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Malmheden's theorem revisited

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ABSTRACT

In 1934 Malmheden [16] discovered an elegant geometric algorithm for solving the Dirichlet problem in a ball. Although his result was rediscovered independently by Duffin (1957) [8] 23 years later, it still does not seem to be widely known. In this paper we return to Malmheden's theorem, give an alternative proof of the result that allows generalization to polyharmonic functions and, also, discuss applications of his theorem to geometric properties of harmonic measures in balls in \mathbb{R}^n .

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1. Introduction

In 1934, Malmheden, a doctoral student of Riesz in Lund, proved that the value of a harmonic function u at any point P in a disk, or in a ball in \mathbb{R}^n , $n \geq 3$, can be computed from its values on the boundary according to the following algorithm. Take an arbitrary chord L through P ; calculate the value at P of the linear function ℓ on L that interpolates the values of u at the endpoints of chord L ; and, finally, calculate the average of the values $\ell(P)$ over all chords L through P (cf. Theorem 2.1 for the precise statement), [16], also see [3–7].

To fix the ideas, if we denote, say in \mathbb{R}^2 , by $A(\theta)$ the value at P obtained by linearly interpolating the boundary values of a harmonic function u in the disk at the endpoints of the chord through P making an angle θ with the positive direction of the x -axis, Malmheden's theorem yields that $u(P) = (1/2\pi) \int_0^{2\pi} A(\theta) d\theta$. It follows then from Malmheden's theorem that $u(P) \leq \max_{\theta} A(\theta)$. In [4,5] Barta attempted to extend the latter inequality to all convex, or even star-shaped regions. In [20], Weinberger showed that no such inequality, even allowing a multiplicative factor on the right, can

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hold at all points of a convex region unless the region is a disk. The analogous assertion is true in higher dimensions. Thus, the converse to Malmheden's theorem also holds; i.e., the assertion of his theorem is correct only if the domain is a ball. (Note, in passing, that the complex-analytic version of Malmheden's theorem is well-known under the name of Bochner–Martinelli formula. It provides holomorphic extensions of analytic functions from the boundaries of *arbitrary* smoothly bounded domains in \mathbb{C}^n and is obtained by averaging one-dimensional Cauchy integral representations—cf., e.g., [11].)

With this paper we hope to return the beautiful result of Malmheden to mathematical folklore since it appears that the result is not widely known. Moreover, we present an alternative approach to Malmheden's theorem that allows us to generalize the result to the polyharmonic functions, cf. [3,6,7]. We also present applications of Malmheden's theorem yielding nice geometric properties of the harmonic measure in balls in \mathbb{R}^n .

Regarding the history of the result, it must be mentioned that Duffin [8] has rediscovered Malmheden's theorem more than two decades later. His proof, although obtained independently, is essentially that of [16]. A slightly different approach to a two-dimensional version of Malmheden's theorem was rediscovered in [18,19], although the authors apparently did not know about the papers [16,8]. Their approach is based on an interpretation of the Poisson integral representation due to Schwarz [1, p. 170]. (Refs. [18,19] were kindly pointed out to us by the referee.) The referee also pointed out yet another approach to Malmheden's theorem via Schwarz' idea that is based on Kelvin's transform.

The paper is organized as follows. In Sections 2 and 3 we give two different proofs of the Malmheden theorem and its generalization to k -dimensional cross-sections of the ball passing through a given point P . The first proof in Section 2 is essentially Malmheden's original proof. The second proof seems to be new and is the crux for further generalizations of Malmheden's procedure to polyharmonic functions. In Section 4 we show the converse, i.e., that balls are the only domains in \mathbb{R}^n for which Malmheden's theorem holds. In Section 5 we discuss Malmheden's theorem in relation to some geometric properties of harmonic measure in the ball. In Section 6 we extend Malmheden's theorem to polyharmonic functions in \mathbb{R}^n . In 2 dimensions this was done by Barta, cf. [3,6,7]; his proof is quite different from ours. A different converse theorem, assuming that Malmheden's algorithm only reproduces harmonic functions at one point and, under additional condition of central symmetry, implying that the domain is a ball is given in Section 7. We finish with some additional remarks in Section 8.

2. A proof of Malmheden's theorem

Let us begin in a more general setting. Let Ω be a convex open set in \mathbb{R}^n with boundary Γ . Let f be a continuous function on Γ , $P \in \Omega$. Draw a chord L through P intersecting Γ in exactly two points Q_1, Q_2 . Denote by f_1, f_2 the values of f at Q_1, Q_2 , respectively. Interpolate the values f_1, f_2 by the unique linear function on L that we denote by ℓ . Clearly, ℓ is a continuous function of P and L . Holding P fixed let $u(P)$ denote the average of $\ell(P)$ over all lines L . $u(P)$ is a continuous function in Ω . Note that if f is a restriction of a linear function onto Γ , then $f=u$ in Ω .

Clearly, $u(P) \rightarrow f(Q_0)$ when $P \rightarrow Q_0, Q_0 \in \Gamma$.

Theorem 2.1 (Malmheden [16], Barta [8]). *If Ω is a ball in \mathbb{R}^n , then u is harmonic in Ω , and hence solves the Dirichlet problem (for the Laplacian) in Ω with data f .*

Proof. In view of the preceding remarks it remains to show that u is harmonic. Let L, P, Q_1, Q_2 be as above, $r_1 = |PQ_1|, r_2 = |PQ_2|$. Then

$$\ell(P) = \frac{r_1 f_2 + r_2 f_1}{r_1 + r_2}. \quad (2.1)$$

First, assume $n=2$. Let θ be the (polar) angle that the chord L makes with the positive direction of the x -axis, so $r_1(\theta + \pi) = r_2(\theta)$ and $f_1(\theta + \pi) = f_2(\theta)$. Then, we have

$$u(P) = \frac{1}{2\pi} \int_0^{2\pi} \ell(P, \theta) d\theta = \frac{1}{\pi} \int_0^{2\pi} \frac{r_2 f_1}{r_1 + r_2} d\theta. \tag{2.2}$$

Let φ be the angle between the (inner) normal \mathbf{n} to Γ at Q_1 and the chord L . Then, comparing infinitesimal arclengths on Γ and the circle of radius r_1 , centered at P , we observe that $r_1 d\theta = \cos\varphi ds$, where ds is the arclength on Γ . Thus, from (2.2) it follows that

$$u(P) = \frac{1}{\pi} \int_0^c \frac{\cos\varphi}{r_1} f_1 ds - \frac{1}{\pi} \int_0^c \frac{\cos\varphi}{r_1 + r_2} f_1 ds, \tag{2.3}$$

where c is the length of Γ .

The first term in (2.3) is harmonic since $\cos\varphi/r_1$ is nothing else but $(\partial/\partial n_{Q_1})\log|P-Q_1|$, so the first integral is the “double layer” potential of f_1 (cf. [13]). Finally, we use the hypothesis that Γ is a circle of radius R . Then $r_1 + r_2 = 2R\cos\varphi$, and the second integral is simply a constant, hence u is harmonic in Ω .

In higher dimensions, the argument is almost identical. For the sake of clarity of notation, we take $n=3$. Then, (2.2) and (2.3) become

$$u(P) = \frac{1}{2\pi} \int_{\Gamma} \frac{r_2 f_1}{r_1 + r_2} d\theta, \tag{2.2'}$$

and, hence,

$$u(P) = \frac{1}{2\pi} \int_{\Gamma} \frac{\cos\varphi}{r_1^2} f_1 dS - \frac{1}{2\pi} \int_{\Gamma} \frac{\cos\varphi}{r_1 + r_2} \frac{f_1}{r_1} dS. \tag{2.3'}$$

Here, dS is the surface area measure on Γ and $d\theta$ is the solid angle measure at P . Once again, the first integral is the double layer potential of f_1 , while when Ω is a ball of radius R , $r_1 + r_2 = 2R\cos\varphi$, so the second integral becomes a “single-layer” potential of f_1 , harmonic in Ω .

(Of course, (2.3') and (2.3) correspond to very well known representations of solutions of the Dirichlet problem in the ball as sums of potentials of double and single layers in dimensions 3 and higher, and as (up to an additive constant) a potential of a double layer in the disk (cf. [13,14]).) \square

Let us sketch here a different proof of Theorem 2.1 that will allow us in Section 6 to generalize Malmheden’s algorithm to biharmonic functions.

2nd Proof of Theorem 2.1. It is more convenient to translate our coordinate system so the origin is now at P while $\Omega = \{x \in \mathbb{R}^n : |x - \mathbf{c}| < 1\}$, where $\mathbf{c} : |\mathbf{c}| < 1$ is the center of the ball. We wish to show that for any homogeneous harmonic polynomial H in $\partial\Omega$, Malmheden’s procedure produces the number $H(0)$. Clearly, this will imply the result since the restrictions of harmonic polynomials to the sphere $\partial\Omega$ are dense in $C(\partial\Omega)$, while linearity of Malmheden’s algorithm yields the result for all harmonic polynomials as long as it holds for homogeneous polynomials. Finally, since Malmheden’s theorem obviously holds for linear polynomials we may assume that $m := \deg H > 1$, and need to verify that the algorithm produces the number $0 = H(P)$. Fix \mathbf{e} a unit vector in \mathbb{R}^n , and let $L = L_{\mathbf{e}}$ be the line through the origin parallel to \mathbf{e} . Let t be the running coordinate on $L_{\mathbf{e}}$. $L_{\mathbf{e}}$ meets $\partial\Omega$ at two points Q_1, Q_2 where t is equal to the roots of the equation

$$|t\mathbf{e} - \mathbf{c}|^2 = 1, \quad t^2 - 2\langle \mathbf{c}, \mathbf{e} \rangle t + (|\mathbf{c}|^2 - 1) = 0. \tag{2.4}$$

Along $L_{\mathbf{e}}$, $H(t\mathbf{e}) = t^m H(\mathbf{e})$. If we denote by $a, b \in \mathbb{R}$ the values of t corresponding to Q_1, Q_2 and determined by (2.4), then the linear interpolant of H along the segment of $L_{\mathbf{e}}$ between Q_1 and Q_2 equals

$$\left\{ \frac{t-b}{a-b} a^m + \frac{t-a}{b-a} b^m \right\} H(\mathbf{e}), \tag{2.5}$$

and is equal at $t=0$ to

$$(ab) \frac{[a^{m-1} - b^{m-1}]}{b-a} H(\mathbf{e}). \tag{2.6}$$

Now, we note from (2.4) that $ab = |\mathbf{c}|^2 - 1$, independent of the chosen vector \mathbf{e} . (Of course, this is a well-known theorem from the Euclidean geometry.) So the cofactor of $H(\mathbf{e})$ in (2.6) is a symmetric polynomial of degree $m-2$ of the roots a, b of (2.4). Hence, by a standard result on symmetric functions, it is a polynomial of degree at most $m-2$ of the coefficients of the quadratic (2.4). Now observe that the only coefficient dependent on \mathbf{e} in (2.4) is that of t and it is of degree 1 in the coordinates e_j of \mathbf{e} . Summarizing, the cofactor of $H(\mathbf{e})$ in (2.6) is a polynomial in e_j of degree at most $m-2$. Hence, the cofactor as a function of \mathbf{e} is orthogonal to the homogeneous harmonic polynomial $H(\mathbf{e})$ over the unit sphere, thus Malmheden’s procedure for H indeed produces $0=H(\mathbf{0})$. The last assertion follows from the fact that a polynomial of any degree k (in our case, $k=m-2$) matches on the sphere a harmonic polynomial of degree $\leq k$, which, of course, is the sum of homogeneous harmonic polynomials. Homogeneous harmonic polynomials of different degrees are orthogonal on the unit sphere, cf. e.g., [10,13,15]. The second proof is now complete. \square

3. An extension of Malmheden’s theorem to cross-sections of higher dimension

Theorem 2.1 has an almost immediate generalization if one replaces chords through a point P by k -dimensional cross-sections. (For the notational convenience we shall often represent P in the coordinate form as $x=(x_1, \dots, x_n)$.) More precisely, let Ω be a convex bounded domain in \mathbb{R}^n with smooth boundary Γ . Let $P(x,y)$, $x \in \Omega$, $y \in \Gamma$, denote the Poisson kernel for Ω , so any harmonic function in Ω , continuous in $\bar{\Omega}$, can be represented by its boundary values $f(y)$ on Γ as

$$u(x) = \int f(y)P(x,y) dS(y). \tag{3.1}$$

Let, as usual, the Green function in Ω be

$$g(x,y) = \begin{cases} -\frac{1}{2\pi} \log|x-y| + u_x(y), & n=2, \\ \frac{1}{\omega_n} |x-y|^{2-n} + u_x(y), & n \geq 3, \end{cases} \tag{3.2}$$

where $u_x(y)$ is harmonic in Ω , and $g(x,y)|_{y \in \Gamma} \equiv 0$. ω_n denotes the area of the unit sphere in \mathbb{R}^n . Then, as is well-known,

$$P(x,y) = \frac{\partial}{\partial n_y} g(x,y), \tag{3.3}$$

when n_y is the inner normal to Γ at $y \in \Gamma$. For $\Omega = \{|x| < 1\}$,

$$P(x,y) = \frac{1}{\omega_n} \frac{1-|x|^2}{|x-y|^n} \tag{3.4}$$

(cf. [13], e.g., for more details).

For $n=1$ there is a formula playing the role of (3.1) and (3.4), where the “domain” is the interval $[-1,1]$. Then

$$\frac{1}{2} \left[f(1) \frac{1-x^2}{1-x} + f(-1) \frac{1-x^2}{1+x} \right] = \frac{1}{2} [f(1)(1+x) + f(-1)(1-x)]$$

is the linear function matching given data f at ± 1 .

Fix $k, 1 \leq k < n$ and consider a k -dimensional plane α through P . $\Omega_\alpha := \Omega \cap \alpha$ is a convex domain in \mathbb{R}^k . Let P^α denote the Poisson kernel for Ω_α . Replacing in (3.1) $P(x,y)$ by P^α , f by $f_\alpha = f|_{\alpha \cap \Gamma}$, and $dS(y)$ by dS_α , the Lebesgue measure on $\alpha \cap \Gamma$, yields the solution of the Dirichlet problem $P^\alpha(f_\alpha)$ for Ω_α with the data f_α with respect to the k -dimensional Laplacian. Fix $x \in \Omega$ and consider the Grassmann manifold $G_x(k,n)$ of all k -dimensional planes α through x . The orthogonal group $SO(n)$ acts naturally on $G_x(k,n)$

and the invariant Haar measure on $SO(n)$ induces the unique normalized measure $dm_k(\alpha)$ on $G_x(k,n)$. $m_k(\alpha)$ is invariant with respect to all rotations of \mathbb{R}^n with the center x .

Now keeping the notations from Section 2, and in view of the note following (3.4) regarding the one-dimensional Poisson formula, we can restate Theorem 2.1 and rewrite (3.1). If $\Omega = \{|x| < 1\}$ is the unit ball, then any harmonic function u in Ω , continuous in $\bar{\Omega}$ with $u|_\Gamma = f$ can be represented via solutions of one-dimensional Dirichlet problems on one-dimensional lines L passing through $x \in \Omega$. Namely, denoting by $f_L := f|_{L \cap \Gamma}$, we have

$$u(x) = \int_{L \in G_x(1,n)} P^L(f_L) dm_1(L). \tag{3.5}$$

This yields an immediate corollary.

Theorem 3.1. *Let $\Omega = \{|x| < 1\}$ be the unit ball, $f \in C(\Gamma)$, $\Gamma = \partial\Omega$ is the unit sphere. The harmonic extension u of f to Ω can be represented in terms of harmonic extensions of f into k -dimensional sections of Ω :*

$$u(x) = \int_{\alpha \in G_x(k,n)} P^\alpha(f_\alpha|_{\Gamma \cap \alpha})(x) dm_k(\alpha). \tag{3.6}$$

Proof. From (3.5) and Fubini’s theorem we get

$$\begin{aligned} u(x) &= \int_{L \in G_x(1,n)} P^L(f_L|_{\Gamma \cap L}) dm_1(L) = \int_{\alpha \in G_x(k,n)} \int_{L \in \alpha, x \in L} P^L(f_L|_{\Gamma \cap L}) dm_1(L) dm_k(\alpha) \\ &= \int_{\alpha \in G_x(k,n)} P^\alpha(f_\alpha|_{\Gamma \cap \alpha}) dm_k(\alpha). \quad \square \end{aligned}$$

4. The converse

Theorem 4.1. *If Ω is a convex domain and the Malmheden procedure described in Section 2, produces for each continuous function f the solution of the Dirichlet problem with data f , then Ω must be a ball.*

For the sake of clarity, we restrict ourselves to the cases $n=2,3$. The extension to higher dimensions is straightforward.

We shall use the notation from Section 2. The hypothesis implies that the function $u(P)$ given by (2.3) and (2.3’) is a harmonic function of P . Note that the first term, as we have observed in Section 2, is ALWAYS harmonic in Ω since it is equal to the double layer potential. Thus the hypothesis yields that the second term is harmonic as a function of point $P \in \Omega$ as well. Therefore the functions

$$(i) \frac{\cos\varphi}{r_1+r_2}, \quad (ii) \frac{\cos\varphi}{r_1(r_1+r_2)} \tag{4.1}$$

are harmonic (as functions of P) in Ω .

Put the origin at the point Q_1 . Let $P=r_1t$, where t is a point on the unit sphere centered at Q_1 .

Let us represent $\Gamma := \partial\Omega$ near Q_2 by its “polar” equation: $\rho = \rho(t)$, $\rho = |Q_1Q_2|$. Then, (4.1) yields that functions

$$(i) \frac{\langle t, n_{Q_1} \rangle}{\rho(t)}, \quad (ii) \frac{1}{r_1} \frac{\langle t, n_{Q_1} \rangle}{\rho(t)}, \tag{4.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product and n_{Q_1} is the (inner) normal to Γ at Q_1 , are harmonic functions of r_1 and t .

In (i), the function depends on the polar angle t only, and being harmonic, forces it to be linear. Yet, it must be single-valued, thus is a constant. Denote it by $1/2R$. Thus, $\rho(t) = 2R \langle t, n_{Q_1} \rangle$ which is an equation of a circle.

For (ii), a simple calculation, or a quick check with [13, p. 141, Ex. 4] yields that const/r_1 (const/r_1^{n-2} , $n \geq 3$, in general) are the only homogeneous harmonic functions of degree -1 ($2-n$, respectively). Hence, again, from (4.2), (ii) we infer that $\langle t, n_{Q_1} \rangle / \rho(t) = \text{const}$, i.e., Γ is a sphere.

Remark 4.2. Weinberger [20] actually showed a stronger converse. To fix the ideas, we shall describe his result for $n=2$.

We keep the same notation as above. If Malmheden's procedure produces the solution of the Dirichlet problem, then putting the origin at point $P \in \Omega$, assuming Ω to be convex, and its boundary Γ to be given in polar coordinates by $\Gamma = \{(r(\theta), \theta) : 0 \leq \theta \leq 2\pi\}$, we can rewrite (2.2) from (2.1) as

$$u(P) = \frac{1}{2\pi} \int_0^{2\pi} \frac{r(\theta)u(r(\theta+\pi), \theta+\pi) + r(\theta+\pi)u(r(\theta), \theta)}{r(\theta) + r(\theta+\pi)} d\theta. \quad (4.3)$$

Eq. (4.3) implies

$$u(P) \leq \max_{\theta} \frac{r(\theta)u(r(\theta+\pi), \theta+\pi) + r(\theta+\pi)u(r(\theta), \theta)}{r(\theta) + r(\theta+\pi)}. \quad (4.4)$$

Weinberger showed that if a weaker version of (4.4) with a constant factor on the right hand side holds for any $P \in \Omega$, Ω being a convex domain, then Ω is a disk. This result refuted the conjecture of Barta [5,6] that (4.4) should hold in general convex domains.

Remark 4.3. Going over the proof of Theorem 4.1, one immediately notes that the hypothesis can be somewhat weakened:

- (i) Instead of convexity, it suffices to assume that the domain Ω is star-shaped with respect to all points Q in a neighborhood of the fixed boundary point $Q_1 \in \partial\Omega$. Equivalently, that placing the origin at any such Q , the (smooth, say C^2) boundary $\partial\Omega$ is given by the "polar" equation $\rho = \rho(t)$, where t runs over the upper hemisphere of the unit sphere.
- (ii) Instead of assuming that Malmheden's algorithm successfully solves the Dirichlet problem for any data f , it suffices to assume that merely for any continuous data supported in a neighborhood of Q_1 . We omit the details.

5. Discussing Malmheden's theorem and harmonic measure

We keep the same notation as in Section 2. If Ω is a convex domain in R^n and P is an interior point, the *metric ratio* R_P associated with the pair (Ω, P) is the function defined on $\partial\Omega$ by: $R_P(Q_1) = |PQ_2|/|Q_1Q_2|$, where Q_2 denotes the second point of intersection with $\partial\Omega$ of the line L joining P to Q_1 . As in Section 2, we note that R_P is a continuous function on $\partial\Omega$ with values strictly between 0 and 1. It is constant (i.e., identically equal to $\frac{1}{2}$) if, and only if, P is a center of symmetry of Ω (i.e., all chords through P are bisected there). Let w_P denote the harmonic measure on $\partial\Omega$ evaluated at P and A_P denotes the subtended angle measure on $\partial\Omega$ with respect to P . Malmheden's theorem then can be reformulated as follows.

Theorem 5.1. *If Ω is a ball, then for any $P \in \Omega$, R_P is the Radon–Nikodym derivative of the harmonic measure w_P with respect to A_P .*

For the proof one only needs to apply Theorem 2.1 to the characteristic functions of arbitrary (say, relatively open) subsets of $\partial\Omega$.

Thus, Malmheden's theorem can be seen as a procedure (when Ω is a ball) to compute harmonic measure from "purely geometric" quantities, lengths and (solid) angles, by integration. Note that the corresponding derivative when A_P is replaced by surface measure on $\partial\Omega$ is given by $C(P)/|PQ_1|^n$, the Poisson kernel (3.4) evaluated at P and Q_1 . For the unit ball centered at the origin O , $C(P) = (1/\omega_n)(1-|OP|^2)$, where ω_n is the surface area of the unit sphere—cf. [10,13]. Modulo the known relation between measures A_w and the surface area measure Malmheden's theorem is thus equivalent to Poisson's formula for the solution to Dirichlet's problem for the ball.

Against this background it is interesting to note the following further connections between harmonic and subtended angle measures in the ball, mediated by Malmheden's theorem. It implies other elegant properties of harmonic measure which, in dimensions > 2 , are rarely noted.

Corollary 5.1. *Let P be a point in the unit ball $\Omega := \{x : |x| < 1\}$, and let a double cone K_P with vertex at P cut out “spherical caps” U and V from the unit sphere S . Let $w_P(E)$ denote the harmonic measure of a set $E \subset S$ evaluated at P . Then, $w_P(U) + w_P(V)$ equals twice the (normalized) solid angle at the vertex P of K_P . (The normalizing factor equals $1/\omega_n$, where ω_n denotes the surface area of the unit sphere in \mathbb{R}^n .)*

Keeping the same notation as in Corollary 5.1, we also have the following.

Corollary 5.2. *If we consider a system of masses consisting of caps U, V , each endowed with the harmonic measure $w_P(U), w_P(V)$, respectively, then the center of mass is at P .*

Remark 5.2. Corollary 5.1 is well known in 2 dimensions—cf. [17, Chapter IV, Section 2]—and can be used to give an even shorter proof of Malmheden’s theorem. Corollary 5.2, even in the two dimensional case, seems not to have been noticed before.

Proof of Corollary 5.1. Fix $P \in \Omega$. Let $J = J_P$ be the involution of S that maps a point x of the unit sphere S onto the second point where the line Px meets S . Let $f \in C(S)$ be self-involutory with respect to J , i.e., $f(x) = f(Jx)$, let us denote by dA_P the normalized solid angle measure at the vertex P . Then, Theorem 2.1 implies that

$$\int_S f(x) dw_P(x) = \int_S f(x) dA_P(x). \tag{5.1}$$

Thus, (5.1) holds for all self-involutory $f \in C(S)$. But the characteristic function $\chi_{U \cup V}$ of the union $U \cup V$ is obviously a bounded pointwise limit of the self-involutory (w.r.t. J) functions in $C(S)$. Hence, by the bounded convergence theorem (5.1) also holds for $\chi_{U \cup V}$. This is the conclusion of Corollary 5.1. \square

Proof of Corollary 5.2. If $\ell(x)$ is a linear polynomial and $f \in C(S), f(x) = f(Jx)$, is self-involutory with respect to J_P , then the Malmheden algorithm applied to $(\ell f)(x)$ (of course, w.r.t. P) produces

$$\ell(P) \int_S f(x) dA_P(x). \tag{5.2}$$

(This is seen at once since Malmheden’s algorithm preserves linear functions while $f(x) = f(J_P x)$).

By Theorem 2.1 we have

$$\ell(P) \int_S f(x) dA_P(x) = \ell(P) \int_S f(x) dw_P(x). \tag{5.3}$$

So, from (5.2) and Theorem 2.1, we therefore infer that

$$\int_S \ell(x) \cdot f(x) dw_P(x) = \ell(P) \int_S f(x) dw_P(x).$$

Hence, for all linear functions ℓ , we have

$$\ell(P) = \frac{\int_S (\ell(x) \cdot f(x)) dw_P(x)}{\int_S f(x) dw_P(x)}. \tag{5.4}$$

Substituting for ℓ the coordinate functions $x_j, j = 1, \dots, n$, we obtain from (5.4) at once that the center of mass of the mass density $f(x) dw_P(x)$ is at P . Taking for f the characteristic function $\chi_{U \cup V}$ of the union $U \cup V$ proves the corollary. \square

Remark 5.3. (i) Corollary 5.1 has an independent proof in 2 dimensions. If P , with a slight abuse of notation, denotes a complex number in $\mathbb{D}, \mathbb{D} = \{|z| < 1\}$, it is straightforward to calculate the involution $J_P(z)$ to be $J_P(z) = (P - z)/(1 - \bar{P}z)$. Consider an arc U on the unit circle \mathbb{T} . Its harmonic measure at P equals the harmonic measure of $V = J_P(U)$ evaluated at 0 (by conformal invariance of the harmonic measure, since $J_P(P) = 0$). The harmonic measure of an arc V evaluated at the origin equals the central angle associated with V normalized by the factor $1/2\pi$. Thus, the harmonic measure $w_P(U)$ at P equals

$$w_P(U) = \frac{|V|}{2\pi}, \tag{5.5}$$

where $|V|$ denotes the length of the arc $V=J_P(U)$. Hence, recalling that $J_P(V)=U$, we conclude that

$$w_P(U)+w_P(V)=\frac{1}{2\pi}(|U|+|V|)=\frac{1}{\pi}(A_P), \tag{5.6}$$

where A_P stands for the angle subtended by either U or V w.r.t. P . (The last equality is, of course, a corollary of Euclid’s theorem about angles formed by the two chords through a point in the disk which states that their radian measure equals to the average of the radian measures of the subtending arcs of the circle.) This gives a different proof of Corollary 5.1 in 2 dimensions.

(ii) From (i), one can easily deduce Theorem 2.1 for $n=2$. Indeed, Corollary 5.1 yields that for every $w \in \mathbb{D}$ with, as before, A_w denoting the (normalized) subtended angle measure as seen from w , and for every analytic polynomial $P(z)$, we have

$$\int_{\mathbb{T}} P(\xi)dA_w(\xi)=\frac{1}{2}(P(0)+P(w)). \tag{5.7}$$

(Recall that $J_w(z)=(w-z)/(1-\bar{w}z)$, $J_w(0)=w$, $J_w(w)=0$.) Obviously, (5.7) then holds for all functions f analytic in \mathbb{D} and continuous in $\bar{\mathbb{D}}$. Let f be any such function. Parametrize the lines through w by their intersection points z with the unit circle \mathbb{T} . It is easy then by solving a system of linear equations to find for the line $\ell := \ell(z)$ the linear interpolant for the values $f(z), f(J_w(z))$. At w it equals

$$w\frac{f(z)-f(J_w(z))}{z-J_w(z)}+\frac{zf(J_w(z))-f(z)J_w(z)}{z-J_w(z)}. \tag{5.8}$$

Clearly for a fixed $w \in \mathbb{D}$ and f analytic, say, in a neighborhood of $\bar{\mathbb{D}}$ (5.8) produces an analytic function $F(z)$. Remembering that J_w maps 0 to w and w to 0 , one easily calculates that $F(0)=F(w)=f(w)$. Hence, applying Malmheden’s procedure to $f(z), z \in \mathbb{T}$, and then using $F(z)$ from (5.8) and (5.7) we obtain the value $f(w)$. Thus, Malmheden’s algorithm applied to, say, any polynomial in z , produces the “correct” value at w .

Separating real and imaginary part yields Malmheden’s theorem in the disk, i.e., for $n=2$.

The following proposition can be viewed as a converse of Corollaries 5.1 and 5.2 combined.

Proposition 5.1. *Let Ω be a bounded convex domain in \mathbb{R}^n and $P \in \Omega$. Assume that the harmonic measure $w_P = w(\cdot, \partial\Omega, P)$ on $\Gamma := \partial\Omega$ at P satisfies the conclusions of Corollaries 5.1 and 5.2. Then, P is a “Malmheden” point for Ω , i.e., Malmheden’s algorithm (cf. Sections 1 and 2) applied to any continuous function $f \in C(\Gamma)$ on Γ produces the value at P of the solution to the Dirichlet problem in Ω with data f on Γ .*

Proof. Let $I(\Gamma) \subset C(\Gamma)$ consist of all self-involutory functions, i.e., such that $f \in C(\Gamma)$ and $f(J(x))=f(x), x \in \Gamma$. (Here, $J=J_P$ is as above, the involution of Γ induced by P .) Denote by $R \subset C(\Gamma)$ the subset of $f \in C(\Gamma)$ for which the Malmheden algorithm produces the desired value at P of the solution of the Dirichlet problem with data f . Our hypothesis implies that $R \supset \{I(\Gamma)\} \cup \{\ell \cdot f, f \in I(\Gamma), \ell$ is a linear function of $x_1, \dots, x_n\}$. We need to show that R contains all polynomials, i.e., is dense in $C(\Gamma)$. Since R is obviously closed the proposition will then follow.

Fix a linear function $\ell(x)$. Denote $\ell^\# = \ell \circ J$. We have $\ell^2 = \ell \cdot (\ell + \ell^\#) - \ell \cdot \ell^\#$ and, since $\ell + \ell^\#, \ell \cdot \ell^\# \in I(\Gamma)$, our hypothesis implies that for any $g \in I(\Gamma), \ell^2 \cdot g = \ell \cdot (\ell + \ell^\#)g - (\ell \cdot \ell^\#)g \in \{I(\Gamma)\} \cup \{\ell \cdot f, f \in I(\Gamma)\} \subset R$. So, $\{\ell^2 \cdot I(\Gamma)\} \subset R$. An induction argument shows that $\{\ell^m \cdot I(\Gamma)\} \subset R$ for any integer m . Since the linear span of the set of powers of linear polynomials contains all polynomials the proposition follows. \square

Remark 5.4. It seems a worthy question whether the two geometric properties of harmonic measure enunciated in Corollaries 5.1 and 5.2 (i.e., the two hypotheses in Proposition 5.1) are actually independent of each other. We suspect they are but have not been able to prove it.

6. Malmheden’s theorem for polyharmonic functions

Let Ω be the unit ball in \mathbb{R}^n . $\Gamma := \partial\Omega$ is the unit sphere, and let $f := f(x)$ be a smooth function, say even real-analytic, in a neighborhood of Γ . Then as is well-known (cf. [10]) the Dirichlet problem for the bi-harmonic operator

$$\begin{cases} \Delta^2 u = 0, \\ u = f, \quad \nabla u = \nabla f \text{ on } \Gamma \end{cases} \tag{6.1}$$

has a unique solution in Ω . Not going into technicalities, the reader may argue as follows.

As is well-known—cf. [2]—any function $u : \Delta^2 u = 0$ in a domain Ω admits the so-called Almansi expansion:

$$u = h_1 + |x|^2 h_2, \tag{6.2}$$

with h_1 and h_2 harmonic functions in Ω , uniquely defined by u . One can trivially adjust (6.2) when $\Omega = \{|x| < 1\}$ is a ball and rewrite (6.2) as

$$u = H_1 + (|x|^2 - 1)H_2, \tag{6.3}$$

H_1, H_2 being harmonic in Ω . Then, to solve the BVP (6.1), we need to solve two consecutive Dirichlet problems: first, for $H_1 (=f \text{ on } \Gamma)$, and then for $H_2 : (\partial/\partial r)(f - H_1) = 2H_2$, where $\partial/\partial r$ stands for the radial derivative. Uniqueness follows by application of Green’s formula making use of the bi-harmonic Green’s function—cf., e.g., [12].

Now to formulate the analogue of Malmheden’s theorem for biharmonic functions (i.e., those for which $\Delta^2 = 0$) we proceed as follows. The one-dimensional analog of the bi-Laplacian is the operator $(d/dx)^4$, whose kernel consists of cubic polynomials. We can extend Malmheden’s procedure for a fixed point $P \in \Omega$ to solutions of (6.1). Again draw a chord L through P intersecting $\Gamma = \partial\Omega$ at points Q_1, Q_2 . Let t denote the real parameter along L , a and b being the values of t at Q_1, Q_2 . Let $C(t)$ denote the (unique) cubic polynomial such that the functionals $C(a), C'(a), C(b), C'(b)$ interpolate the corresponding values of the data $f_L(t) := f|_L$. (For derivatives of f_L at a and b we take the values of the directional derivatives of the data f along L at those points.) $C_L(P)$ is then the value of C at P . Holding $P \in \Omega$ fixed, let $u(P)$ denote the average of $C_L(P)$ over all lines L . $u(P)$ is, of course, a continuous function in B . Also, if $f(x)$ is a cubic polynomial, then $u = f$ in Ω . Clearly, $u(P) \rightarrow f(Q_0), \nabla u(P) \rightarrow \nabla f(Q_0)$ when $P \rightarrow Q_0, Q_0 \in \Gamma$.

Theorem 6.1 (Barta [3,7]). *If Ω is a ball in \mathbb{R}^n , u is biharmonic in Ω , and hence solves (6.1).*

Proof. It is convenient to translate the coordinate system so Ω is a unit ball centered at some point c , while P is now the origin. By a standard approximation argument we may assume that the data $f(x)$ is a polynomial since the latter are dense in the space $C^1(\Gamma)$ of continuously differentiable functions on Γ . Moreover, we may also assume that $f(x)$ is a biharmonic polynomial since the latter are also dense in the space of smooth biharmonic functions in the C^1 -metric. Lastly, by linearity of the Malmheden operator, we may assume that $f(x)$ is in fact a homogeneous biharmonic polynomial of degree $m > 3$.

Let us separate the following one-dimensional interpolation lemma.

Lemma 6.1. *Let $a < 0 < b, m > 3$ and $C(t) = C_m(t)$ be the (unique) cubic polynomial interpolating the values of t^m and its first derivative at a and b . Then, $C_m(0)$ is a homogeneous symmetric polynomial of degree m . Moreover,*

$$C_m(0) = (ab)^2 q_{m-4}(a,b), \tag{6.4}$$

where q_{m-4} is a homogeneous polynomial of a and b of degree $m - 4$.

Assuming the lemma and in view of our chain of reductions we need only show that for $f(x) := H(x)$, $H(x)$ being a homogeneous biharmonic polynomial of degree $m \geq 4$, the Malmheden algorithm produces the number $0 = H(P)$ (point P , recall, is at the origin). Let L, a, b be as described in the

paragraph preceding Theorem 6.1. Note that ab is a constant that depends on P only as already noted in Section 2 (cf. the argument below (2.6)). From this, (6.4) and since H is a homogeneous polynomial it follows, by repeating the argument following (2.6), that for each line L through $P = \mathbf{0}$ defined by the directional vector $\mathbf{e} \in \Gamma$, Malmheden’s procedure produces

$$\text{const}q_{m-4}(\mathbf{e})H(\mathbf{e}) \tag{6.5}$$

where the constant only depends on the point P .

Averaging (6.5) over all chords C , i.e., over all directional unit vectors $\mathbf{e} \in \Gamma$ produces 0 as required. To verify this last assertion we simply note, since H is homogeneous, (6.2) implies that

$$H(x) = h_1(x) + |x|^2 h_2(x), \tag{6.6}$$

where h_1, h_2 are homogeneous harmonic polynomials of degrees m and $m - 2$ respectively. q_{m-4} , as is true for any polynomial—cf. e.g., [15]—can be matched on the unit sphere Γ by a harmonic polynomial h_q , $\text{deg } h_q \leq m-4 < m-2$. From the well-known orthogonality of spherical harmonics (see, e.g., [15,2,10,13]), we obtain our last assertion and, hence, Theorem 6.1 follows modulo Lemma 6.1. \square

Proof of Lemma 6.1. Let $C_m(t) := C(t) = At^3 + Bt^2 + Ct + D$ match the function t^m together with its first derivatives at a and b , $a < 0 < b$.

Writing down four equations corresponding to these four interpolating conditions we see at once (by Cramer’s Rule) that all the coefficients are *rational* functions of a, b of degree at most $m+6$ symmetric with respect to a and b (interchanging a and b merely permutes equations in the system).

Furthermore, solving this linear system of equations via Cramer’s Rule we also observe that each coefficient A, \dots, D is a rational function of a and b for which the degree of the numerator is at most $m+6$, while the degree of the denominator which is the determinant of the system is precisely 6. Now these rational functions actually cannot have any finite poles for some complex values of a and b since the Hermite interpolation polynomial $C_m(t)$ exists and is unique for all complex values a, b . Thus, all the zeros of denominators in the rational expressions for the coefficients A, \dots, D of $C_m(t)$ must cancel out. Hence, all the coefficients A, \dots, D of $C_m(t)$ are actually polynomials of degree $(m+6) - 6 = m$. Finally, since when a or $b = 0$, $D = D(a, b)$ must have a double zero at the origin, it follows that $D(a, b) = (ab)^2 q_{m-4}(a, b)$, where q_{m-4} is a symmetric polynomial in a, b of degree $\leq m-4$. The lemma is proved and the proof of Theorem 6.1 is now complete. \square

Remark 6.2. (i) Obviously, the idea of the above argument originated from our second proof of Malmheden’s original result—Theorem 2.1—given in Section 2.

(ii) Lemma 6.1 has a natural extension to higher order differential operators $(d/dt)^{2k}$, $k \geq 3$. Accordingly, with rather obvious modifications Theorem 6.1 extends to polyharmonic operators Δ^k —cf. [3,7].

(iii) Theorem 3.1 also readily extends to polyharmonic operators.

(iv) We do not know whether the converse to Theorem 6.1, similar to Theorem 4.1, also holds for polyharmonic functions.

7. Another converse to Malmheden’s theorem

Let, as before, Ω be a convex, bounded domain in \mathbb{R}^n , $\Gamma = \partial\Omega$, $P \in \Omega$. If Malmheden’s procedure with respect to P as described in Section 2 applied to any, say, harmonic polynomial h yields the value $h(P)$, it does not seem to imply that Ω is a ball. The problem, of course, is that it does not allow us to locate the center of the ball. However if one assumes that not only Malmheden’s procedure applied to harmonic functions u in Ω yields the correct value $u(P)$, but also Ω is centrally symmetric with respect to P , one can conclude that Ω is a ball centered at P .

The driving force for this result is the following useful observation regarding harmonic measures.

Lemma 7.1. *Let $\Omega_1 \neq \Omega_2$ be two smoothly bounded domains of equal volume in \mathbb{R}^n , and assume also that $\Omega_1 \cap \Omega_2 \neq \emptyset$ and $\partial\Omega_1 \cap \partial\Omega_2 \neq \emptyset$. Suppose that Ω_1 and Ω_2 are both star-shaped with respect to the point $O \in \Omega_1 \cap \Omega_2$. Let w_1, w_2 be harmonic measures on $\partial\Omega_1, \partial\Omega_2$, respectively, evaluated at O . Let $E_1 = \partial\Omega_1 \setminus \Omega_2$ be the portion of $\partial\Omega_1$ that lies outside Ω_2 , while $F_2 = \partial\Omega_2 \cap \Omega_1$ is the portion of $\partial\Omega_2$ that lies inside Ω_1 . Similarly, define $E_2 = \partial\Omega_2 \setminus \Omega_1$, and $F_1 = \partial\Omega_1 \cap \Omega_2$.*

Then,

$$w_1(E_1) < w_2(F_2), \tag{7.1}$$

and, similarly,

$$w_2(E_2) < w_1(F_1). \tag{7.2}$$

(Both inequalities are strict.)

Proof. We just prove (7.1); the proof of (7.2) is identical.

Let $U = \Omega_1 \cap \Omega_2$, then $\partial U \supset F_1 \cup F_2$. Note that in view of the hypothesis U is also star-shaped w.r.t. O and so, in particular, is connected. On ∂U the (harmonic) functions $w_1(E_1; x), w_2(F_2; x)$ satisfy (by the maximum principle) the following:

$$\begin{aligned} w_1(E_1; x)|_{F_1} &= 0, & w_1(E_1; x)|_{F_2} &\leq 1, \\ w_2(F_2; x)|_{F_1} &\geq 0, & w_2(F_2; x)|_{F_2} &= 1. \end{aligned} \tag{7.3}$$

Moreover, each one of the two inequalities in (7.3) is *strict* at least on some set of positive harmonic measure. Thus,

$$w_2(F_2; x) > w_1(E_1; x) \tag{7.4}$$

on ∂U . The maximum principle—cf., e.g., [10,13,17]—for bounded harmonic functions yields that (7.4) holds everywhere in U . This proves (7.1). \square

Theorem 7.1. *Let Ω be star-shaped domain in \mathbb{R}^n and $O \in \Omega$ be a point in Ω . Assume also that Ω is centrally symmetric w.r.t. O . If the Malmheden algorithm applied to O reproduces the values at O of all functions harmonic in Ω and continuous in $\bar{\Omega}$, then Ω is a ball centered at O .*

Proof. First, observe that Lebesgue bounded convergence theorem immediately extends the hypothesis to all bounded harmonic functions whose boundary values are pointwise limits of a sequence of uniformly bounded continuous functions on $\partial\Omega$. Thus, in particular, the conclusion applies to harmonic measures of smoothly bounded open subsets of $\partial\Omega$ (or, to subarcs in 2 dimensions). Let B be the ball centered at O with same volume as Ω . If $B = \Omega$ there is nothing to prove. Then, $B \neq \Omega$ and since $\text{Vol}(B) = \text{Vol}(\Omega)$, $\partial B \cap \partial\Omega \neq \emptyset$. Since the Malmheden algorithm applies to O , and Ω is centrally symmetric with respect to O , the proof of Corollary 5.1 implies that the harmonic measure on $\partial\Omega$ evaluated at O is identical with the normalized solid angle measure subtended from O . Thus, applying Lemma 7.1 to the configuration $\Omega_1 = \Omega, \Omega_2 = B$ and $O \in \Omega \cap B$, we arrive at a contradiction, since both harmonic measures on $\partial\Omega$ and ∂B at O equal to the solid angle measure subtended from O and hence, must be the same for the sets $E_1 = \partial\Omega \setminus B$ and $F_2 = \partial B \cap \Omega$, respectively, for $E_2 = \partial B \setminus \Omega$ and $F_1 = \partial\Omega \cap B$. This contradicts (7.1)–(7.2).

Therefore, Ω must equal B and the theorem is proved. \square

8. Concluding remarks

- (i) Theorems 2.1 and 3.1 admit a nice probabilistic interpretation, e.g., in \mathbb{R}^3 . Informally, it reduces to the following. Consider three “Brownian travelers” departing from a point P in the unit ball B in \mathbb{R}^3 . The first moves according to the laws of standard Brownian motion, the second chooses at random a plane through P and follows the Brownian motion in that plane; the third chooses at random a line through P and follows the Brownian motion on that line. All of these travelers will reach the unit sphere $S := \partial B$ with probability 1. Malmheden’s theorem asserts that for any portion $E \subset S$, the probability that the first contact with S occurs in the set E is the same for all

three travelers, i.e., the observer registering the exiting travelers has no way of knowing how they arrived to the unit sphere from P .

- (ii) Theorem 2.1 and, a more general Theorem 3.1, certainly suggest connections to integral geometry. Indeed, the Malmheden algorithm reminds of the inversion formula for Radon transform which reconstructs functions from their integrals over hyperplanes, or more generally, k -planes. According to this formula, the value of a function at a point coincides, after applying a certain power (a half integer in even dimensions) of the Laplace operator to the average of the Radon data through the point.

The Radon inversion formula delivers representation of functions as continuous sums (i.e., integrals) of the so-called plane waves, i.e., of functions which are constant on families of parallel planes. Similarly, Theorem 3.1 can be interpreted in an analogous manner with the role of plane waves played by “harmonic k -waves”. Here, by harmonic k -waves we understand functions which are harmonic on parallel k -dimensional planes.

More precisely, fix the natural number k , $1 \leq k \leq n$. Denote by Δ_k the partial Laplace operator acting only on the first k variables:

$$\Delta_k = \sum_{j=1}^k \frac{\partial^2}{\partial x_j^2}.$$

For every rotation $\omega \in SO(n)$ we denote by Δ_k^ω the “rotated” partial Laplacian:

$$(\Delta_k^\omega g)(x) := \Delta_k(g \circ \omega^{-1})(\omega x).$$

It is not hard to show then that the Laplace operator coincides with the average of the rotated partial Laplacians, i.e.,

$$\Delta g(x) = \int_{SO(n)} \Delta_k^\omega g(x) d\omega,$$

where $d\omega$ denotes the normalized Haar measure on the orthogonal group. Malmheden theorem establishes a similar link between the solutions of the corresponding Laplace equations:

Theorem 8.1. *Let $\Omega := \{x : |x| < 1\}$ be the unit ball and $f \in C(\partial\Omega)$ and, for every $\omega \in SO(n)$, let u_ω denote the (unique) solution of the boundary value problem:*

$$\begin{cases} \Delta_k^\omega u_\omega(x) = 0, & x \in \Omega, \\ u_\omega(x) = f(x), & x \in \partial\Omega. \end{cases}$$

Then the function

$$u(x) = \int_{SO(n)} u_\omega(x) d\omega \tag{8.1}$$

solves the boundary value problem

$$\begin{cases} \Delta u(x) = 0, & x \in \Omega, \\ u(x) = f(x), & x \in \partial\Omega. \end{cases}$$

Proof. By construction, the function $u_\omega \circ \omega^{-1}, \omega \in SO(n)$ is harmonic with respect to the first variables x_1, \dots, x_k , i.e., it is harmonic on every cross-section of the ball by a k -dimensional plane parallel to the k -dimensional plane $\Pi = \{x_{k+1} = \dots = x_n = 0\}$. Then, u_ω is harmonic on each k -plane parallel to $\omega\Pi$. Let $x \in \Omega$ and let $\Pi_{\omega,x}$ be the k -dimensional plane parallel to $\omega\Pi$ and passing through x . Since $u_\omega = f$ on the unit sphere, u_ω is the harmonic extension of f into $\Pi_{\omega,x} \cap \Omega$. When ω runs over the whole orthogonal group $SO(n)$, $\Pi_{\omega,x}$ runs over all k -dimensional planes passing through x and Theorem 3.1 simply claims that the average, with respect to all rotations ω , of the values $u_\omega(x)$ equals to the value at x of the harmonic extension of the function f into the ball Ω , i.e., $u(x)$. This proves (8.1). \square

The extended Malmheden’s theorem for polyharmonic functions—cf., e.g., Theorem 6.1 for biharmonic functions with $k=1$ —allows a similar interpretation. In this case, the boundary

conditions will be corresponding complete sets of Cauchy data for the polyharmonic equation $\Delta^N u = 0$, i.e., the prescribed values of functions and their first $N - 1$ normal derivatives.

It would be interesting to investigate further whether a similar decomposition perhaps holds for other differential operators with constant coefficients and rotational symmetry, i.e., some operators of the form $P(\Delta)$ where P is a polynomial.

(iii) There are various levels at which the converse to the Malmheden theorem can be formulated. Theorem 4.1 (arguably, the most natural one) assumes that Malmheden’s procedure produces the solution to the Dirichlet problem with arbitrary data at all points of a convex domain Ω . The conclusion is then that Ω is a ball. Theorem 7.1 is an attempt to obtain the same conclusion under weaker assumptions: Ω is assumed to be star-shaped and Malmheden’s procedure is only assumed to produce the desired results at one point. However, it required an extra assumption of central symmetry. There are several other venues of interest one may pursue here. To fix the ideas, let Ω be a convex domain in \mathbb{R}^n , $P \in \Omega$, a fixed point. $J = J_P$, as before, is the involution of the boundary $\Gamma := \partial\Omega$ determined by chords through P . As in Section 5, $I = I_P(\Gamma)$ denotes the subspace of functions in $C(\Gamma)$ invariant under J . We have noted earlier in Section 5, that if P is a Malmheden point (an M-point for short) of Ω i.e., Malmheden’s algorithm applied to every $f \in C(\Gamma)$ produces the value $D_P f$ at P of the solution to the Dirichlet problem for the Laplacian with data f , then the following hold.

- (a) The two measures on Γ , dA_P is the normalized subtended angle from P and $w_P := w(\cdot, \Gamma, P) :=$ the harmonic measure on Γ at P produce the same results acting on all functions in $I_P(\Gamma)$. (Incidentally, this is equivalent to the relation between the solid angles of double cones with vertex at P and the harmonic measure of the two surface portions they cut out of Γ —cf. Section 5, Corollary 5.1).
- (b) For every function $f \in I_P(\Gamma)$ and every linear polynomial ℓ , it follows that $D_P(\ell \cdot f) = \ell(P) \cdot D_P(f)$ —cf. Corollary 5.2, Proposition 5.1.

Proposition 5.1 proves the converse, namely: if Ω, P are such that (a) and (b) hold, then P is an M-point of Ω . However, we do not know the exact relationship between (a) and (b).

(iv) Property (a) of a domain Ω is on its own somewhat a mystery and holds, perhaps, the key to a deeper understanding of Malmheden’s theorem. Theorem 7.1 yields that (a), together with the additional hypothesis that Ω is symmetric about point P , imply that Ω is a ball centered at P . Without this assumption the conclusion fails. A similar problem in two dimensions with the subtended angle measure replaced by a weighted arclength measure was settled in [9, Theorem 3.29(i)], where it was shown that for those problems there are other solutions besides circles. We must exercise similar caution for our problem as well, since there are star-shaped domains Ω (e.g., bicircular curves) in the plane for which the subtended angular measure from a point $A \in \Omega$ equals to the harmonic measure w_B evaluated at another point $B \in \Omega$.

The following example is due to the third author (H. S. Shapiro, 2006, unpublished).

Proposition 8.1. *There exists a planar domain Ω , star-shaped with respect to the origin, and a point $P \neq 0$, $P \in \Omega$, so that for any subarc $U \subset \Gamma$, $\Gamma := \partial\Omega$, the angle that U subtends at the origin is equal to $2\pi w_P(U, \Omega, P) = 2\pi w_P(U)$. (As in previous sections, $w_P(U)$ denotes the harmonic measure on Γ evaluated at P .)*

Proof. Consider the polynomial $q(z) := az^2 + z + a$, $0 < a < \frac{1}{2}$, q is univalent in the unit disk \mathbb{D} and maps it conformally onto a simply connected domain Ω . For $z \in \mathbb{T}$, $z = e^{i\theta}$, we have $q(z) = z[a(z + 1/z) + 1] = (1 + 2a\cos\theta)z$. Since $1 + 2a\cos\theta > 0$ for all θ , q preserves arguments of each $z \in \mathbb{T}$. Thus Ω is star-shaped and, moreover, for any subarc $U \subset \Gamma$, the angle subtended by U at the origin O is equal to the angle that its preimage $U' = q^{-1}(U)$, $U' \subset \mathbb{T}$ subtends at O . The latter, of course, equals to the length $(U') = 2\pi w(U', \mathbb{D}, O) = 2\pi w_O(U')$, the harmonic measure of U' in \mathbb{D} evaluated at the origin. By the conformal invariance of harmonic measure, $2\pi w(U', \mathbb{D}, O) = 2\pi w(U, \Omega, q(O))$ and setting $P = q(O) (= a)$ we are done. \square

B. Gustafsson (a personal communication) has recently characterized all simply connected domains $\Omega \subset \mathbb{C}$, where the subtended angle measure from an interior point $A \in \Omega$ equals a linear combination of the harmonic measures at finitely many other (fixed) interior points. But this is the beginning of another tale.

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