ADVANCES IN MATHEMATICS 56, 28–38 (1985)

Geometric Properties of Heisenberg-Type Groups

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INTRODUCTION

Gauges, or equivalently, left-invariant pseudodistances on the Heisenberg group, have been used for a long time. It was, however, only in 1978 that Cygan [3] noted that one of these natural gauges actually induces a distance, i.e., a left-invariant metric space structure on the group.

Peter Greiner posed the problem of studying the notion of arc length associated to this metric; in particular, he asked whether there is a kind of infinitesimal metric giving rise to the same arc length.

Section 2 of the present paper gives an answer to this question. It is shown that there is a certain "contravariant Riemannian metric" in the sense of [10] (roughly speaking, a Riemannian metric in which some vectors can have infinite length), studied previously by Gaveau [6, 7], which induces the arc length in question.

We will work in the greater generality of H-type groups introduced recently by Kaplan [11]. This class includes among others the nilpotent parts N in the Iwasawa decomposition $G = KAN$ of semisimple Lie groups of real rank one. In this especially interesting case the "contravariant Riemannian metric" (M_0) which we construct is characterized (up to a factor) by being left-invariant under N, invariant under the centralizer M of A in K, and transforming under A by a character. It is therefore the direct generalization of the standard metric on $\mathbf{R}^n$, which is characterized by the same properties when $\mathbf{R}^n$ is regarded as the N-part of $SO(n, 1)$.

In general, when an H-type group is not Abelian, (M_0) arises as the limit of a family of ordinary Riemannian metrics (M_c) $(c > 0)$. In Section 3 we consider geodesic arcs with respect to these metrics. For (M_0) on the Heisenberg group these were studied earlier by Gaveau [6, 7] and for (M_1) by Debiard [5] and Kaplan [12]. Besides some slight generalization of the previous results our goal here is to show that the (M_c)-geodesics joining two fixed points tend uniformly to the (M_0)-geodesic as $c \to 0$. From this one gets

* Partially supported by N.S.F. Grant MCS-79-20062.

0001-8708/85 $7.50

a very simple proof of the fact that the shortest (M_0)-geodesics keep their minimizing property even in the class of piecewise C^1 curves.

In a short final section it will be shown that with the aid of the metrics (M_c) one can construct simple examples which, similarly to the examples of Urakawa [14], show that there is no Faber–Krahn-type inequality connecting the Riemannian volume and the lowest eigenvalue of the Laplacian in the class of compact Riemann manifolds.

1. Lie Groups of Type H

Lie groups of type H; were introduced by Kaplan in [11]. They are defined as follows.

Let v, z be real Euclidean spaces and let j: $z \to \operatorname{End} v$ be a linear map satisfying

$$|j(Y)X| = |Y|\,|X|, \tag{1.1}$$

$$j(Y)^2 = -|Y|^2 I. \tag{1.2}$$

Now let $n = v + z$ and let the Lie algebra structure on n be defined so that z is the center and, for $X, X' \in v$ and all $Y \in z$,

$$\langle Y, [X, X']\rangle = \langle j(Y)X, X'\rangle. \tag{1.3}$$

An equivalent definition [11] is to say that $n = v + z$ is a two-step nilpotent Lie algebra with center z and such that for all $X \in v$ with $|X| = 1$, $\operatorname{ad}(X)$ is a surjective isometry of the orthocomplement $v \ominus \ker \operatorname{ad}(X)$ onto z.

N is a group of type H if it is simply connected and its Lie algebra is of type H. It is easy to see that such groups have a particularly simple representation theory, since the orbits of the coadjoint representation are parametrized by v (one-point orbits) and $z - \{0\}$ (orbits that are affine subspaces isomorphic with v, permuted transitively by the authomorphism group of N). It is known that groups of type H have a natural gauge defined by

$$\exp(X + Y) = (|X|^4 + 16\,|Y|^2)^{1/4} \tag{1.4}$$

for $X \in v$, $Y \in z$; by [4] this gauge satisfies

$$|gh| \leqslant |g| + |h| \tag{1.5}$$

for all $g, h \in N$.

The following result is contained in [12]; we give here a simple proof which is independent of classification.

PROPOSITION 1.1. *The nilpotent part in the Iwasawa decomposition of a semisimple Lie group of real rank one is of type H.*

Proof. Let $g = k + a + n$ be the Iwasawa decomposition at the Lie algebra level, let θ be the Cartan involution, let α and 2α denote the positive restricted roots, $g_\alpha, g_{2\alpha}$ the corresponding root spaces, and p, q their respective dimensions. We may assume $p, q = 0$, for otherwise the proposition is trivial. Denoting the Killing form by B, we define $H_\alpha \in a$ by the condition $\alpha(H) = B(H, H_\alpha)$ for all $H \in a$. As in [13], we define the number $b > 0$ by

$$2b^2 = \alpha(H_\alpha) = B(H_\alpha, H_\alpha). \tag{1.6}$$

(Incidentally, an easy computation shows that $4b^2 = (p + 4q)^{-1}$.)

Now we set $z = g_{2\alpha}$, $v = g_\alpha$, we define the Eulidean structure on $n = v + z$ by

$$\langle Z, Z' \rangle = -4b^2 B(Z, \theta Z') \tag{1.7}$$

and for all $Y \in z$, $X \in v$ we set

$$j(Y)X = \operatorname{ad}(Y)\,\theta X. \tag{1.8}$$

We have to verify (1.1)–(1.3); this is done by computations similar to those in [9, pp. 54–55] and is based on the identity

$$[Y, \theta Y] = B(Y, \theta Y)\, H_{2\alpha}$$

for $Y \in z$. In fact, using the Jacobi identity we find

$$\begin{aligned} j(Y)^2 X &= [Y, \theta[Y, \theta X]] = [Y, [\theta Y, X]] \\ &= [[Y, \theta Y], X] = B(Y, \theta Y)[H_{2\alpha}, X] = -2\alpha(H_\alpha) B(Y, \theta Y) X, \end{aligned}$$

which proves (1.2). The invariance of the Killing form together with the final part of the computation just performed give

$$B(j(Y)X, j(Y)X) = B(\theta X, [Y, [\theta Y, X]]) = -2\alpha(H_\alpha) B(Y, \theta Y) B(X, \theta X)$$

proving (1.1). Finally (1.3) follows from the invariance of the Killing form:

$$\begin{aligned} \langle Y, [X, X'] \rangle &= -4b^2 B(Y, \theta[X, X']) \\ &= -4b^2 B([Y, \theta X], \theta X') = \langle j(Y) X, X' \rangle. \end{aligned}$$

Remark. The gauge defined by (1.4) is in this case equal to

$$2(b^4 B(X, \theta X)^2 + 4b^2 B(Y, \theta Y))^{1/4},$$

which, up to the factor 2 agrees with the gauge occurring in a natural way in [9, Theorem 1.14].

2. Arc Length and the Infinitesimal Metric

Let N be a group of type H; we will follow the notations of Section 1. With the aid of the gauge (1.4) we define a distance function d on N by

$$d(g, h) = |h^{-1}g|. \tag{2.1}$$

(The triangle inequality follows from (1.5).)

If $\gamma(s)$ ($s \in [a, b]$) is a curve in N, we consider the associated arc length: Letting

$$a = s_0 < s_1 < \cdots < s_n = b \tag{2.2}$$

we write $\gamma_k = \gamma(s_k)$, $d_k = d(\gamma_k, \gamma_{k-1})$, $\Delta s_k = s_k - s_{k-1}$. The length of γ is

$$l(\gamma) = \sup \sum_{k=1}^{n} d_k$$

with the "sup" taken over all partitions (2.2) and is the same as "lim sup" for $\operatorname{Max}\{\Delta s_k\} \to 0$.

Since the exponential map is bijective, any curve can be written in the form $\gamma(s) = \exp(X(s) + Y(s))$ with some curves $X(x)$, $Y(s)$ in v and z, respectively. We identify v, z with their tangent spaces and denote by $\dot{X}(s)$, $\dot{Y}(s)$ the tangent vectors of $X(s)$, $Y(s)$.

Proposition 2.1. *Suppose $\gamma(s) = \exp(X(s) + Y(s))$ ($s \in [a, b]$) is a C^2-curve. Then $l(\gamma) < +\infty$ if and only if*

$$\dot{Y} - \tfrac{1}{2}[X, \dot{X}] = 0. \tag{2.3}$$

If (2.3) holds, we have

$$l(\gamma) = \int_a^b |\dot{X}(s)| \, ds. \tag{2.4}$$

Proof. Writing $X_k = X(s_k)$, the Campbell–Hausdorff formula gives

$$d_k = |\exp(X_k - X_{k-1} + Y_k - Y_{k-1} - \tfrac{1}{2}[X_{k-1}, X_k])|.$$

Using the notation $\Delta X_k = X_k - X_{k-1}$, etc., by (1.4) this can be rewritten as

$$d_k = (|\Delta X_k|^4 + 16\,|\Delta Y_k - \tfrac{1}{2}[X_{k-1}, \Delta X_k]|^2)^{1/4}.$$

By Taylor's formula, with an obvious notation we have

$$\Delta X_k = \dot{X}_{k-1}\,\Delta s_k + \ddot{X}_{k-1}\,\Delta s_k^2 + o(\Delta s_k^2)$$

and similarly for ΔY_k, with $o(\Delta s_k^2)$ being uniform over $[a, b]$. Therefore,

$$d_k = \left\{ |\ddot{X}_{k-1} + o(1)|^4 + \left| \frac{1}{s_k} \left(\dot{Y}_{k-1} - \frac{1}{2}\,[X_{k-1}, \dot{X}_{k-1}] \right) \right.\right.$$
$$\left.\left. + \frac{1}{2} \left(\ddot{Y}_{k-1} - \frac{1}{2}\,[X_{k-1}, \ddot{X}_{k-1}] \right) + o(1) \right|^2 \right\}^{1/4} \Delta s_k.$$

Suppose that (2.3) holds. Then, differentiating, we also have

$$\ddot{Y} - \tfrac{1}{2}[X, \ddot{X}] = 0$$

and $\sum d_k$ tends to the limit (2.4) as $\mathrm{Max}\{\Delta s_k\} \to 0$. On the other hand, if (2.3) fails, then on some subinterval $\dot{Y} - \frac{1}{2}[X, \dot{X}]$ will stay close to a constant non-zero vector, and hence $\sum d_k$ will be unbounded. This finishes the proof.

To proceed further, for every $c > 0$ we introduce a Riemannian metric (M_c) on N, by setting for $X \in v$, $Y \in z$,

$$\|X + Y\|_c = (|X|^2 + c^{-2}\,|Y|^2)^{1/2}.$$

($X + Y \in n$ is regarded here as a left-invariant vector field.) As pointed out in [13], in the case of the rank one Iwasawa groups these are the only left-invariant metrics invariant under the action of the subgroup M (but they do not transform by a character of A, except in the degenerate case where N is Abelian).

We define (M_0) as the limit for $c \to 0$; this is a "contravariant Riemannian metric" in the sense of [10], the length of $X + Y$ is infinite in case $Y = 0$. So the length $l_0(\gamma)$ with respect to (M_0) of a curve γ is finite if and only if the tangent $\dot{\gamma}(s)$ is in v for all s.

We denote the length with respect to (M_c) by $l_c(\gamma)$.

PROPOSITION 2.2. *For any C^1-curve $\gamma(s) = \exp(X(s) + Y(s))$, $s \in [a, b]$, we have, for $c > 0$,*

$$l_c(\gamma) = \int_a^b \left(|\dot{X}|^2 + \frac{1}{c^2} \left| \dot{Y} - \frac{1}{2}\,[X, \dot{X}] \right|^2 \right)^{1/2} ds.$$

Furthermore, $l_0(\gamma) < +\infty$ if and only if (2.3) holds for all s. In this case

$$l_0(\gamma) = \int_a^b |\dot{X}(s)|\, ds.$$

Proof. It is known that the differential $\exp_*$ of the exponential map at a point Z is given by

$$\frac{I-\exp(-\operatorname{ad} Z)}{\operatorname{ad} Z}=I-\frac{1}{2}\operatorname{ad} Z+\cdots$$

(cf., e.g., [8, p. 95], where there is also a translation term which we do not need since we regard the Lie algebra as the set of left-invariant vector fields). The tangent vector $\dot{\gamma}(s)$ is the image of $\dot{X}(s)+\dot{Y}(s)$ under $\exp_*$ at $X(s)+Y(s)$, so

$$\dot{\gamma}(s)=\dot{X}+\dot{Y}-\tfrac{1}{2}[X,\dot{X}]$$

all other brackets being equal to zero. The proposition now follows from the definitions.

Remark. It follows that if a curve satisfies (2.3), then

$$l_c(\gamma)=l_0(\gamma)$$

for all $c>0$.

3. Geodesics

We consider the behaviour of the (M_c)-geodesics as $c\to 0$. Since the geodesics for (M_1) and for (M_0) have been studied earlier (at least in the case of the Heisenberg group) [12, 5–7], we omit most of the computations.

We will use the notations

$$m(x)=x-\sin x,$$

$$\mu(x)=\frac{d}{dx}\log m(x)=\frac{1-\cos x}{x-\sin x}.$$

We note that $\mu(x)=\theta(x/2)^{-1}$ with the θ of Gaveau [6, p. 112].

For (M_0) we define a geodesic arc joining e to the point $\exp(X_1+Y_1)$ as an extremal of the variation problem associated to minimizing the arc length. More exactly, this will be a curve $\gamma(s)=\exp(X(s)+Y(s))$, $s\in[0,1]$, with $X(0)=Y(0)=0$, $X(1)=X_1$, $Y(1)=Y_1$, minimizing the integral

$$\int_0^1 |\dot{X}|\,ds$$

under the condition

$$2Y_1 = \int_0^1 [X, \dot{X}]\, ds.$$

($Y(s)$ is then automatically determined by (2.3).) The Euler equation of this problem is obtained by introducing a z-valued Lagrange multiplier λ and adding $\langle \lambda, [X, \dot{X}] \rangle$ to X. A simple computation using (1.3) gives the Euler equation

$$\frac{\partial}{\partial s} \frac{\dot{X}}{|\dot{X}|} = 2j(\lambda)\dot{X}.$$

This equation can be integrated explicitly, similarly to the equation (17) in [12], and one finds the following geodesic arcs:

If $Y_1 = 0$ (this corresponds to $\lambda = 0$), then

$$X(s) = sX_1, \qquad Y(s) = 0. \tag{3.1}$$

If $Y_1 \neq 0$, then, with the notation $Y_1' = Y_1/|Y_1|$,

$$\begin{aligned} X(s) &= (\exp a_0 s j(Y_1') - I)\, W_0, \\ Y(s) &= \tfrac{1}{2} m(a_0 s)\, |W_0|^2 Y_1'. \end{aligned} \tag{3.2}$$

Here a_0 is a positive solution of

$$\mu(a_0) = \frac{|X_1|^2}{4\,|Y_1|^2}. \tag{3.3}$$

(there are finitely many solutions if $X_1 \neq 0$, with exactly one in $(0, 2\pi)$; if $X_1 = 0$, the solutions are $a_0 = 2k\pi$, $k = 1, 2, \ldots$.) W_0 is determined in the case $X_1 \neq 0$ by

$$X_1 = (\exp a_0\, j(Y_1') - I)\, W_0 \tag{3.4}$$

while in the case $X_1 = 0$, W_0 is subject only to the condition

$$2\,|Y_1| = m(a_0)\,|W_0|^2 \tag{3.5}$$

and is otherwise arbitrary.

The length of the geodesic arc is

$$a_0\,|W_0|. \tag{3.6}$$

(Note that in the case $X_1 = 0$, $a_0 = 2k\pi$, this is $2(k\pi\,|Y_1|)^{1/2}$.)

To find the geodesic arcs with respect to (M_c) one can use Kaplan's computations for (M_1). The right-hand side of formula (10) in [12] giving the Riemannian connection has to be multiplied by $1/c^2$ and corresponding modifications have to be made in the subsequent computations. One finds that the (M_c)-geodesic arcs joining e to $\exp(X_1 + Y_1)$ are given as follows.

(i) If $Y_1 = 0$, by (3.1).

(ii) If $Y_1 \neq 0$ and $X_1 \neq 0$, by

$$\begin{aligned} X(s) &= (\exp a_c\, sj(Y_1') - I)\, W_c, \\ Y(s) &= (c^2 a_c s + \tfrac{1}{2}\, |W_c|^2 m(a_c s))\, Y_1' \end{aligned} \tag{3.7}$$

where a_c is a solution of

$$\mu(a_c) = \frac{|X_1|^2}{4(|Y_1| - c^2 a_c)} \tag{3.8}$$

satisfying $0 < a_c < |Y_1|/c^2$ (there always exists exactly one solution in $(0, 2\pi)$, and for c small enough there exist as many as for (3.3)); W_c is determined by

$$X_1 (\exp a_c j(Y_1') - I)\, W_c.$$

(iii) If $X_1 = 0$ and $Y_1 \neq 0$, there is a geodesic arc given by

$$X(s) = 0, \qquad Y(s) = sY_1, \tag{3.9}$$

and there may be others, given by (3.7) and the solutions $a_c = 2k\pi$ $(k = 1, 2, \ldots, m)$ of (3.8) (i.e., of $\mu(a_c) = 0$), such that $a_c < |Y_1|/c^2$; in this case W_c is subject only to the condition

$$|Y_1| - c^2 a_c = \tfrac{1}{2} m(a_c)\, |W_c|^2.$$

The length of the arc (3.7) is given by

$$a_c(|W_c|^2 + c^2)^{1/2}.$$

In the special case of $X_1 = 0$, denoting by γ_k the arc (3.7) corresponding to $a_c = 2k\pi$, this gives

$$l_c(\gamma_k) = 2[k\pi(|Y_1| - k\pi c^2)]^{1/2}$$

and one sees at once that

$$l_c(\gamma_1) < \cdots < l_c(\gamma_m) < l_c(\gamma_0)$$

exactly as in the case considered by Debiard [5].

The formulas of this section, together with translation invariance imply the following result.

PROPOSITION 3.1. *Let $n_1, n_2 \in N$, and let γ_0 be an (M_0)-geodesic joining them. Then for all $c > 0$ there exists an (M_c)-geodesic arc γ_c joining n_1 to n_2, such that*

$$\lim_{c \to 1} \gamma_c(s) = \gamma_0(s)$$

uniformly in s, and

$$\lim_{c \to 1} l_c(\gamma_c) = l_0(\gamma_0).$$

COROLLARY. *Given n_1, $n_2 \in N$, an (M_0)-geodesic arc of minimal l_0-length has minimal l_0-length even in the class of piecewise C^1 curves joining n_1 to n_2.*

Proof. Immediate from the well-known corresponding property of Riemannian metrics and from the remark after Proposition 2.2.

4. AN EXAMPLE IN RIEMANNIAN GEOMETRY

Let M be an n-dimensional compact connected C^∞-manifold. For any Riemannian metric on M let μ_1 denote the smallest non-zero eigenvalue of the Laplace–Beltrami operator and let vol M be the Riemannian volume. In [14] Urakawa showed that as the metric varies, the quantity $\psi = \mu_1(\text{vol}\, M)^{2/n}$ has no absolute upper bound, thereby answering a question of Berger. In [1] a general class of metrics furnishing similar examples was studied. Here we wish to point out that our family of metrics (M_c) projected to compact quotients of the group N also fit into this scheme. Furthermore, with the aid of some harmonic analysis on N one can make all the calculations directly and so obtain what is probably the simplest possible example of the phenomenon in question.

For this let N be a group as in Section 2, and let $\{X_i\}$, $\{Y_i\}$ be orthonormal bases of v, z, respectively. For a co-compact subgroup Γ of N we still denote by (M_c) the projection of (M_c) to $\Gamma \backslash N$. Writing $\Delta_1 = \sum X_i^2$, $\Delta_2 = \sum Y_i^2$, it is clear that the Laplace–Beltrami operator for the metric (M_c) is

$$\Delta_c = \Delta_1 + c^2 \Delta_2.$$

Since Δ_1 and Δ_2 commute, the eigenspaces of Δ_c are direct sums of the joint eigenspaces of Δ_1 and Δ_2.

For the case where N is the Iwasawa group of a semisimple Lie group of real rank one, the joint diagonalization of Δ_1 and Δ_2 can be found in [13]. Since we are only interested in describing the simplest example, we may restrict ourselves to the case where N is the three-dimensional Heisenberg group in its usual presentation as upper-triangular matrices, and we may take for Γ the subgroup with integral entries.

It is clear from [13] that the eigenvalues of Δ_2 on $\Gamma\backslash N$ are (up to an irrelevant factor coming from the normalization of the metric) the numbers λ^2 ($\lambda = 0, 1, 2,...$) with eigenfunctions of the form $\Phi(X)\exp(i\lambda\,|Y|)$. If λ is 0, Δ_1 acts on $\Phi(X)$ as the ordinary Laplacian of $\mathbf{Z}^2\backslash\mathbf{R}^2$, so its eigenvalues are $m^2 + n^2$ ($m, n \in \mathbf{Z}$). If $\lambda \neq 0$, the corresponding eigenvalues of Δ_1 which give bounded eigenfunctions on N are found in [13] to be $\lambda b(p+4n)$ ($n = 0, 1, 2,...$), where b and p are as in our Section 1. It follows that $\mu_1 = \min\{1, bp + c^2\}$.

Obviously, in the metric (M_c) we have $\mathrm{vol}(\Gamma\backslash N) = c^{-1}$, so

$$\psi = c^{-2/3} \min\{1, bp + c^2\},$$

which is unbounded as c varies, proving our statement.

We may note that the phenomenon of the crossing of eigenvalues observed in [14, 1] and leading to μ_1 having exceptionally high multiplicity for a certain value of c also occurs in our case.

Acknowledgments

I wish to express my gratitude to Professors P. Greiner and I. Kupka for useful conversations on the subject of this paper.

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