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Maximum density for the Sierpinski carpet

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ABSTRACT

We prove that there exists a closed convex set obtaining the maximum density for the Sierpinski carpet S. That is, there exists a closed convex set $V \subset E_0$, with $|V| > 0$, such that $\sup\left\{\frac{\mu(U)}{|U|^s} : U \subset E_0, \text{ is closed}\right\} = \frac{\mu(V)}{|V|^s}$, where E_0 is defined in the introduction and μ denotes the unique self-similar probability measure on S. We give a reasonable description about the shape of V.

0. Introduction and the main result

The following famous fractal set is from [1]. Begin with a unit square E_0 (with the inside) of side 1. Subdivide it into 9 smaller squares of side length $\frac{1}{3}$ by trisecting the sides. For the next approximation E_1, the trema to be removed is the center square. That means 8 small squares remain. (The boundaries of these 8 squares must also remain, so that the set will be compact.) Inductively, for $n \geq 1$ continue in this way, at the nth stage replacing each square of E_{n-1} by its 8 smaller squares of side length $\left(\frac{1}{3}\right)^n$ to get E_n. We obtain $E_0 \supset E_1 \supset \cdots \supset E_n \supset \cdots$. The non-empty set $S = \bigcap_{n=0}^{\infty} E_n$ is called the Sierpinski carpet. For each $n \geq 0$, E_n consists of 8^n squares with side length 3^{-n}. Any one of such squares is called a 3^{-n}-basic square. The Hausdorff dimension of S is $s = \dim_H(S) = \log_3 8$.

The Sierpinski carpet can be obtained as an iterated function system construction. It is made up of 8 parts, each similar to the whole with contraction ratio $\frac{1}{3}$. Suppose that 8 similar contraction maps are f_i, $(i = 1, 2, \ldots, 8)$. (see Fig. 0.1). Let $S_n = \{f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_n}(S) : 1 \leq i_1, i_2, \ldots, i_n \leq 8\}$.

Let μ denote the unique self-similar probability measure on S. It is easy to know that for any Borel set $U \subset R^2$, $\mu(U) = \frac{H^s(S \cap U)}{H^s(S)}$. For any subset $U \subset R^2$, we define the density $d(U) = \frac{\mu(U)}{|U|^s}$.

In [2], it was showed that the maximum density $\sup\{d(J) : J \subset [0, 1]\}$ for a linear Cantor set is attained in the field of sets generated by some stage basic intervals. However, it is not true in higher dimensional Euclidean spaces. In this paper, we prove that

Theorem 0.1. *For the Sierpinski carpet, there exists a closed convex set* $V \subset E_0$, *with* $|V| > 0$, *such that* $\sup\{d(U) : U \subset E_0, \text{ is closed}\} = d(V)$.

1. Some lemmas

Let $D \subset R^n$ be a non-empty set. $E \subset R^n$ is a self-similar set defined by m similar contracting maps $f_i : D \to D$, with contracting ratios, $0 < c_i < 1$, $(i = 1, 2, \ldots, m)$ and satisfying open set condition, that is, there exists a non-empty open set U for which we have $f_i[U] \cap f_j[U] = \phi$ for $i \neq j$ and $U \supseteq f_i[U]$ for all i. Then

$$\dim_H(E) = s, \quad 0 < H^s(E) < +\infty,$$

where s satisfies $\sum_{i=1}^{m} C_i^s = 1$, $\dim_H(E)$ and $H^s(E)$ denote the Hausdorff dimension and measure of E, respectively.

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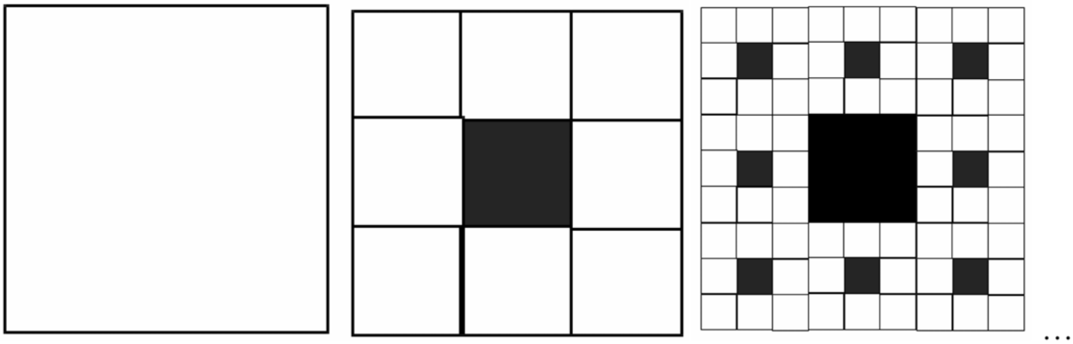

Fig. 0.1. The construction of the Sierpinski carpet.

Lemma 1.1 ([3]). *Suppose that E is a self-similar set satisfying the open set condition and* $s = \dim_H(E)$, *then for any measurable set U, we have*

$$H^s(E \cap U) \leq |U|^s.$$

Using the definition of the Hausdorff measure and the self-similarity of the Sierpinski carpet, we can obtain the following

Lemma 1.2.

$$\begin{aligned} H^s(S) = H^s_\delta(S) &= \inf\left\{\sum_i |U_i|^s : \{U_i\} \text{ is a } \delta\text{-closed cover of } S\right\} \\ &= \inf\left\{\sum_i |U_i|^s : \{U_i\} \text{is a closed cover of } S\right\} \\ &= \inf\left\{\sum_i |U_i|^s : \bigcup_i U_i = S, U_i \text{ is closed}\right\}. \end{aligned}$$

Lemma 1.3. $\sup\left\{\frac{\mu(U)}{|U|^s} : U \subset S \text{ is closed}\right\} = \frac{1}{H^s(S)}$.

Proof. Set $L = \sup\left\{\frac{\mu(U)}{|U|^s} : U \subset S \text{ is closed}\right\}$.

By Lemma 1.1, we have $\frac{\mu(U)}{|U|^s} = \frac{H^s(S\cap U)}{H^s(S)|U|^s} \leq \frac{1}{H^s(S)}$, so $L \leq \frac{1}{H^s(S)}$.

For each closed cover $\{U_i\}$ of K with $\bigcup_i U_i = S$, by the definition of L, we have $\frac{\mu(U_i)}{|U_i|^s} \leq L$, so

$$H^s(S) = H^s\left(\bigcup_i (S \cap U_i)\right) \leq \sum_i H^s(S \cap U_i) \leq LH^s(S)\sum_i |U_i|^s.$$

By Lemma 1.2, $1 \leq LH^s(S)$. Therefore $L \geq \frac{1}{H^s(S)}$. □

Lemma 1.4. *Let* $W \subset S$ *be a non-empty closed set with* $|W| < \frac{1}{27}$. *Then there exists a non-empty closed set* $W' \subset S$ *with* $|W'| \geq \frac{1}{27}$ *such that*

$$\frac{\mu(W)}{|W|^s} = \frac{\mu(W')}{|W'|^s}.$$

Proof. Since $|W| < \frac{1}{27}$, W at most intersects with three elements of $\{f_i(S), (i = 1, 2, \ldots, 8)\}$.

Case (1). W intersects three elements, $\Delta^1_1, \Delta^1_2, \Delta^1_3$ of $\{f_i(S), (i = 1, 2, \ldots, 8)\}$

When $|\Delta^1_1 \cup \Delta^1_2 \cup \Delta^1_3| \geq 1$, $|W| \geq \frac{1}{3}$. This contradicts $|W| < \frac{1}{27}$.

So $|\Delta^1_1 \cup \Delta^1_2 \cup \Delta^1_3| < 1$, (See Fig. 1.1)

If W at least intersects four elements of $\{f_i \circ f_j(S), (i, j = 1, 2, \ldots, 8)\}$, then $|W| \geq \frac{1}{9}$. This contradicts $|W| < \frac{1}{27}$. So W at most intersects three elements $\Delta^1_{21}, \Delta^1_{22}, \Delta^1_{23}$ of $\{f_i \circ f_j(S), (i, j = 1, 2, \ldots, 8)\}$ and $|\Delta^1_{21} \cup \Delta^1_{22} \cup \Delta^1_{23}| < \frac{1}{3}$ (See Fig. 1.1).

By self-similarity, there exist a positive integer t and $\Delta^1_t, \Delta^2_t, \Delta^3_t \in S_t$ such that $3^t\Delta^1_t = \Delta^1_1$, $3^t\Delta^2_t = \Delta^2_1$, $3^t\Delta^3_t = \Delta^3_1$ and $W \subset \Delta^1_t \cup \Delta^2_t \cup \Delta^3_t$, W intersects four elements of S_{t+1} or W intersects three elements $\Delta^1_{t+1}, \Delta^2_{t+1}, \Delta^3_{t+1}$, of S_{t+1} but

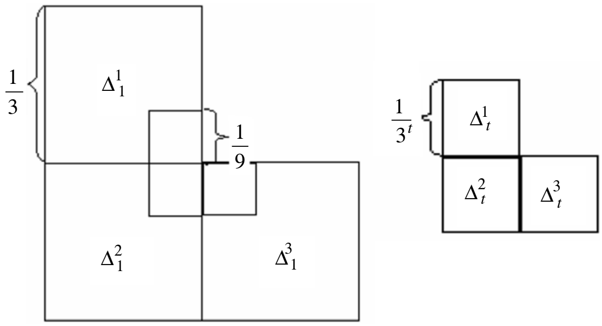

Fig. 1.1.

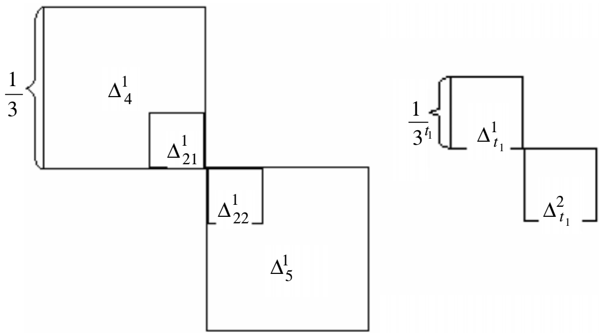

Fig. 1.2.

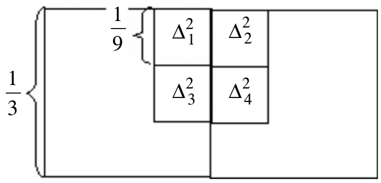

Fig. 1.3.

$|\Delta^1_{t+1} \cup \Delta^2_{t+1} \cup \Delta^3_{t+1}| \geq \frac{1}{3^t}$. Set $W' = 3^t W$. Note that

$$\frac{\mu(W)}{|W|^s} = \frac{\mu(3^t W)}{|3^t W|^s}.$$

We have $|W'| \geq \frac{1}{9} > \frac{1}{27}$ and $\frac{\mu(W)}{|W|^s} = \frac{\mu(W')}{|W'|^s}$.

Case (2). W intersects two elements, Δ^1_4, Δ^1_5 of $\{f_i(S), (i = 1, 2, \ldots, 8)\}$.

Since $|W| < \frac{1}{27}$, Δ^1_4 and Δ^1_5 have a common intersection point orΔ^1_4 and Δ^1_5 have a common side.

(a) Δ^1_4 and Δ^1_5 have a common intersection point (See Fig. 1.2)

Similar to Case (1), by self-similarity, there exist a positive integer t_1 and $\Delta^1_{t_1}$, $\Delta^2_{t_1}$, $\in S_{t_1}$ such that $3^{t_1}\Delta^1_{t_1} = \Delta^1_4$, $3^{t_1}\Delta^2_{t_1} = \Delta^1_5$, and $W \subset \Delta^1_{t_1} \cup \Delta^2_{t_1}$, W at least intersects three elements of S_{t_1+1} or W intersects two elements $\Delta^1_{t_1+1}$, $\Delta^2_{t_1+1}$, of S_{t+1} but $|\Delta^1_{t_1+1} \cup \Delta^2_{t_1+1}| \geq \frac{1}{3^{t_1}}$. Set $W' = 3^{t_1} W$. Note that

$$\frac{\mu(W)}{|W|^s} = \frac{\mu(3^{t_1} W)}{|3^{t_1} W|^s}.$$

We have $|W'| \geq \frac{1}{9} > \frac{1}{27}$ and $\frac{\mu(W)}{|W|^s} = \frac{\mu(W')}{|W'|^s}$.

(b) Δ^1_4 and Δ^1_5 have a common side (See Fig. 1.3).

Since $|W| < \frac{1}{27}$, W at most intersects with four elements of $\{f_i \circ f_j(S), (i, j = 1, 2, \ldots, 8)\}$

(b_1) W intersects four elements Δ^2_1, Δ^2_2, Δ^2_3, Δ^2_4 of $\{f_i \circ f_j(S), i, j = 1, 2, \ldots, 8\}$. Since $|W| < \frac{1}{27}$, $|\Delta^3_1 \cup \Delta^3_2 \cup \Delta^3_3 \cup \Delta^3_4| < \frac{1}{9}$ (See Fig. 1.4).

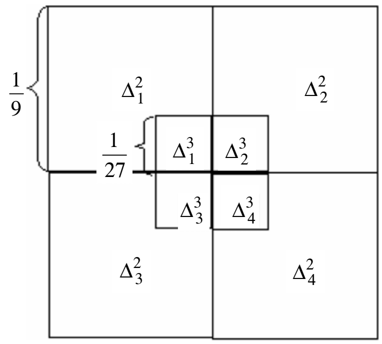

Fig. 1.4.

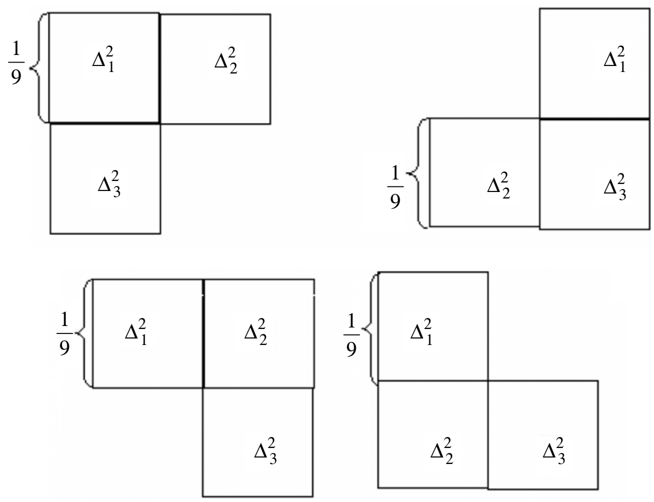

Fig. 1.5.

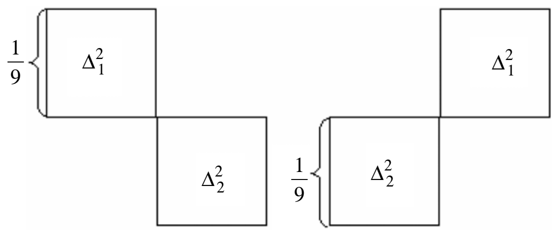

Fig. 1.6.

Similar to Case (1), by self-similarity, there exists a positive integer t_2,
Set $W' = 3^{t_2}W$. We have $|W'| \geq \frac{1}{27}$ and $\frac{\mu(W)}{|W|^s} = \frac{\mu(W')}{|W'|^s}$.
(b_2) W intersects three elements of $\{f_i \circ f_j(S), (i, j = 1, 2, \ldots, 8)\}$ (See Fig. 1.5), by self-similarity, this belongs to Case (1).
(b_3) W intersects two elements of Δ_1^2, Δ_2^2

$$\{f_i \circ f_j(S), (i, j = 1, 2, \ldots, 8)\}.$$

When Δ_1^2, Δ_2^2 have a common intersection point (See Fig. 1.6), by self-similarity, this belongs to Case (2)-(a).
When Δ_1^2, Δ_2^2 have a common side, by self-similarity, this belongs to Case (2)-(b).
Case (3). If W only intersects a element of $\{f_i(S), (i = 1, 2, \ldots, 8)\}$, by self- similarity, there exist a positive integer l and $i_1^0, i_2^0, \ldots i_l^0 \in \{1, 2, \ldots, 8\}$ such that $W \subset E_{i_1^0 i_2^0 \cdots i_l^0} = f_{i_1^0} \circ f_{i_2^0} \circ \cdots \circ f_{i_l^0}(S)$ and W intersects at least two of $E_{i_1^0 i_2^0 \cdots i_l^0 1}, E_{i_1^0 i_2^0 \cdots i_l^0 2}, \ldots, E_{i_1^0 i_2^0 \cdots i_l^0 8}$. Therefore $f_{i_l^0}^{-1} \circ \cdots \circ f_{i_2^0}^{-1} \circ f_{i_1^0}^{-1}(W)$ intersects at least two of $f_1(S), f_2(S), \ldots, f_8(S)$. Note that $\frac{\mu(W)}{|W|^s} = \frac{\mu(f_{i_l^0}^{-1} \circ \cdots \circ f_{i_2^0}^{-1} \circ f_{i_1^0}^{-1}(W))}{|f_{i_l^0}^{-1} \circ \cdots \circ f_{i_2^0}^{-1} \circ f_{i_1^0}^{-1}(W)|^s}$.

By Case (1) and Case (2), there exists $W' \subset S$ with $|W'| \geq \frac{1}{27}$ such that

$$\frac{\mu(W)}{|W|^s} = \frac{\mu(W')}{|W'|^s}. \quad \square$$

From Lemmas 1.3 and 1.4, it follows that

Lemma 1.5. $\sup\left\{\frac{\mu(U)}{|U|^s} : U \subset S, |U| \geq \frac{1}{27}, U \text{ is closed}\right\} = \frac{1}{H^s(S)}$.

2. The proof of the main theorem

The following definitions are from [4].

If $E \subset R^n$, the $\delta-$*parallel body* of E is the closed set of points within distance δ of E that is, $[E]_\delta = \{x \in R^n : \inf_{y\in E} |x - y| \leq \delta\}$.

The Hausdorff metric δ is defined on the collection of all non-empty compact subsets of R^n by $h(E, F) = \inf\{\delta : E \subset [F]_\delta \text{and} F \subset [E]_\delta\}$.

Let $B(R^n)$ be the sets of all non-empty compact sets in R^n. Then the set $B(R^n)$ equipped with the above Hausdorff metric h becomes a complete metric space.

The convex hull of $U \subset R^n$ is the intersection of all the convex sets which contain U, and it is denoted by conv U.

Lemma 2.1. *Let $\{A_n\}$ be a sequence of non-empty compact subsets of R^n. If $\{A_n\}$ converges to $A \subset R^n$ with the Hausdorff metric, then*
(i) $\lim_{n\to\infty} |A_n| = |A|$,
(ii) $\lim_{n\to\infty} \sup \mu(A_n) \leq \mu(A)$.

Proof. (i) Because $\{A_n\}$ converges to A with the Hausdorff metric, for $\forall\varepsilon > 0$, $\exists N > 0$, when $n \geq N$, $h(A_n, A) \leq \varepsilon$, i.e. $A_n \subset A_\varepsilon$ and $A \subset (A_n)_\varepsilon$. Thus for $n \geq N$, we have $|A_n| \leq |A| + 2\varepsilon$ and $|A| \leq |A_n| + 2\varepsilon$, So (i) holds.

(ii) From (i), $A_n \subset A_\varepsilon$, so $\mu(A_n) \leq \mu(A_\varepsilon)$.

Therefore

$$\lim_{n\to\infty} \sup \mu(A_n) \leq \lim_{\varepsilon\to 0} \mu(A_\varepsilon) = \mu(A). \quad \square$$

Proof of Theorem 0.1. By Lemma 1.5, there exists a closed set sequence $\{U_i\}$ in S, such that $|U_i| \geq \frac{1}{27}$ and $\frac{\mu(U_i)}{|U_i|^s} \to \frac{1}{H^s(S)}$, $i \to \infty$. Since S is compact, it follows that U_i is uniformly bounded. By the Blaschke selection theorem (see Theorem 3.16 of [4]), there exists a subsequence $\{U_{i_k}\}$of $\{U_i\}$such that $\{U_{i_k}\}$ converges to a non-empty compact set $W \subset S$ with the Hausdorff metric. Without loss of generality, we suppose U_i converges to W in the Hausdorff metric. By (i) of Lemma 2.1, $|W| \geq \frac{1}{27}$. By (ii) of Lemma 2.1,

$$\lim_{i\to\infty} \sup \mu(U_i) \leq \mu(W).$$

By the definition of upper limit, there exists a subsequence $\{U_{i_j}\}$of $\{U_i\}$such that $\lim_{i\to\infty} \sup \mu(U_i) = \lim_{j\to\infty} \mu(U_{i_j})$. Thus by Lemma 1.5, we have

$$\frac{1}{H^s(S)} \geq \frac{\mu(W)}{|W|^s} \geq \frac{\lim\limits_{i\to\infty} \sup \mu(U_i)}{\lim\limits_{i\to\infty} |U_i|^s} = \frac{\lim\limits_{j\to\infty} \mu(U_{i_j})}{\lim\limits_{j\to\infty} |U_{i_j}|^s} = \lim_{j\to\infty} \frac{\mu(U_{i_j})}{|U_{i_j}|^s} = \frac{1}{H^s(S)}.$$

Set $V = \text{conv}(W)$ which is the convex hull of W. $V \subset E_0$ is a closed convex set and $|V| = |W|$, $\mu(W) \leq \mu(V)$. So,

$$\frac{1}{H^s(S)} = \frac{\mu(W)}{|W|^s} \leq \frac{\mu(V)}{|V|^s}.$$

By Lemma 1.1,

$$H^s(S) = \frac{H^s(V \cap S)}{\mu(V)} \leq \frac{|V|^s}{\mu(V)}.$$

So,

$$\frac{1}{H^s(S)} = \frac{\mu(V)}{|V|^s}.$$

That is $\sup\left\{\frac{\mu(U)}{|U|^s} : U \subset S \text{is closed}\right\} = \frac{1}{H^s(S)} = \frac{\mu(V)}{|V|^s}$, where $V \subset E_0$ is a closed convex set, $|V| = W| \geq \frac{1}{27}$.

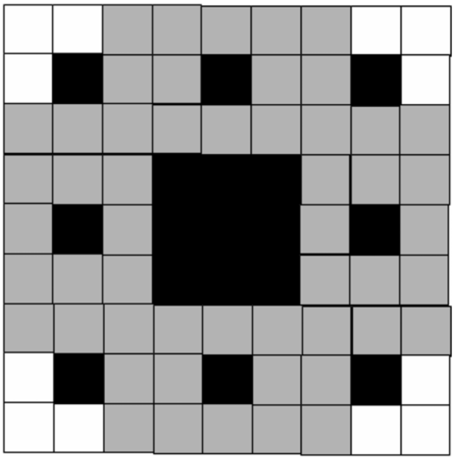

Fig. 3.1.

By Lemma 1.1,

$$\frac{1}{H^s(S)} \geq \sup\left\{\frac{\mu(U)}{|U|^s} : U \subset E_0 \text{ is closed}\right\}$$
$$\geq \sup\left\{\frac{\mu(U)}{|U|^s} : \quad U \subset S \text{ is closed}\right\},$$
$$= \frac{1}{H^s(S)} = \frac{\mu(V)}{|V|^s}. \quad \square$$

3. A description of the shape of *V*

For $n \geq 1$, let $S_n = \{f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_n}(S) : 1 \leq i_1, i_2, \ldots, i_n \leq 8\}$.
The following proposition is from [5].

Proposition 3.1. *For $n \geq 1$, $1 \leq k \leq 8^n$, let $\Delta_1\Delta_2, \ldots, \Delta_k \in S_n$ and μ be the common self-similar probability measure on S, $\mu(f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_n}(K)) = \left(\frac{1}{8}\right)^n$.*

Let $b_k = \min_{\Delta_1,\ldots,\Delta_k \in S_n}\left\{\frac{|\bigcup_{i=1}^k \Delta_i|^s}{k4^{-n}}\right\}$, where the minimum is taken for all possible union of k elements of S_n and $a_n = \min_{1\leq k\leq 8^n}\{b_k\}$. Then for $n \geq 1$, a_n decreases and $\lim_{n\to\infty} a_n = H^s(S)$.

Suppose that

$$a_n = \min_{1\leq k\leq 8^n} \min_{\Delta_1,\ldots,\Delta_k \in S_n}\left\{\frac{|\bigcup_{i=1}^{k} \Delta_i|^s}{k8^{-n}}\right\} = \frac{|U_{k_n}|^s}{k_n 8^{-n}},$$

where the U_{k_n} is the union of some k_n elements of S_n.

Since

$$\lim_{n\to\infty} a_n = H^s(S), \qquad \frac{\mu(V)}{|V|^s} = \frac{1}{H^s(S)} = \frac{1}{\lim\limits_{n\to\infty} a_n}.$$

Our basic idea is that by computing the values of a_n, we get the components of U_{k_n} and the shapes of conv (U_{k_n}). *It is easy to know that* conv $(U_{k_n}) \to V$ *with the Hausdorff metric. So by means of the shape of* conv (U_{k_n}), *we can know the shape of V.*

It is easy to get that $a_1 = \sqrt{2}^s$, $a_2 = \frac{\sqrt{106}^s}{52}$ (See Fig. 3.1). Note that U_{k_2} consists of 52 squares with side length $\frac{1}{9}$. In [6], *two conjectures that $a_3 = \frac{\sqrt{\left(\frac{5}{9}\right)^2+1}^s}{\frac{444}{512}}$ and $a_4 = \frac{\left(\frac{2\sqrt{2}}{27}\right)^s}{\frac{5}{512}}$ are given (see* Figs. 3.2 *and* 3.3*). Note that U_{k_3} consists of 444 light black squares with side length $\frac{1}{27}$ and U_{k_4} consists of 40 squares with side length $\frac{1}{81}$ in the small circle with diameter $\left(\frac{2\sqrt{2}}{27}\right)^s$ in* Fig. 3.3.

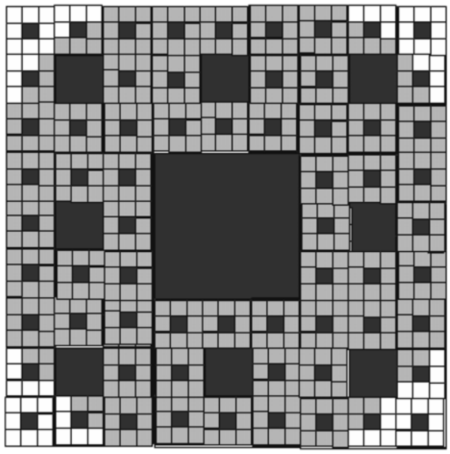

Fig. 3.2.

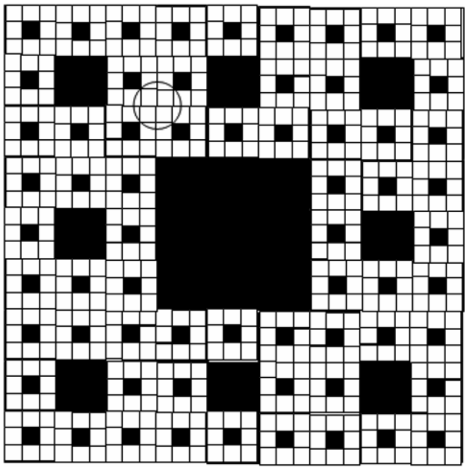

Fig. 3.3.

Therefore conv (U_{k_1}) *is a square.* conv (U_{k_2}) *is an octagon (see Fig. 3.1). We conjecture that* conv (U_{k_3}) *is a polygon with* 12 *sides and* conv (U_{k_3}) *is an octagon (see Fig. 3.3). Note that* conv (U_{k_3}) *is not symmetric. We conjecture that for* $n \geq 4$, conv (U_{k_n}) *is always in the small circle of Fig. 3.3.*

Acknowledgment

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