm{dom}(\kappa)$, and $\nu_m = \kappa_{g(n)}$, for all $n \in \mathrm{dom}(\nu)$.
(2) ν is *many-one equivalent* to κ, if $\nu \leqslant_m \kappa$ and $\kappa \leqslant_m \nu$.

A subset X of S is *completely enumerable*, if there is an r.e. set W_n such that $\nu_i \in X$ if and only if $i \in W_n$, for all $i \in \mathrm{dom}(\nu)$. Set $C_n = X$, for any such n and X, and let C_n be undefined, otherwise. Then C is a numbering of the class of all completely enumerable subsets of S. If W_n is recursive, X is said to be *completely recursive*. X is *enumerable*, if there is an r.e. set $A \subseteq \mathrm{dom}(\nu)$ such that $X = \{\nu_i \mid i \in A\}$. A relation $R \subseteq S \times S$ is *completely enumerable*, if there is an r.e. set A such that $(\nu_i, \nu_j) \in R$ if and only if $\langle i, j \rangle \in A$, for all $i, j \in \mathrm{dom}(\nu)$.

Numbered sets form a category [10]. Morphisms are the effective maps, where for two nonempty sets S and S' with numberings ν and ν', respectively, a map $F : S \rightharpoonup S'$ is called *effective*, if there is a function $f \in P^{(1)}$ such that for all $n \in \nu^{-1}(\mathrm{dom}(F))$, $f(n)\downarrow \in \mathrm{dom}(\nu')$ and $F(\nu_n) = \nu'_{f(n)}$. We say in this case that f *realizes* F or that f is a *realizer* of F.

Obviously, only countable sets have numberings. In order to study the computability of mappings between sets which may have the cardinality of the continuum, Weihrauch considered sets the elements of which are no longer represented by numbers, but by arithmetic functions.

Let $\mathbb{B}$ be the set of all functions $p : \omega \to \omega$, and let T be a nonempty set. A *representation* δ of T is a partial map $\delta : \mathbb{B} \rightharpoonup T$(onto) with domain $\mathrm{dom}(\delta)$. As in the case of numberings the value of δ at $p \in \mathrm{dom}(\delta)$ is denoted by δ_p and/or $\delta(p)$.

Definition 2. Let δ and η be representations of the set T.
(1) $\delta \leqslant_c \eta$, read δ is (*computably*) *reducible* to η, if there is a recursive operator [23] Γ such that for all $p \in \mathrm{dom}(\delta)$, $\Gamma(p) \in \mathrm{dom}(\eta)$, and $\delta_p = \eta_{\Gamma(p)}$.
(2) $\delta \equiv_c \eta$, read δ is (*computably*) *equivalent* to η, if $\delta \leqslant_c \eta$ and $\eta \leqslant_c \delta$.

For $p \in \mathbb{B}$ define $\mathbb{M}_p$ to be the set of all numbers i with $i + 1 \in \mathrm{range}(p)$. Then $\mathbb{M}$ is a representation of the powerset of ω.

Let T' be a further nonempty set and δ' be a representation of T'. A map $F : T \rightharpoonup T'$ is called *computable*, if there is a recursive operator Γ with $\Gamma(p) \in \mathrm{dom}(\delta')$ and $F(\delta_p) = \delta'_{\Gamma(p)}$, for all $p \in \delta^{-1}(\mathrm{dom}(F))$.

There is a natural way to introduce computability on a represented set. An element z of T is said to be *computable*, if there is some $p \in R^{(1)}$ with $z = \delta_p$. Let T_c be the set of all computable elements of T. Obviously, the computability of an element is invariant under the equivalence of representations. Associated with each representation δ of T is a canonical numbering v^δ of the corresponding set T_c of computable elements, called the *derived numbering*: $v_i^\delta = \delta(\varphi_i)$, if $\varphi_i \in \mathrm{dom}(\delta)$; otherwise v_i^δ is undefined. As is easy to see, the derived numberings of two equivalent representations of T are many-one equivalent.

Lemma 3. *Let δ and δ', respectively, be representations of the sets T and T'. Moreover, let $F : T \rightharpoonup T'$ be computable. Then $F(T_c) \subseteq T'_c$ and the restriction of F to T_c is effective with respect to the derived numberings.*

The proof of this lemma is given in [32], where Weihrauch also asks when the converse is true, i.e.:

> *Given an effective map $F : T_c \to T'_c$, is there a computable map $\bar{F} : T \rightharpoonup T'$ which extends F?*

As follows from examples in the literature, there is no such extension in general. In Section 4 we present a condition, which in the case of countably based separable topological T_0-spaces and certain natural representations characterizes the effective maps that have a computable extension.

2. Topological spaces and standard representations

Let $\mathscr{T} = (T, \tau)$ be a topological T_0-space with a countable basis $\mathscr{B}$. For any subset X of T, $\mathrm{int}_\tau(X)$, $\mathrm{cl}_\tau(X)$ and $\mathrm{ext}_\tau(X)$, respectively, are the interior, the closure and the exterior of X.

Let $B : \omega \to \mathscr{B}$(onto) be a total numbering of $\mathscr{B}$. In the applications we have in mind the basic open sets can be described in some finite way. The indexing B is then obtained by an encoding of the finite descriptions. Moreover, in these cases there is a canonical relation between the (code numbers of the) descriptions which is stronger than the usual set inclusion between the described sets. This relation is r.e., which, in general, is not true for set inclusion. One has to use this stronger relation in order to derive the result we talked about in the introduction.

Definition 4. Let $\prec_B$ be a transitive binary relation on ω. We say that
(1) $\prec_B$ is a *strong inclusion*, if for all $m, n \in \omega$ from $m \prec_B n$ it follows that $B_m \subseteq B_n$.
(2) $\mathscr{B}$ is a *strong basis*, if $\prec_B$ is a strong inclusion and for all $z \in T$ and $m, n \in \omega$ with $z \in B_m \cap B_n$ there is a number $a \in \omega$ such that $z \in B_a$, $a \prec_B m$ and $a \prec_B n$.

Example 5. Let S be a nonempty set with a family $\{V_n \mid n \in \omega\}$ of nonempty (basic) predicates. Moreover, let D be a standard coding of all finite sets of natural numbers. Define

$$\bar{V}_n = \bigcap \{V_a \mid a \in D_n\}.$$

Then $\{\bar{V}_n \mid n \in \omega\}$ is basis of a topology on S. Set

$$m \prec_{\bar{V}} n \quad \Leftrightarrow \quad D_n \subseteq D_m.$$

Obviously, $\prec_{\bar{V}}$ is an r.e. strong inclusion and $\{\bar{V}_n \mid n \in \omega\}$ is a strong basis.

If one follows Smyth [25] and thinks of open sets as properties of data objects, then strong inclusion relations can be considered as ‘definite refinement’ relations. For what follows we assume that $\prec_B$ is an r.e. strong inclusion with respect to which $\mathscr{B}$ is a strong basis.

Since in a T_0-space every point y is uniquely determined by (a base of) its neighbourhood filter $\mathscr{N}(y)$, there is a canonical way to define representations of such spaces.

Definition 6. Let $\mathscr{H}$ be a filter. A nonempty subset $\mathscr{F}$ of $\mathscr{H}$ is said to be a *strong base* of $\mathscr{H}$ if the following two conditions hold:
(1) $(\forall m, n \in \omega)[B_m, B_n \in \mathscr{F} \Rightarrow (\exists a \in \omega) B_a \in \mathscr{F} \wedge a \prec_B m \wedge a \prec_B n]$.
(2) $(\forall m \in \omega)[B_m \in \mathscr{H} \Rightarrow (\exists a \in \omega) B_a \in \mathscr{F} \wedge a \prec_B m]$.

Definition 7. For $p \in \mathbb{B}$ let
(1) ξ_p be the unique point $y \in T$ such that $B(\mathbb{M}_p)$ is a strong base of $\mathscr{N}(y)$,
(2) ρ_p be the unique point $z \in T$ such that $\mathbb{M}_p$ is the set of all numbers n with $z \in B_n$,

if there are such points, and let ξ_p and ρ_p be undefined, otherwise. The map ξ is called *standard representation* of T.

Lemma 8. $\rho \equiv_c \xi$.

Proof. As $\mathscr{B}$ is a strong basis, we have for every point $z \in T$ that the set of all B_n with $z \in B_n$ is a strong base of $\mathscr{N}(z)$. Thus, the identity on $\mathbb{B}$ witnesses that $\rho \leqslant_c \xi$.

In order to show that $\xi \leqslant_c \rho$, let $p \in \mathbb{B}$ and consider the set

$$A[p] = \{n \in \omega \mid (\exists a \in \mathbb{M}_p) a \prec_B n\}.$$

Obviously, $A[p]$ is r.e. in p. Hence, there is some recursive operator Θ with $\mathbb{M}_{\Theta(p)} = A[p]$. Assume that $p \in \mathrm{dom}(\xi)$ and let $n \in A[p]$. Then $\xi_p \in B_n$. Now, to the contrary, let the number m be so that $\xi_p \in B_m$. Since $B(\mathbb{M}_p)$ is a strong base of $\mathscr{N}(\xi_p)$, there is a number $a \in \mathbb{M}_p$ with $a \prec_B m$. It follows that $m \in A[p]$. This shows that $\xi_p \in B_e$ if and only if $e \in A[p]$. Thus $\xi_p = \rho_{\Theta(p)}$. $\square$

A representation ζ of T with $\zeta \equiv_c \xi$ is called *admissible*. For the remainder of this paper we assume that the spaces under consideration are admissibly represented. Moreover, the computability of a point is always meant with respect to such a representation.

Observe that the notions of a standard representation and of admissibility used in this paper differ slightly from Weihrauch's: In [32] filter bases are not required to be strong.

Let us now introduce some well known examples of T_0-spaces with a strong basis. We first give the definition of the space and then discuss properties of their admissible representations and the corresponding derived numberings.

Example 9 (*Effective metric spaces*). Let $\mathscr{M} = (M, d)$ be a separable metric space with dense subset M_0. For the purposes of this paper it is sufficient to impose only rather weak effectively requirements on the space (cf. [32, 16]). $\mathscr{M}$ is said to be *weakly effective*, if there exists a total indexing β of M_0 such that the set $\{\langle i,j,a\rangle \mid d(\beta_i,\beta_j) < v_a^{\mathbb{Q}}\}$ is r.e. Here $v^{\mathbb{Q}}$ is a canonical indexing of the rational number set. If, in addition, also $\{\langle i,j,a\rangle \mid d(\beta_i,\beta_j) > v_a^{\mathbb{Q}}\}$ is r.e., $\mathscr{M}$ is called *effective*. Since the less-than relation on the computable real numbers is completely enumerable [19], both conditions are satisfied if the restriction of the distance function d to M_0 has only computable values and is effective. (Here, computability is understood with respect to the standard representation of the real number set $\mathbb{R}$, and $\mathbb{R}_c$ is enumerated by the corresponding derived numbering. As basis of the usual topology on $\mathbb{R}$ the set of all open intervals with rational end points is taken, enumerated in a canonical way.)

As is well known, the collection of all sets $B_{\langle i,m\rangle} = \{y \in M \mid d(\beta_i, y) < 2^{-m}\}$ $(i, m \in \omega)$ is a basis of the canonical Hausdorff topology Δ on M. Define

$$\langle i,m\rangle \prec_B \langle j,n\rangle \;\Leftrightarrow\; d(\beta_i,\beta_j) + 2^{-m} < 2^{-n}.$$

Using the triangular inequation it is easily verified that $\prec_B$ is a strong inclusion and the collection of all $B_{\langle i,m\rangle}$ is a strong basis. If $\mathscr{M}$ is weakly effective, we moreover have that $\prec_B$ is also r.e.

Another widely known fact is that each element of M is the limit of a normed Cauchy sequence of elements of the dense subset M_0, where a sequence $(y_a)_{a\in\omega}$ of elements of M_0 is said to be *normed* if $d(y_m, y_n) < 2^{-m}$ for all $m, n \in \omega$ with $m \leqslant n$. For $p \in \mathbb{B}$ define γ_p to be the limit of the sequence $(\beta_{p(a)})_{a\in\omega}$, if this sequence is normed and the limit exists. In any other case let γ_p be undefined. The map γ is called *Cauchy representation* of M.

Lemma 10. *Let $\mathscr{M}$ be weakly effective. Then the Cauchy representation is admissible.*

Proof. For $p \in \mathbb{B}$ let

$$A[p] = \{\langle a,m,n\rangle \mid a \in \mathbb{M}_p \wedge a \prec_B m \wedge a \prec_B p(n) - 1\}.$$

Obviously, $A[p]$ is r.e. in p. Hence, there is some recursive operator Ω with $\mathbb{M}_{\Omega(p)} = A[p]$. Define the operator $\Phi : \mathbb{B} \rightharpoonup \mathbb{B}$ by

$$\Phi(p)(0) = p(0),$$

$$\Phi(p)(n+1) = 1 + \pi_1^3(\Omega(p)(\mu e : \pi_2^3(\Omega(p)(e) - 1) = \Phi(p)(n) - 1 \wedge \pi_3^3(\Omega(p)(e) - 1) = n + 1) - 1).$$

Clearly, Φ is recursive.

Now, assume that $p \in \mathrm{dom}(\xi)$. Then $B(\mathbb{M}_p)$ is a strong base of $\mathscr{N}(\xi_p)$. Therefore it follows by induction that $\Phi(p)(n)$ is defined, for all $n \in \omega$. By the definition of Φ we moreover have that $\mathbb{M}_{\Phi(p)} \subseteq \mathbb{M}_p$ and $\Phi(p)(n+1) - 1 \prec_B \Phi(p)(n) - 1$. Set $\hat{\Phi}(p)(n) = \pi_1(\Phi(p)(n) - 1)$. Then we obtain from the definition of $\prec_B$ that $(\beta_{\hat{\Phi}(p)(n)})_{n \in \omega}$ is a normed Cauchy sequence of elements of the dense subset M_0 which converges to ξ_p. Thus $\xi_p = \gamma_{\hat{\Phi}(p)}$. This shows that $\xi \leqslant_c \gamma$.

For the proof that also $\gamma \leqslant_c \xi$ let

$$\Psi(p)(n) = 1 + \langle p(n+1), n \rangle.$$

Then Ψ is recursive. Moreover, we have for $p \in \mathrm{dom}(\gamma)$ and $n \in \omega$ that

$$\begin{aligned} & d(\beta_{\pi_1(\Psi(p)(n+1)-1)}, \beta_{\pi_1(\Psi(p)(n)-1)}) + 2^{-\pi_2(\Psi(p)(n+1)-1)} \\ & \quad = d(\beta_{p(n+2)}, \beta_{p(n+1)}) + 2^{-(n+1)} \\ & \quad < 2^{-(n-1)} + 2^{-(n+1)} \\ & \quad = 2^{-\pi_2(\Psi(p)(n)-1)}. \end{aligned}$$

Thus $\Psi(p)(n+1) - 1 \prec_B \Psi(p)(n) - 1$, for all $n \in \omega$. In order to obtain that $B(\mathbb{M}_{\Psi(p)})$ is a strong base of $\mathscr{N}(\gamma_p)$ it remains to show that for each $B_m \in \mathscr{N}(\gamma_p)$ there is some $a \in \omega$ with $\Psi(p)(a) - 1 \prec_B m$.

Let $\gamma_p \in B_m$ and let a be the smallest number n with $2^{1-n} \leqslant 2^{-\pi_2(m)} - d(\gamma_p, \beta_{\pi_1(m)})$. Since $(\beta_{p(n)})_{n \in \omega}$ is normed and converges to γ_p, we have that

$$\begin{aligned} & d(\beta_{\pi_1(\Psi(p)(a)-1)}, \beta_{\pi_1(m)}) + 2^{-\pi_2(\Psi(p)(a)-1)} \\ & \quad = d(\beta_{p(a+1)}, \beta_{\pi_1(m)}) + 2^{-a} \\ & \quad \leqslant d(\beta_{p(a+1)}, \gamma_p) + d(\gamma_p, \beta_{\pi_1(m)}) + 2^{-a} \\ & \quad < 2^{-(a+1)} + 2^{-a} + d(\gamma_p, \beta_{\pi_1(m)}) \\ & \quad < 2^{-\pi_2(m)}. \end{aligned}$$

It follows that $\gamma_p = \xi_{\Psi(p)}$, i.e., $\gamma \leqslant_c \xi$. □

Note that an analogous result has been shown by Weihrauch [32] for separable metric spaces without isolated points. But as has already been mentioned his notion of a standard representation is slightly different from ours.

A standard example of an effective metric space is *Baire space*, that is, the set $\mathbb{B}$ of all total arithmetic functions with the metric

$$d^B(p,q) = \begin{cases} 0 & \text{if } p = q, \\ 2^{-\mu a: p(a) \neq q(a)} & \text{otherwise.} \end{cases}$$

The functions that are eventually zero form a dense subset. Let α be a canonical encoding of these functions (cf. e.g. [32, p. 399]). The indexing of the basic open balls considered above is denoted by H in this case. With the help of Lemma 10 it is readily verified that the identity mapping on $\mathbb{B}$ is an admissible representation. Thus, the computable elements are exactly the total recursive functions.

Example 11 (*Effective continuous directed-complete partial orders*). Let $\mathscr{Q} = (Q, \sqsubseteq)$ be a partial order with smallest element $\perp$. A nonempty subset S of Q is *directed* if for all $y_1, y_2 \in S$ there is some $u \in S$ with $y_1, y_2 \sqsubseteq u$. $\mathscr{Q}$ is a *directed-complete* partial order (dcpo) if every directed subset S of Q has a least upper bound $\sup S$ in Q. Let $\ll$ denote the *way-below relation* on Q, i.e., let $y_1 \ll y_2$ if for directed subsets S of Q the relation $y_2 \sqsubseteq \sup S$ always implies the existence of a $u \in S$ with $y_1 \sqsubseteq u$.

A subset Z of Q is a *basis* of $\mathscr{Q}$, if for any $y \in Q$ the set $Z_y = \{z \in Z \mid z \ll y\}$ is directed and $y = \sup Z_y$. A dcpo that has a basis is called *continuous*. As is well known, on each dcpo there is a canonical T_0 topology: the *Scott topology*. A subset X of Q is open if X is upwards closed with respect to $\sqsubseteq$ and with each $u \in X$ there is some $y \in X$ with $y \ll u$. If $\mathscr{Q}$ is continuous, this topology is generated by the sets $O_z = \{y \in Q \mid z \ll y\}$ with $z \in Z$. Moreover, Z is dense in $\mathscr{Q}$.

Let $\mathscr{Q}$ be continuous, then it is called *effective*, if there exists a total indexing β of Z such that the restriction of the way-below relation to Z is completely enumerable. Set $B_n = O_{\beta(n)}$ and define

$$m \prec_B n \iff \beta_n \ll \beta_m.$$

Then $\prec_B$ is an r.e. strong inclusion and the collection of all B_n is a strong basis. □

A widely known example of an effective continuous dcpo is the space of all partial arithmetic functions ordered by graph inclusion. The nowhere defined function is the smallest element and the subspace of all functions with finite domain is a basis. With respect to a canonical indexing of this basis (cf. e.g. [32, p. 444]) the computable elements are just the partial recursive functions in this case.

As is well known, on T_0-spaces there is a canonical partial order, the specialization order, which we denote by $\leqslant_\tau$.

Definition 12. Let $\mathscr{T} = (T, \tau)$ be a T_0-space, and $y, z \in T$. $y \leqslant_\tau z$ if $\mathscr{N}(y) \subseteq \mathscr{N}(z)$.

Note that in the case of a dcpo with the Scott topology the specialization order coincides with the partial order.

Example 13 (*Effective A-spaces*). A- and f-spaces have been introduced by Eršov [7–9, 11] as a more topologically oriented approach to domain theory. They are not required to be complete. An example of an f-space which is not directed-complete can be found in [14].

Let $\mathscr{Y}=(Y,\sigma)$ be a topological T_0-space. For a subset X of Y, $\mathrm{int}(X)$ is its interior. Moreover, for $y,z\in Y$ define $y\ll z$ if $z\in\mathrm{int}(\{u\in Y\mid y\leqslant_\sigma u\})$. Then $\mathscr{Y}$ is an *A-space* if there is a subset Y_0 of Y satisfying the following three properties:
(1) Any two elements of Y_0 which are bounded in Y with respect to the specialization order have a least upper bound in Y_0.
(2) The collection of sets $\mathrm{int}(\{u\in Y\mid y\leqslant_\sigma u\})$, for $y\in Y_0$, is a basis of topology σ.
(3) For any $y\in Y_0$ and $u\in Y$ with $y\ll u$ there is some $z\in Y_0$ such that $y\ll z$ and $z\ll u$.
Any subset Y_0 of Y with these properties is called *basic subspace*.

The A-space $\mathscr{Y}$ with basic subspace Y_0 is *effective*, if there is a total indexing β of Y_0 such that the restriction of the relation $\ll$ to Y_0 is completely enumerable. For $m,n\in\omega$ set $B_n=\mathrm{int}(\{u\in Y\mid \beta_n\leqslant_\rho u\})$ and define

$$m\prec_B n \quad\Leftrightarrow\quad \beta_n\ll\beta_m.$$

Then $\prec_B$ is an r.e. strong inclusion. Moreover, the collection of all B_n is a strong basis.

Example 14 (*Effective f-spaces*). Let $\mathscr{Y}=(Y,\sigma)$ be again an arbitrary topological T_0-space. An open set V is an *f-set*, if V is nonempty and there is a some element $z_V\in V$ such that $V=\{y\in Y\mid z_V\leqslant_\sigma y\}$. The uniquely determined element z_V is called *f-element*. $\mathscr{Y}$ is an *f-space*, if the following two conditions hold:
(1) If U and V are f-sets with nonempty intersection, then $U\cap V$ is also an f-set.
(2) The collection of all f-sets is a basis of topology σ.
An f-space is *effective*, if the set of all f-elements has a total numbering β such that the restriction of the specialization order to this set and the boundedness of two f-elements are completely recursive and there is a function $su\in R^{(2)}$ such that in the case that β_n and β_m are bounded, $\beta_{su(n,m)}$ is their least upper bound.

Obviously, every f-space is an A-space with basic subspace the set of all f elements. Moreover, for $y,z\in Y$ such that y or z is an f-element, $y\ll z$ if and only if $y\leqslant_\sigma z$.

A common property of the above examples is that the spaces contain a dense indexed subset such that for each basis open set B_n one can effectively and uniformly generate all dense points included in it.

Definition 15. A T_0-space $\mathscr{T}=(T,\tau)$ with a countable basis $\mathscr{B}$ and a total indexing B of $\mathscr{B}$ is *effectively separable*, if there exists a dense subset T_0 of T and a total numbering β of T_0 such that $\{\langle i,n\rangle\mid\beta_i\in B_n\}$ is r.e.

A further property of the spaces in Examples 11, 13 and 14 is that their canonical topology has a basis with every basic open set being an upper set which has a lower bound in a certain neighbourhood of it.

Definition 16. We say that $\mathscr{T}$ is *pointed*, if there is a subset Pt of T and a numbering χ of Pt such that for all $n \in \omega$ with $B_n \neq \emptyset$, $n \in \mathrm{dom}(\chi)$, $\chi_n \leqslant_\tau z$, for all $z \in B_n$, and $\chi_n \in B_m$ if and only if $n \prec_B m$, for all $m \in \omega$.

Let $\mathrm{hl}_n(B_n) = \bigcap\{B_m \mid n \prec_B m\}$. Then $\chi_n \in \mathrm{hl}_n(B_n)$. Note that

$$B_n \subseteq \{z \in T \mid \chi_n \leqslant_\tau z\} \subseteq \mathrm{hl}_n(B_n).$$

As it is readily verified, Pt is dense in $\mathscr{T}$. Since strong inclusion is assumed to be r.e., we obtain that if $\mathscr{T}$ is pointed then it is also effectively separable.

Obviously, every effective continuous dcpo with the Scott topology, every effective A-space and every effective f-space is pointed.

For pointed spaces it can be shown that every element y is the least upper bound of all χ_n with $y \in B_n$, where the least upper bound is taken with respect to the specialization order [26]. Thus, we have for the representation ρ that $\rho_p = \sup \chi(\mathbb{M}_p)$ $(p \in \mathrm{dom}(\rho))$.

Let us now return to the general case of a topological T_0-space $\mathscr{T}$ with a countable strong basis $\mathscr{B}$. For every point $z \in T$ the set of all indices n with $z \in B_n$ is a filter with respect to $\prec_B$.

Definition 17. A nonempty set I of natural numbers is a *filter* with respect to $\prec_B$ if the following two conditions hold:

(1) $(\forall m, n \in I)(\exists a \in I) a \prec_B m \wedge a \prec_B m$.

(2) $(\forall n \in \omega)(\forall m \in I)[m \prec_B n \Rightarrow n \in I]$.

If only the first stipulation is fulfilled, I is called *filtered* with respect to $\prec_B$.

For an important class of spaces every such filter is generated by the neighbourhood filter of a point.

Definition 18. An enumeration $(B_{f(a)})_{a \in \omega}$ with $f : \omega \to \omega$ such that

$$\mathrm{range}(f) \subseteq \mathrm{dom}(B)$$

is said to be *normed* if f is decreasing with respect to $\prec_B$. If f is recursive, it is also called *recursive* and any Gödel number of f is said to be an *index* of it.

In case $(B_{f(a)})$ enumerates a strong base of the neighbourhood filter of some point, we say it *converges* to that point.

Definition 19. A T_0-space $\mathscr{T} = (T, \tau)$ with countable strong basis $\mathscr{B}$ is *strongly complete*, if each normed enumeration of nonempty basic open sets converges.

In the case of a weakly effective metric space, it follows from the proof of Lemma 10 that the space is strongly complete if and only if every normed sequence of elements of the dense subset converges.

Lemma 20. *Let $\mathscr{T}$ be strongly complete. Then for every filter I with respect to $\prec_B$ such that B_n is nonempty, for each $n \in I$, there is some point $z \in T$ so that I is the set of all indices n with $z \in B_n$.*

Proof. Since I is a filter, a sequence $(a_i)_{i \in \omega}$ of elements of I can be constructed which is decreasing with respect to $\prec_B$ such that for every $m \in I$ there is some index i with $a_i \prec_B m$. Then, by the strong completeness of $\mathscr{T}$, there is a point $z \in T$ so that the set of all B_{a_i} $(i \in \omega)$ is a strong base of the neighbourhood filter of z. It follows that for each n with $z \in B_n$ there is some index i with $a_i \prec_B n$, from which we obtain that $n \in I$, as $a_i \in I$. The converse property that $z \in B_n$, for every $n \in I$, is obvious from the construction of the a_i. $\square$

Every T_0-space with a countable strong basis is embeddable in a strongly complete space.

Proposition 21. *There is a strongly complete T_0-space $\hat{\mathscr{T}} = (\hat{T}, \hat{\tau})$ and a computable embedding $\imath : T \to \hat{T}$ such that for every other strongly complete T_0-space $\tilde{\mathscr{T}} = (\tilde{T}, \tilde{\tau})$ and any computable map $F : T \rightharpoonup \tilde{T}$ there is a computable map $G : \hat{T} \rightharpoonup \tilde{T}$ with $F = G \circ \imath$. If, in addition, $\emptyset \notin \mathscr{B}$ then* range($\imath$) *is dense in $\hat{\mathscr{T}}$.*

Proof. Let $\hat{T}$ be the set of all filters with respect to $\prec_B$, and for $m, n \in \omega$ define $\hat{B}_n$ to be the collection of all filters in $\hat{T}$ that contain n and set

$$m \prec_{\hat{B}} n \Leftrightarrow m \prec_B n.$$

Then $\prec_{\hat{B}}$ is a strong inclusion relation and the collection $\hat{\mathscr{B}}$ of all sets $\hat{B}_n$ is a strong basis of a topology $\hat{\tau}$ on $\hat{T}$. Moreover, we have for the representation $\hat{\rho}$ of $\hat{T}$ that $\hat{\rho}_p = \mathbb{M}_p$, if $\mathbb{M}_p$ is a filter with respect to $\prec_B$. Otherwise $\hat{\rho}$ is undefined.

For the verification that $\hat{\mathscr{T}}$ is strongly complete let $(a_i)_{i \in \omega}$ be a sequence of indices which is decreasing with respect to $\prec_{\hat{B}}$ such that $\hat{B}_{a_i}$ is not empty, for each $i \in \omega$. Then the sequence is also decreasing with respect to $\prec_B$. Hence, the set I of all $m \in \omega$ with $a_i \prec_B m$, for some index i, is a filter with respect to $\prec_B$. Obviously, $I \in \hat{B}_{a_i}$, for all $i \in \omega$. It remains to show that the collection of all $\hat{B}_{a_i}$ is a strong base of the neighbourhood filter of I. The first condition in Definition 6 holds since I is filtered. For the second requirement let $n \in \omega$ with $I \in \hat{B}_n$. Then $n \in I$, which implies that for some index i, $a_i \prec_B n$.

For $z \in T$ let $\imath(z)$ be the set of all indices n with $z \in B_n$. Then $\imath(z)$ is a filter with respect to $\prec_B$. Since $\mathscr{T}$ is T_0, the map $\imath$ is one-to-one. Moreover $\imath \circ \rho = \hat{\rho}$, which shows that it is computable.

Now, assume that the empty set is not basic open and let $n \in \omega$. Then there is some point $z \in B_n$. Hence $n \in \imath(z)$, which implies that $\imath(z) \in \hat{B}_n$. Thus, $\imath(T)$ is dense in $\hat{T}$.

Let $\tilde{\mathscr{T}} = (\tilde{T}, \tilde{\tau})$ be a further strongly complete topological T_0-space with a strong basis $\tilde{\mathscr{B}}$ and $F : T \rightharpoonup \tilde{T}$ be a computable map. For $I \in \imath(\text{dom}(F))$, say $I = \imath(z)$, define $G(I) = F(z)$. Then for $p \in \mathbb{B}$ with $\rho_p \in \text{dom}(F)$ we have that $G(\hat{\rho}_p) = G(\imath(\rho_p)) = F(\rho_p)$, which shows that computability of G. $\square$

As is well known, (the restriction of) each recursive operator (to Baire space) is effectively continuous. The same can be shown for computable maps. We shall see that both notions coincide. To this end we show that the representation ρ is both effectively continuous and effectively open.

Definition 22. Let $\mathscr{T}=(T,\tau)$ and $\mathscr{T}'=(T',\tau')$, respectively, be T_0-spaces with countable bases $\mathscr{B}$ and $\mathscr{B}'$ and total numberings B and B' of $\mathscr{B}$ and $\mathscr{B}'$. A map $F: T \rightharpoonup T'$ is called

(1) *Effectively continuous*, if there is a function $h \in R^{(1)}$ such that for all $n \in \omega$, $F^{-1}(B'_n) = \bigcup\{B_a \cap \mathrm{dom}(F) \mid a \in W_{h(n)}\}$.

(2) *Effectively open*, if there is a function $g \in R^{(1)}$ such that for all $n \in \omega$, $F(B_n) = \bigcup\{B'_a \cap \mathrm{range}(F) \mid a \in W_{g(n)}\}$.

Assume that F is effectively continuous and let this be witnessed by $h \in R^{(1)}$. For disjoint basic open sets B'_m and B'_n one always has that $F^{-1}(B'_m)$ and $F^{-1}(B'_n)$ are disjoint as well. But this does not necessarily mean that also $\bigcup\{B_a \mid a \in W_{h(m)}\}$ and $\bigcup\{B_a \mid a \in W_{h(n)}\}$ are disjoint. We call h *consistent* if for all $m,n \in \omega$

$$B'_m \cap B'_n = \emptyset \;\Rightarrow\; \bigcup\{B_a \mid a \in W_{h(m)}\} \cap \bigcup\{B_a \mid a \in W_{h(n)}\} = \emptyset.$$

Lemma 23. *The representation ρ is effectively continuous.*

Proof. For $p \in \mathrm{dom}(\rho)$ and $n \in \omega$ we have

$$\begin{aligned} \rho_p \in B_n &\Leftrightarrow n+1 \in \mathrm{range}(p) \\ &\Leftrightarrow (\exists i,j,a \in \omega)\, p \in H_{\langle i,a\rangle} \wedge j \leqslant a \wedge \alpha_i(j) = n+1. \end{aligned}$$

Let the function $h \in R^{(1)}$ be such that $W_{h(n)} = \{m \mid (\exists j \leqslant \pi_2(m))\alpha_{\pi_1(m)}(j) = n+1\}$. Then it witnesses the effective continuity of ρ. $\square$

It follows that all admissible representations of T are effectively continuous.

Lemma 24. *The representation ρ is effectively open.*

Proof. Let $z \in T$ and $i,n \in \omega$. Then we have

$$\begin{aligned} z \in \rho(H_{\langle i,n\rangle}) &\Leftrightarrow (\forall a \leqslant n)\,[\alpha_i(a) > 0 \Rightarrow z \in B_{\alpha_i(a)-1}] \\ &\Leftrightarrow (\exists m \in \omega) z \in B_m \wedge (\forall a \leqslant n)\,[\alpha_i(a) > 0 \Rightarrow m \prec_B \alpha_i(a) - 1]. \end{aligned}$$

The last equation holds as $\mathscr{B}$ is a strong basis. It follows that ρ is effectively open. $\square$

Now, let $\mathscr{T}'=(T',\tau')$ be a further T_0-space with countable basis $\mathscr{B}'$ and a total numbering B' of $\mathscr{B}'$.

Proposition 25. *Let $F : T \rightharpoonup T'$. Then the following statements hold*:
(1) *If F is computable then it is effectively continuous.*
(2) *If F is effectively continuous, then a computable map $\bar{F} : T \rightharpoonup T'$ can be constructed which extends F.*

Proof. Without restriction we assume that T and T', respectively, are represented by ρ and ρ'.

(1) Let the recursive operator Γ witness the computability of F. Then $F \circ \rho = \rho' \circ \Gamma$ on $\rho^{-1}(\mathrm{dom}(F))$. As a consequence we have for $n \in \omega$ that $F^{-1}(B'_n) = \rho(\Gamma^{-1}(\rho'^{-1}(B'_n))) \cap \mathrm{dom}(F)$, which implies the effective continuity of F.

(2) Suppose that the function $h \in R^{(1)}$ witnesses the effective continuity of F. For $p \in \mathbb{B}$ set $A[p] = \{a \in \omega \mid W_{h(a)} \cap \mathbb{M}_p \neq \emptyset\}$. Then $A[p]$ is r.e. in p. Hence, there is some recursive operator Γ with $\mathbb{M}_{\Gamma(p)} = A[p]$. Obviously, for $p, q \in \mathbb{B}$ with $\mathbb{M}_p = \mathbb{M}_q$ we have that $\mathbb{M}_{\Gamma(p)} = \mathbb{M}_{\Gamma(q)}$. It follows for $p, q \in \mathrm{dom}(\rho) \cap \Gamma^{-1}(\mathrm{dom}(\rho'))$ with $\rho_p = \rho_q$ that $\rho'_{\Gamma(p)} = \rho'_{\Gamma(q)}$.

For $y \in T$ set $\bar{F}(y) = \rho'_{\Gamma(p)}$, for some $p \in \rho^{-1}(y) \cap \Gamma^{-1}(\mathrm{dom}(\rho'))$, if this set is not empty, and let $\bar{F}(y)$ be undefined, otherwise. Then $\bar{F}$ is computable.

It remains to show that on $\mathrm{dom}(F)$, $\bar{F}$ coincides with F. Let to this end $p \in \rho^{-1}(\mathrm{dom}(F))$. Then $a \in A[p]$ if and only if $F(\rho_p) \in B'_a$. Hence $\Gamma(p) \in \mathrm{dom}(\rho')$ and $\bar{F}(\rho_p) = \rho'_{\Gamma(p)} = F(\rho_p)$. □

As is widely known, the set $R^{(1)}$ of all unary total recursive functions is dense in Baire space. It follows that, if it is also dense in the domain of an admissible representation (with respect to the induced topology), then the subset of all computable points is dense in T.

The next result displays an important case in which the extension $\bar{F}$ constructed in the above proposition is total. Moreover, it is uniquely determined in this case.

Lemma 26. *Let $\mathscr{T}$ be pointed and $\mathscr{T}'$ be strongly complete. Moreover, let $F : T \rightharpoonup T'$ be effectively continuous with $Pt \subseteq \mathrm{dom}(F)$. Then the computable extension $\bar{F}$ of F constructed in Proposition 25 is total and uniquely determined.*

Proof. Let $y \in T$ and $p \in \rho^{-1}(y)$. We first show that $A[p]$ is a filter with respect to $\prec_{B'}$.

Let to this end $a, c \in A[p]$. Then there are $m \in W_{h(a)}$ and $n \in W_{h(c)}$ such that $y \in B_m \cap B_n$. Since $\mathscr{B}$ is a strong basis, there is some $i \in \omega$ with $y \in B_i$, $i \prec_B m$ and $i \prec_B n$. It follows that $\chi_i \leqslant_\tau y$ and $\chi_i \in B_m \cap B_n$. Hence $F(\chi_i) \in B'_a \cap B'_c$. As $\mathscr{B}'$ is also a strong basis, this implies that there exists an index e such that $F(\chi_i) \in B'_e$, $e \prec_{B'} a$ and $e \prec_{B'} c$. Thus $\chi_i \in B_b$, for some $b \in W_{h(e)}$. Since open sets are upwards closed under the specialization order, we obtain that $y \in B_b$ too, i.e., $e \in A[p]$.

Next, assume that $a \in A[p]$ and $c \in \omega$ with $a \prec_{B'} c$. Then there is some $m \in W_{h(a)}$ such that $y \in B_m$. By the strongness of $\mathscr{B}$ we obtain some index i with $y \in B_i$ and $i \prec_B m$. It follows that $\chi_i \leqslant_\tau y$ and $\chi_i \in B_m$. Thus $F(\chi_i) \in B'_a$. But $B'_a \subseteq B'_c$, which implies that $\chi_i \in B_n$, for some $n \in W_{h(c)}$. As a consequence we have that also $y \in B_n$. Hence $c \in A[p]$.

As we have just seen, all basic open sets B_c with $c \in A[p]$ are nonempty. By Lemma 20 we therefore obtain that there is a point $z \in T'$ such that $A[p]$ is the set of all indices a with $z \in B'_a$. Thus $\Gamma(p) \in \mathrm{dom}(\rho')$, which shows that $\bar{F}(y)$ is defined, for every $y \in T$.

It remains to show that $\bar{F}$ is uniquely determined. As has already been mentioned, it is shown in [26] that for every $y \in T$, $y = \sup\{\chi_n \mid y \in B_n\}$. In a similar way it can be proved for continuous maps $G : T \to T'$ that $G(y) = \sup\{G(\chi_n) \mid y \in B_n\}$. With this it follows that F has at most one computable extension. □

In [28] T_0-spaces consisting only of computable elements are considered and a condition is presented forcing effective maps between such spaces to be effectively continuous. We shall generalize this result in our study when effective maps between the subspaces of computable elements of two admissibly represented T_0-spaces have a computable extension. The numberings of the spaces in that paper are assumed to satisfy very natural conditions relating topology with effectivity. As we shall see next, for numberings which are derived from admissible representations these conditions are provable.

Definition 27. Let x be a numbering of T_c. We say that:
(1) x is *computable* if there is some r.e. set L such that for all $i \in \mathrm{dom}(x)$ and $n \in \omega$, $\langle i, n\rangle \in L$ if and only if $x_i \in B_n$.
(2) x *allows effective limit passing* if there is a function $pt \in P^{(1)}$ such that, if φ_m is decreasing with respect to $\prec_B$ and the set of all $B(\varphi_m(a))$ $(a \in \omega)$ is a strong base of the neighbourhood filter of some point $y \in T_c$, then $pt(m)\downarrow \in \mathrm{dom}(x)$ and $x_{pt(m)} = y$.
(3) x is *acceptable* if it allows effective limit passing and is computable.

Lemma 28. *Let the numbering x of T_c be derived from an admissible representation of T. Then x is acceptable.*

Proof. As it is easily verified, computability and the property of allowing effective limit passing are both invariant under many-one equivalence. Therefore, we can assume that x is derived from the representation ρ and/or the standard representation. In the first case it is obvious that x is computable, in the second that x allows effective limit passing.

In the sequel we always assume that x is a numbering of T_c which is derived from an admissible representation of T, similarly for x' and T'. Then it follows for every basic open set that one can effectively list all its computable elements. In general there is no way to do the same with respect to its exterior. An exception are effective metric spaces and effective f-spaces, respectively, since

$$x_a \in \mathrm{ext}_\Delta(B_{\langle i,m\rangle}) \;\Leftrightarrow\; (\exists \langle j,n\rangle \in \mathbb{M}(\varphi_a))d(\beta_i, \beta_j) > 2^{-m} + 2^{-n}$$

and

$$x_a \in \mathrm{ext}_\sigma(B_m) \iff (\exists n \in \mathbb{M}(\varphi_a))\neg(\exists c)\beta_m, \beta_n \leqslant_\sigma \beta_c.$$

As follows from the definition of admissible representations, for each computable point one can effectively enumerate a strong base of basic open sets of its neighbourhood filter. The next result, which is proved in [28], shows that this can be done in a normed way.

Lemma 29. *There are functions $q \in R^{(1)}$ and $p \in R^{(2)}$ such that for all $i \in \mathrm{dom}(x)$ and all $n \in \omega$ with $x_i \in B_n$, $q(i)$ and $p(i,n)$ are indices of normed recursive enumerations of basic open sets which converge to x_i. Moreover, $\varphi_{p(i,n)}(0) \prec_B n$.*

Let us now assume that $\mathscr{T}$ is effectively separable with dense subset T_0 and total numbering β of T_0. Moreover, without restriction, suppose that the numbering x is derived from the representation ρ. Then the function $g \in R^{(1)}$ with $\mathbb{M}(\varphi_{g(i)}) = \{n \mid \beta_i \in B_n\}$ witnesses that $\beta \leqslant_m x$.

In [28] a countably based countable T_0-space $\hat{\mathscr{T}} = (\hat{T}, \hat{\tau})$ with numberings $\hat{x}$ and $\hat{B}$, respectively, of $\hat{T}$ and the topological basis $\hat{\mathscr{B}}$ such that for some function $g \in R^{(1)}$ with $\mathrm{range}(g) \subseteq \mathrm{dom}(\hat{x})$ the set $\hat{x}(\mathrm{range}(g))$ is dense in $\hat{\mathscr{T}}$ is called *recursively separable*. Moreover, $\hat{\mathscr{T}}$ is said to be *effectively pointed*, if there is a function $pd \in P^{(1)}$ such that for all $n \in \mathrm{dom}(\hat{B})$ with $\hat{B}_n \neq \emptyset$, $pd(n)\downarrow \in \mathrm{dom}(\hat{x})$, $\hat{x}_{pd(n)} \in \mathrm{hl}_n(\hat{B}_n)$ and $\hat{x}_{pd(n)} \leqslant_{\hat{\tau}} z$, for all $z \in \hat{B}_n$.

Let τ_c be the induced topology on T_c. Then it follows that $\mathscr{T}_c = (T_c, \tau_c)$ is recursively separable, if $\mathscr{T}$ is effectively separable. Moreover, $\mathscr{T}_c$ is effectively pointed, if $\mathscr{T}$ is pointed.

As is well known, each function $f \in P^{(1)}$ defines a partial effective map $F^f : T_c \rightharpoonup T_c'$ with $F^f(x_i) = x'_{f(i)}$, if f behaves extensional on x_i, which means that for all $j \in \mathrm{dom}(x)$ one has

$$x_j = x_i \Rightarrow f(j)\downarrow \in \mathrm{dom}(x') \wedge x'_{f(j)} = x'_{f(i)}.$$

In any other case, F^f is undefined. Observe that, if $F : T_c \rightharpoonup T_c'$ is an effective map with realizer $f \in P^{(1)}$, then the effective map F^f defined by f is an extension of F.

For the following definition as well as the remainder of this paper we assume that the spaces under consideration are represented by ρ and that x is the corresponding derived numbering. Note that in this case we can always presume that an effective map is realized by a total recursive function.

Definition 30. Let $f \in R^{(1)}$. Then the set

(1) $\mathrm{WE}(f) = \{z \in T_c \mid (\forall i, j \in x^{-1}(\{z\}))\varphi_{f(i)} \in R^{(1)} \wedge \mathbb{M}(\varphi_{f(i)}) = \mathbb{M}(\varphi_{f(j)})\}$ is called the *weak extensionality domain* of f.

(2) $\mathrm{SE}(f) = \{x_i \in \mathrm{WE}(f) \mid \mathbb{M}(\varphi_{f(i)})$ is a filter with respect to $\prec_{B'} \wedge \emptyset \notin B'(\mathbb{M}(\varphi_{f(i)}))\}$ is called the *strong extensionality domain* of f.

Obviously, $\mathrm{dom}(F) \subseteq \mathrm{SE}(f) \subseteq \mathrm{WE}(f)$.

If $\mathbb{M}(\varphi_{f(i)})$ is a filter with respect to $\prec_{B'}$ and $B'(\mathbb{M}(\varphi_{f(i)}))$ does not contain the empty set, then there are no disjoint pairs of basic open sets in $B'(\mathbb{M}(\varphi_{f(i)}))$. We say that f is *correct* if for all $i \in x^{-1}(\mathrm{WE}(f))$ and $m, n \in \mathbb{M}(\varphi_{f(i)})$, B'_m intersects B'_n.

Lemma 31. *Let $f \in R^{(1)}$ and $F : T_c \rightharpoonup T'_c$ be the corresponding effective map. If $\mathscr{T}$ is strongly complete then* $\mathrm{SE}(f) = \mathrm{dom}(F)$.

Proof. Let $x_i \in \mathrm{SE}(f)$. Then $\mathbb{M}(\varphi_{f(i)})$ is a filter with respect to $\prec_{B'}$, which means that it is both filtered and upwards closed with respect to $\prec_{B'}$. Because of the first property a sequence $(a_m)_{m\in\omega}$ of elements of $\mathbb{M}(\varphi_{f(i)})$ can be constructed, which is decreasing with respect to $\prec_{B'}$, such that for every $c \in \mathbb{M}(\varphi_{f(i)})$ there is some index m with $a_m \prec_{B'} c$. All B'_{a_m} are nonempty. By the strong completeness of $\mathscr{T}'$ we therefore obtain that there is some point $y \in T'$ so that $B'(\mathbb{M}(\varphi_{f(i)}))$ is a strong base of $\mathscr{N}'(y)$. Since $\mathbb{M}(\varphi_{f(i)})$ is upwards closed with respect to $\prec_{B'}$, it follows that $\mathbb{M}(\varphi_{f(i)})$ is the set of all numbers n with $y \in B'_n$. Thus $y = \rho'(\varphi_{f(i)}) = x'_{f(i)}$, i.e., $x_i \in \mathrm{dom}(F)$. □

Because of Proposition 21 this means that the sets $\mathrm{SE}(f)$ are the natural (maximal) domains of effective maps.

Since $\mathrm{dom}(F) \subseteq \mathrm{WE}(f)$, we can also consider F as a partial map from $\mathrm{WE}(f)$ into T'. With respect to the induced topology τ^f the space $\mathscr{T}^f = (\mathrm{WE}(f), \tau^f)$ also satisfies the general assumptions made in this paper. Set $B^f_n = B_n \cap \mathrm{WE}(f)$ and let $m \prec_{B^f} n$, if $m \prec_B n$. Then $\prec_{B^f}$ is an r.e. strong inclusion with respect to which the set of all B^f_n is a strong basis. Moreover, the restriction of representation ρ to $\mathrm{WE}(f)$ is the representation ρ^f of set $\mathrm{WE}(f)$, correspondingly for the derived numberings.

3. A counterexample

By Proposition 25 it suffices to study the problem whether a (partial) effective map is effectively continuous. The following example of Friedberg [13], which was originally stated in a slightly different way, shows that even for total maps this is not the case, in general.

Proposition 32. *There is an effective metric space $\mathscr{M}$, an effective continuous directed-complete partial order $\mathscr{S}$, and a map $F : M_c \to S_c$ which is effective, but not continuous.*

Proof. Let $\mathscr{M} = (M, d)$ be Baire space and $\mathscr{S} = (S, \sqsubseteq)$ be Sierpinski space, i.e., $S = \{\bot, \top\}$ with $\bot \sqsubseteq \top$. S is a basis of itself. Number its elements by $\beta'_0 = \bot$ and $\beta'_n = \top$, for $n \geqslant 1$. Then $\mathscr{S}$ is an effective continuous dcpo, and we have that $B'_0 = S$ and $B'_n = \{\top\}$, for $n \geqslant 1$.

Obviously, all elements of S are computable. Moreover, $M_c = R^{(1)}$. As is readily verified, the restriction of the Gödel numbering φ to $R^{(1)}$ is acceptable. Note that all acceptable numberings of M_c are many-one equivalent [28]. Thus, φ is many-one equivalent to the derived numbering x of the representation ρ of M. It follows that there is some function $k \in P^{(1)}$ such that $k(i)\downarrow$ and $x_i = \varphi_{k(i)}$, for $i \in \mathrm{dom}(x)$.

Define

$$\Omega = \{i \in \omega \mid [(\forall a \leqslant i)\varphi_i(a) = 0] \wedge (\exists c)[\varphi_i(c) \neq 0$$
$$\wedge (\forall a < c)\varphi_i(a) = 0 \wedge (\exists j < c)(\forall b \leqslant c)\varphi_i(b) = \varphi_j(b)]\}.$$

Then Ω is r.e. and for all $i, j \in \varphi^{-1}(R^{(1)})$ with $\varphi_i = \varphi_j$ we have that $i \in \Omega$ if and only if $j \in \Omega$. Moreover, let $f \in R^{(1d)}$ with

$$\varphi_{f(i)}(n) = \begin{cases} 2 & \text{if } k(i) \in \Omega_n, \\ 1 & \text{otherwise.} \end{cases}$$

For the representation ρ' of S we then obtain that $\rho'(\varphi_{f(i)}) = \top$, if $k(i) \in \Omega$, and $\rho'(\varphi_{f(i)}) = \bot$. otherwise. Set $F(x_i) = x'_{f(i)}$. The numbering x' is derived from ρ'.

Now, assume that F is continuous. Since $F(\lambda n.0) = \top$ and $\{\top\}$ is open in the Scott topology on S, by the definition of the Baire metric there is some $m > 0$ such that for all $g \in R^{(1)}$ with $g(a) = 0$, for $a < m$, we have that $F(g) = \top$. Set $n = \max\{\varphi_i(m) + 1 \mid i < m \wedge \varphi_i(m)\downarrow\}$, if the set is not empty, and let $n = 1$, otherwise. Furthermore, define

$$\hat{g}(a) = \begin{cases} 0 & \text{if } a \neq m, \\ n & \text{otherwise.} \end{cases}$$

Then $\hat{g} \in R^{(1)}$. Since $\hat{g}(a) = 0$, for all $a < m$, we have that $F(\hat{g}) = \top$. On the other hand, since $\hat{g}(m) \neq 0$ and for every Gödel number $j < m$, $\varphi_j(m) \neq \hat{g}(m)$, it follows from the definition of F that $F(\hat{g}) \neq \top$. Thus, F cannot be continuous. □

Because of Proposition 25 it follows that the map F cannot be extended to a computable map. This shows that the converse of Lemma 3 is not true, in general, and hence that working with computable elements only and maps which are computed by transforming the programs that compute the approximations of the points results in a more general computability notion than working with maps which are computed by transforming the approximations itself. From the literature also other examples of effective maps that are not effectively continuous are known [15, 17, 22, 35, 36]. The importance of Friedberg's example lies in the fact that it provides an example of a total, effective, and discontinuous map on a natural space.

Since the metric topology on Baire space is induced by the Scott topology on the effective continuous dcpo of all partial arithmetic functions, we have the following consequence.

Corollary 33. *There are effective continuous directed-complete partial orders $\mathscr{Q}$ and $\mathscr{S}$, and a partial map $F : Q_c \rightharpoonup S_c$ which is effective, but not continuous. In addition, the subspace* $\mathrm{dom}(F)$ *is recursively separable.*

As is well known, all total effective maps between the subspaces of computable elements of two effective continuous dcpo's are effectively continuous, and vice versa [6, 34]. The above result shows that this is no longer true in the case of partial maps.

4. Enforcing continuity

In this section we present a condition which forces a (partial) effective map between the subspaces of computable elements of two countably based T_0-spaces to be effectively continuous, and hence, to be extendable to a computable map. As we shall see on the contrary, this requirement is satisfied by every effectively continuous map between such spaces. The only extra stipulation in this case is that the domain space is effectively separable and the requirement is already met by the identity map on the subspace of its computable points.

The problem of when a total effective map is effectively continuous has been studied by the author in [30, 28], where it is shown that effective continuity is always enforced, if the effective map has a witness for noninclusion and its domain is recursively separable. The following definition differs from that given in [28] in that now the second requirement is independent of the first and is thus of a more global nature.

Definition 34. A partial map $F : T_c \rightharpoonup T'_c$ has a (*global*) *witness for noninclusion*, if there exist functions $s \in R^{(2)}$ and $r \in P^{(3)}$ such that $\text{range}(s) \subseteq \text{dom}(C)$ and for all $i, m, n \in \omega$ the following hold:
(1) If $i \in x^{-1}(\text{dom}(F))$ with $F(x_i) \in B'_m$, then $x_i \in C_{s(i,m)} \cap \text{dom}(F) \subseteq F^{-1}(B'_m)$.
(2) If $W_{s(i,m)} \neq \emptyset$ and $F(B_n) \nsubseteq B'_m$, then $r(i,n,m)\downarrow \in \text{dom}(x)$ and $x_{r(i,n,m)} \in \text{hl}_n(B_n) \setminus C_{s(i,m)}$.

The pair of functions (s, r) is called *witness for noninclusion* of F.

Note that such a condition appears quite naturally. As follows from [1–4], the effective continuity of effective maps can only be proved indirectly. But assuming that a map is not continuous means assuming that there is some basic open set in the codomain into which no basic open set in the domain is mapped. And since in most cases only the elements of a basic open set, but not of its complement are effectively listable, a witness for this not being mapped into cannot be found effectively, in general, if the map is only effective.

In applications often not the map $F : T_c \rightharpoonup T'_c$, but its restriction to a subspace $\hat{\mathscr{T}}$ of $\mathscr{T}$ with $\text{dom}(F) \subseteq \hat{T}_c$ has a witness for noninclusion. We say in this case that F has a *witness for noninclusion with respect to* $\hat{\mathscr{T}}$.

Proposition 35. *Let $F : T_c \rightharpoonup T'_c$ be effective with a realizer $f \in R^{(1)}$ and let it have a witness for noninclusion with respect to $\mathscr{T}^f$. Then F must be effectively continuous.*

Proof. Without restriction assume that $T = \text{WE}(f)$. Let the functions $s \in R^{(2)}$ and $r \in P^{(3)}$ be a witness for noninclusion of F, $p \in R^{(2)}$ and $q \in R^{(1)}$ be as in Lemma 29, and $\mathrm{pt} \in P^{(1)}$ witness that the indexing x allows effective limit passing. Then there is some r.e. set W_b such that for $i \in x^{-1}(\mathrm{dom}(F))$ and $m \in \omega$, as well as any index n of a converging normed recursive enumeration of basic open sets, $\langle n,i,m\rangle \in W_b$ if and only if $F(x_i) \in B'_m$ and $x_{\mathrm{pt}(n)} \in C_{s(i,m)}$, namely

$$W_b = \{\langle n,i,m\rangle \mid m \in \mathbb{M}(\varphi_{f(i)}) \wedge \mathrm{pt}(n) \in W_{s(i,m)}\}.$$

Set $\hat{g}(n,i,m) = \mu c : \varphi_b(\langle n,i,m\rangle)\downarrow_c$ and let $h \in R^{(3)}$ be such that

$$\varphi_{h(n,i,m)}(a) = \begin{cases} \varphi_{q(i)}(a) & \text{if} \varphi_b(\langle n,i,m\rangle)\uparrow_a, \\ \varphi_{p(r(i,\varphi_{q(i)}(\hat{g}(n,i,m)),m),\varphi_{q(i)}(\hat{g}(n,i,m)-1))}(a - \hat{g}(n,i,m)) & \text{otherwise.} \end{cases}$$

By the recursion theorem there is then a function $d \in R^{(2)}$ with

$$\varphi_{h(d(i,m),i,m)} = \varphi_{d(i,m)}.$$

Let $g(i,m) = \hat{g}(d(i,m),i,m)$, and suppose that $g(i,m)\uparrow$, for some $i,m \in \omega$ so that $F(x_i)$ is defined and $F(x_i) \in B'_m$. Then $d(i,m)$ is an index of a normed recursive enumeration of basic open sets converging to x_i. By the acceptability of x it follows that $\mathrm{pt}(d(i,m))\downarrow \in \mathrm{dom}(x)$ and $x_{\mathrm{pt}(d(i,m))} = x_i$. Moreover, since $x_i \in \mathrm{dom}(F)$ and $F(x_i) \in B'_m$, we have that $x_i \in C_{s(i,m)}$. Thus $x_{\mathrm{pt}(d(i,m))} \in C_{s(i,m)}$, which implies that $g(i,m)\downarrow$. This contradicts our assumption. Therefore $g(i,m)\downarrow$, for all $i \in x^{-1}(\mathrm{dom}(F))$ and $m \in \omega$ with $F(x_i) \in B'_m$.

Assume next that $F(B(\varphi_{q(i)}(g(i,m)))) \nsubseteq B'_m$, for some $i,m \in \omega$ such that $g(i,m)\downarrow$ and for all $a_1 < a_2 \leqslant g(i,m)$, $\varphi_{q(i)}(a_1)\downarrow$, $\varphi_{q(i)}(a_2)\downarrow$ as well as

$$\varphi_{q(i)}(a_2) \prec_B \varphi_{q(i)}(a_1).$$

Then $W_{s(i,m)}$ is nonempty and hence $r(i,\varphi_{q(i)}(g(i,m)),m)\downarrow \in \mathrm{dom}(x)$. Moreover,

$$\begin{aligned} x_{r(i,\varphi_{q(i)}(g(i,m)),m)} &\in \mathrm{hl}_{\varphi_{q(i)}(g(i,m))}(B_{\varphi_{q(i)}(g(i,m))})\backslash C_{s(i,m)} \\ &\subseteq B_{\varphi_{q(i)}(g(i,m)-1)})\backslash C_{s(i,m)}. \end{aligned} \tag{1}$$

The last inclusion holds, since $\varphi_{q(i)}(g(i,m)) \prec_B \varphi_{q(i)}(g(i,m)-1)$. According to Lemma 29, $p(r(i,\varphi_{q(i)}(g(i,m)),m), \varphi_{q(i)}(g(i,m)-1))$ is an index of a normed recursive enumeration of basic open sets which converges to $x_{r(i,\varphi_{q(i)}(g(i,m)),m)}$. In addition

$$\varphi_{p(r(i,\varphi_{q(i)}(g(i,m)),\varphi_{q(i)}(g(i,m)-1))}(0) \prec_B \varphi_{q(i)}(g(i,m)-1),$$

which implies that $d(i,m)$ is also an index of a normed recursive enumeration of basic open sets converging to $x(r(i,\varphi_{q(i)}(g(i,m)),m))$. Therefore $\mathrm{pt}(d(i,m))\downarrow \in \mathrm{dom}(x)$ and $x(\mathrm{pt}(d(i,m))) = x(r(i,\varphi_{q(i)}(g(i,m)),m))$. Because of (1) it follows that $x_{\mathrm{pt}(d(i,m))} \notin C_{s(i,m)}$. Hence $g(i,m)\uparrow$, which contradicts our assumption. Thus

$$F(B_{\varphi_{q(i)}(g(i,m))}) \subseteq B'_m,$$

for all $i,m \in \omega$ such that $g(i,m)\downarrow$ and for all $a_1 < a_2 \leqslant g(i,m)$, $\varphi_{q(i)}(a_1)\downarrow$, $\varphi_{q(i)}(a_2)\downarrow$ as well as $\varphi_{q(i)}(a_2) \prec_B \varphi_{q(i)}(a_1)$.

For the next step let $v \in R^{(2)}$ such that

$$W_{v(m)} = \{i \mid g(i,m)\downarrow \wedge (\forall a_1, a_2)[a_1 < a_2 \leqslant g(i,m) \Rightarrow \varphi_{q(i)}(a_1)\downarrow \wedge \varphi_{q(i)}(a_2)\downarrow \wedge \varphi_{q(i)}(a_2) \prec_B \varphi_{q(i)}(a_1)]\}.$$

We want to prove that for all $m \in \omega$

$$F^{-1}(B'_m) = \mathrm{dom}(F) \cap \bigcup\{B_{\varphi_{q(i)}(g(i,m))} \mid i \in W_{v(m)}\},$$

which shows that F is effectively continuous.

Let $i \in x^{-1}(\mathrm{dom}(F))$ such that $F(x_i) \in B'_m$. As we have seen above, $g(i,m)\downarrow$ in this case. Moreover, $i \in W_{v(m)}$ and $x_i \in \mathrm{dom}(F) \cap B(\varphi_{q(i)}(g(i,m)))$. For the verification of the converse inclusion let $i, m \in \omega$ with $i \in W_{v(m)}$. Then it follows from the second assertion shown above that $F(B(\varphi_{q(i)}(g(i,m)))) \subseteq B'_m$, i.e., $\mathrm{dom}(F) \cap B(\varphi_{q(i)}(g(i,m))) \subseteq F^{-1}(B'_m)$. $\square$

It has already been noted that in order to derive the analogous result for total effective maps, in [28] we had to assume that the domain space is recursively separable. Here, we did not need this extra supposition, which shows the modified notion of a witness for noninclusion is stronger than that used in [28].

As follows from Proposition 25 and Lemma 3, effectively continuous maps between the subspaces of computable elements of two countably based T_0-spaces are effective. Let us see next, whether they can be realized in such a way that the additional requirement in the above proposition holds. To verify this we need that the identity map on T_c has a witness for noninclusion.

Definition 36. Let $\bar{\mathscr{T}} = (\bar{T}, \bar{\tau})$ be a countable T_0-space with countable basis $\bar{\mathscr{B}}$, and let $\bar{x}$ and $\bar{B}$ be numberings of $\bar{T}$ and $\bar{B}$, respectively. A pair of functions (s, r) with $s \in R^{(2)}$ and $r \in P^{(3)}$ is a *realizer for noninclusion* of $\bar{\mathscr{T}}$, if range(s) $\subseteq \mathrm{dom}(C)$ and for all $i, m, n \in \omega$ the following hold:

(1) If $i \in \mathrm{dom}(\bar{x})$ with $\bar{x}_i \in \bar{B}_m$, then $\bar{x}_i \in C_{s(i,m)} \subseteq \bar{B}_m$.

(2) If $W_{s(i,m)} \neq \emptyset$ and $\bar{B}_n \nsubseteq \bar{B}_m$, then $r(i,n,m)\downarrow \in \mathrm{dom}(\bar{x})$ and $\bar{x}_{r(i,n,m)} \in \mathrm{hl}_n(\bar{B}_n) \backslash C_{s(i,m)}$.

In the case that $\bar{\mathscr{T}}$ is the subspace of computable points of a T_0-space $\mathscr{T}$, we also say that $\mathscr{T}$ has a realizer for noninclusion if $\bar{\mathscr{T}}$ has one.

Proposition 37. *Let $\mathscr{T}$ be effectively separable. Then every effectively continuous map $F: T_c \rightharpoonup T'_c$ is effective with a realizer $f \in R^{(1)}$ so that the following hold*:

(1) $\mathrm{WE}(f)$ *contains an enumerable subset X with* $\mathrm{dom}(F) \subseteq \mathrm{cl}_\tau(X)$.

(2) *If the effective continuity of F is witnessed by a consistent function, then f is correct.*

(3) *If, in addition, $\mathscr{T}$ has a realizer for noninclusion, then F also has a witness for noninclusion with respect to $\mathscr{T}^f$.*

Proof. Let $h \in R^{(1)}$ witness that F is effectively continuous and be $k \in R^{(1)}$ with

$$\varphi_{k(i)}(a) = \begin{cases} 0 & \text{if } \mathbb{M}(\varphi_i) \text{ intersects } W_{h(a)}, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Then we have for $i, j \in \omega$ with $\mathbb{M}(\varphi_i) = \mathbb{M}(\varphi_j)$ that $\text{dom}(\varphi_{k(i)}) = \text{dom}(\varphi_{k(j)})$. Let $v \in R^{(1)}$ such that $\varphi_{v(i)} \in R^{(1)}$ and $\mathbb{M}(\varphi_{v(i)}) = \text{dom}(\varphi_i)$ and set $f = v \circ k$. Obviously, f realizes F. Moreover, $\text{WE}(f) = T_c$. Since $\mathcal{T}$ is effectively separable, T_c contains an enumerable dense subset X. It follows that $\text{dom}(F) \subseteq \text{cl}_\tau(X)$. As is readily verified, f is correct if the function h is consistent.

Now, let $A \subseteq \omega$, $\bar{s} \in R^{(2)}$ and $\bar{r} \in P^{(3)}$, respectively, witness that X is enumerable and $\mathcal{T}$ has a realizer for noninclusion. Moreover, for $i, m \in \omega$ let $\langle \bar{\imath}, \bar{m}, \bar{n} \rangle$ be the first element enumerated in $\{\langle i', m', n' \rangle \mid n' \in W_{h(m')} \cap \mathbb{M}(\varphi_{i'})\}$ with $\bar{\imath} = i$ and $\bar{m} = m$. Set $t(i, m) = \bar{n}$. Then there is some function $s \in R^{(2)}$ such that $W_{s(i,m)} = W_{\bar{s}(i, t(i,m))}$. Obviously $\text{range}(s) \subseteq \text{dom}(C)$.

In order to see whether the first condition in Definition 34 holds, let $m \in \omega$ and $i \in x^{-1}(\text{dom}(F))$ with $F(x_i) \in B'_m$. Then we have that $W_{h(m)}$ intersects $\mathbb{M}(\varphi_i)$. Hence $t(i, m)\downarrow \in \mathbb{M}(\varphi_i)$, which means that $x_i \in B_{t(i,m)}$. It follows that $x_i \in C_{s(i,m)} \cap \text{dom}(F)$. By construction $t(i, m) \in W_{h(m)}$. Therefore $C_{s(i,m)} \cap \text{dom}(F) \subseteq F^{-1}(B'_m)$.

It remains to verify the second requirement. Set $r(i, n, m) = \bar{r}(i, n, t(i, m))$. Then we have for $i, m, n \in \omega$ such that $W_{s(i,m)}$ is not empty and $F(B_n) \nsubseteq B'_m$ that $t(i, m)\downarrow$ and $B_n \nsubseteq B_{t(i,m)}$, which implies that $\bar{r}(i, n, t(i, m))\downarrow \in \text{dom}(x)$ and $x_{\bar{r}(i,n,t(i,m))} \in \text{hl}_n(B_n) \setminus C_{\bar{s}(i,t(i,m))}$. $\square$

Note that $\mathcal{T}$ has a realizer for noninclusion if $\mathcal{T}$ is effectively separable, the topology τ is semi-regular, i.e., $B_n = \text{int}_\tau(\text{cl}_\tau(B_n))$, for all $n \in \omega$, and the sets $\text{ext}_\tau(B_n)$ are completely enumerable, uniformly in n.

To see this, let X be the dense subset of T_c and let $A \subseteq \omega$ witness its enumerability. Moreover, let $\bar{L} \subseteq \omega$ be such that for $i \in \text{dom}(x)$ and $m \in \omega$, $\langle i, m \rangle \in \bar{L}$ exactly if $x_i \in \text{ext}_\tau(B_m)$. Since the numbering x is computable, $B \leqslant_m C$. Let this be witnessed by $g \in R^{(1)}$. Set $s(i, m) = g(m)$. Then $\text{range}(s) \subseteq \text{dom}(C)$. The first condition in Definition 36 is obviously satisfied. For the second requirement let $\langle \bar{n}, \bar{m}, \bar{a} \rangle$ be the first element enumerated in

$$\{\langle n', m', a' \rangle \mid a' \in E \wedge n' \in \mathbb{M}(\varphi_{a'}) \wedge \langle a', m' \rangle \in \bar{L}\}$$

with $\bar{n} = n$ and $\bar{m} = m$, where $n, m \in \omega$, and define $r(i, n, m) = \bar{a}$.

If $B_n \nsubseteq B_m$, we have that B_n intersects the complement of B_m, which implies that it intersects its exterior, since τ is semi-regular. Hence, we can find a point x_a of the dense subset X in $B_n \cap \text{ext}_\tau(B_m)$. Thus $r(i, n, m)\downarrow \in \text{dom}(x)$ and $x_{r(i,n,m)} \in B_n \setminus B_m$, i.e., $x_{r(i,n,m)} \in \text{hl}_n(B_n) \setminus C_{s(i,m)}$.

As we have already seen, for every effective metric space and every effective f-space the exterior of B_n is completely enumerable, uniformly in n. The canonical topology of every metric space is semi-regular. In [29] it is shown that the class of f-spaces with a semi-regular topology contains important subclasses.

Now, combining the above two results we obtain the characterization we were looking for.

Theorem 38. *Let $\mathcal{T}$ be effectively separable and have a realizer for noninclusion. Then a map $F: T_c \rightharpoonup T'_c$ is effectively continuous, if and only if it is effective with a realizer $f \in R^{(1)}$ so that F has a witness for noninclusion with respect to $\mathcal{T}^f$.*

For total maps F, $\mathrm{WE}(f)$ coincides with T_c. By this way we obtain the ensuing special case.

Corollary 39. *Let $\mathcal{T}$ be effectively separable and have a realizer for noninclusion. Then a map $F: T_c \to T'_c$ is effectively continuous, if and only if it is effective and has a witness for noninclusion.*

A similar result has been derived in [28]. (Note that the semi-regularity requirement made there can be replaced by the assumption that topology τ has a realizer for noninclusion with respect to itself.)

If we put the above characterization together with Proposition 25, we have the following rather general answer to Weihrauch's problem.

Theorem 40. *Let $\mathcal{T}$ be effectively separable and have a realizer for noninclusion. Then exactly those effective maps $F: T_c \rightharpoonup T'_c$ have a computable extension $\bar{F}: T \rightharpoonup T'$ which are realized by a function $f \in R^{(1)}$ so that F has a witness for noninclusion with respect to $\mathcal{T}^f$.*

5. Special cases

There are important special cases in which total effective maps *have* a witness for noninclusion. As follows from [30, 28] this holds if the domain of F is the subset of all computable elements of a pointed space, or the codomain is the subset of all computable elements of an effective metric space and the domain is recursively separable. We shall now investigate whether similar results can be obtained for partial effective maps.

Lemma 41. *Let $\mathcal{M}'$ be an effective metric space. Then every effective map $F: T_c \rightharpoonup M'_c$ with a correct realizer $f \in R^{(1)}$ such that* $\mathrm{WE}(f)$ *contains an enumerable subset X with* $\mathrm{dom}(F) \subseteq \mathrm{cl}_\tau(X)$ *has a witness for noninclusion with respect to $\mathcal{T}^f$.*

Proof. Without restriction, let $T = \mathrm{WE}(f)$. Moreover, for $i, m \in \omega$, let $\langle \bar{\imath}, \bar{m}, \bar{a} \rangle$ be the first element enumerated in

$$\{\langle i', m', a' \rangle \mid a' \prec_{B'} m' \wedge a' \in \mathbb{M}(\varphi_{f(i')})\}$$

with $\bar{\imath} = i$ and $\bar{m} = m$. Set $t(i,m) = \bar{a}$. Then there is some function $s \in R^{(2)}$ such that $W_{s(i,m)} = \{ j \mid t(i,m) \in \mathbb{M}(\varphi_{f(j)})\}$. By the definition of $\mathrm{WE}(f)$ we have that $W_{s(i,m)}$ is an index set with respect to the numbering x of T_c. Thus $\mathrm{range}(s) \subseteq \mathrm{dom}(C)$.

Now, suppose that $i \in x^{-1}(\mathrm{dom}(F))$ and $m \in \omega$ with $F(x_i) \in B'_m$. Since the collection of all B'_n is a strong basis of the canonical topology Δ' on M', there is then some index a such that $a \prec_{B'} m$ and $F(x_i) \in B'_a$. Thus $t(i,m)\downarrow$ and $F(x_i) \in B'_{t(i,m)}$, which implies that $x_i \in C_{s(i,m)} \cap \mathrm{dom}(F)$. In addition, we have for $z \in C_{s(i,m)} \cap \mathrm{dom}(F)$ that $F(z) \in B'_{t(i,m)} \subseteq B'_m$.

Next, we define the function r. As has already been mentioned, T_c contains an enumerable dense subset. Let this be witnessed by the set $E \subseteq \omega$. Moreover, let

$$A = \{\langle\langle i,m\rangle, \langle j,n\rangle\rangle \mid \delta(\beta_i, \beta_j) > 2^{-m} + 2^{-n}\}.$$

Then B_c and $B_{c'}$ are disjoint, for any pair $\langle c, c'\rangle \in A$. Now, for $i, m, n \in \omega$, be $\langle \bar{\imath}, \bar{n}, \bar{m}, \bar{a}\rangle$ the first element enumerated in

$$\{\langle i', n', m', a'\rangle \mid a' \in E \wedge n' \in \mathbb{M}(\varphi_{a'}) \wedge (\exists c \in \mathbb{M}(\varphi_{f(a')}))\langle c, t(i', m')\rangle \in A\}$$

with $\bar{\imath} = i$, $\bar{n} = n$ as well as $\bar{m} = m$. Define $r(i,n,m) = \bar{a}$.

If $i, n, m \in \omega$ such that $W_{s(i,m)}$ is not empty and $F(B_n) \nsubseteq B'_m$, then $t(i,m)\downarrow$ and by the definition of the strong inclusion for effective metric spaces we have that $\mathrm{cl}_{\Delta'}(B'_{t(i,m)}) \subseteq B'_m$. Hence $F(B_n) \nsubseteq \mathrm{cl}_{\Delta'}(B'_{t(i,m)})$. Thus, there is some point $y \in B_n \cap \mathrm{dom}(F)$ with $F(y) \in \mathrm{ext}_{\Delta'}(B'_{t(i,m)})$, meaning that there exists some basic open set B'_c which is disjoint with $B'_{t(i,m)}$ and contains $F(y)$. Let Z be the set of all points x_j for which c is enumerated in $\mathbb{M}(\varphi_{f(j)})$. Then Z is completely enumerable. Moreover, it intersects B_n and since the realizer f is correct, it is also disjoint with $C_{s(i,m)}$. With [28, Lemma 6.9] it therefore follows that Z also intersects $B_n \cap X$. Thus $r(i,n,m)\downarrow \in \mathrm{dom}(x)$ and $x_{r(i,n,m)} \in B_n \setminus C_{s(i,m)}$. □

As follows from an example of Myhill [17, p. 293, Remark 2] and/or Pour-El [22], the above theorem is false without the extra condition on $\mathrm{WE}(f)$ (see also [21, p. 211, Proposition II.4.9]). (Since the Myhill/Pour-El map is not continuous, it follows from Proposition 35 that the map is only realizable by a recursive function f so that it has no witness for noninclusion with respect to $\mathscr{T}^f$.)

We have already seen that every effective metric space has a realizer for noninclusion. By definition, these spaces are effectively separable. From results in [33] it moreover follows for effectively continuous maps between effective metric spaces that the effective continuity can always be witnessed by a consistent recursive function. With Propositions 25, 35 and 37 we therefore obtain a characterization which tells us exactly which (partial) effective maps between effectively given metric spaces are extendable to a computable map.

Theorem 42. *Let $\mathscr{M}$ and $\mathscr{M}'$ be effective metric spaces. Then exactly those effective maps $F : M_c \rightharpoonup M'_c$ have a computable extension $\bar{F} : M \rightharpoonup M'$ which are realized by a correct function $f \in R^{(1)}$ so that* $\mathrm{WE}(f)$ *has an enumerable subset X with* $\mathrm{dom}(F) \subseteq \mathrm{cl}_{\Delta}(X)$.

A similar result has been obtained by Hertling [16] with respect to the Cauchy representation. Both results improve that of Weihrauch [32] and hence the well-known theorems by Moschovakis [20] and Ceĭtin [5], which all give only a sufficient condition for an effective map to have a computable extension: the subspace dom(F) is required to have an enumerable dense subset.

In [23, p. 370, Ex. 15–30(b),(c)] an effective map $F : R^{(1)} \to \omega$ is given which is *not* the restriction of a *total* computable map between the effective metric spaces $\mathbb{B}$ and ω. This shows that the above theorem as well as Theorem 40 cannot be improved, in the sense that even if F is total, we cannot expect, in general, that the computable extension $\bar{F}$ is a total map.

Let us now consider the case that the domain of a given effective map is a subset of computable elements of a pointed space. As we have previously seen, pointed spaces $\mathscr{T}$ are effectively separable with dense base Pt. Moreover $Pt \subseteq T_c$. From Corollary 33 we know that, in general, we cannot expect a partial effective map between pointed spaces to be continuous, even if the domain of the map is recursively separable. The situation is different, however, if the domain of the map is itself effectively pointed, which means that without restriction we can consider the map to be a total map on the computable elements of a pointed space. In this case it has a witness for noninclusion.

Lemma 43. *Let $\mathscr{T}$ be pointed. Then every total effective map $F : T_c \to T'_c$ has a witness for noninclusion.*

Proof. As $\mathscr{T}$ is pointed, we have that T_c is effectively pointed. Let this be witnessed by the function $\mathrm{pd} \in P^{(1)}$. Then, for $i, n, m \in \omega$, set $r(i,n,m) = \mathrm{pd}(n)$ and let $s \in R^{(2)}$ with $W_{s(i,m)} = \{ j \mid m \in \mathbb{M}(\varphi_{f(j)}) \}$, where $f \in R^{(1)}$ realizes F. Since F is total, $W_{s(i,m)}$ is a index set with respect to the numbering x of T_c. Thus range(s) $\subseteq$ dom(C). Obviously, $C_{s(i,m)} = \{ z \in T_c \mid F(z) \in B'_m \}$. As follows from [28, Lemma 6.4], F is monotone with respect to the specialization order. With both properties it is now readily verified that (s,r) is a witness for noninclusion of F. □

As a consequence we have that every pointed space has a realizer for noninclusion. Therefore, by applying Theorem 40 and Lemma 26, we obtain the following coincidence of effective and computable maps between pointed spaces.

Theorem 44. *Let $\mathscr{T}$ and $\mathscr{T}'$ be pointed. Then every total effective map $F : T_c \to T'_c$ has a* (*possibly partial*) *computable extension $\bar{F} : T \rightharpoonup T'$ and*, *conversely*, *the restriction of each computable map $G : T \rightharpoonup T'$ to the computable elements of T is effective. If*, *in addition*, *$\mathscr{T}'$ is strongly complete*, *the every such extension $\bar{F}$ is uniquely determined by F and total.*

Obviously, every effective continuous dcpo is strongly complete. As a special case we thus obtain the well-known Myhill/Shepherdson Theorem (cf. [32, 34]). But the result also holds for maps between effective A- and f-spaces. In addition, as a consequence of Lemma 26, Theorem 40 and Lemma 43, we gain that the total effective maps between the subspaces of all computable elements of a pointed space and a complete weakly effective metric space are exactly the extensions of uniquely determined total computable maps between these spaces.

6. Conclusion

In this paper we have given an answer to Weihrauch's question whether and, if not always, when an effective map between subspaces of computable elements of two represented sets is extendable to a computable map, in the context of effectively presented topological T_0-spaces. Considering topological spaces is not a restriction in dealing with this problem, as any representation induces a topology on the represented set. In the case of the representations considered here this topology is equivalent to the given one.

We characterized the effective maps that have a computable extension. The additional requirement that is satisfied by the extendable maps allows the generation of negative information. In the context of effectively given topological spaces, one can, in general, only list the (computable) elements of basic open sets, but not those of their complements. Such information is necessary in order to force an effective map to have a computable extension. In special cases, such as effective metric spaces, in which this information can always be obtained, the extra condition can be verified.

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