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Polynomial approximation of C^M functions by means of boundary values and applications: A survey

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Abstract

We collect classical and more recent results on polynomial approximation of sufficiently regular real functions defined in bounded closed intervals by means of boundary values only. The problem is considered from the point of view of the existence of explicit formulas, interpolation to boundary data, bounds for the remainder and convergence of the polynomial series. Applications to some problems of numerical analysis are pointed out, such as nonlinear equations, numerical differentiation and integration formulas, special associated differential boundary value problems. Some polynomial expansions for smooth enough functions defined in rectangles or in triangles of $\mathbb{R}^2$ as well as in cuboids or in tetrahedrons in $\mathbb{R}^3$ and their applications are also discussed.

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1. The problem

Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 1$, be a bounded closed polygonal domain. We denote by $C^M(\Omega)$ the class of continuous functions with continuous partial derivatives in Ω up to a fixed order M. We denote by $\mathscr{P}_{\boldsymbol{x}}^M$ the space of polynomials in $\boldsymbol{x} = (x_1, \ldots, x_N)$ of degree not greater than M. It will be clear from the context when M denotes a positive integer number or $M = (m_1, \ldots, m_N)$ with $m_i \in \mathbb{N}$, $i = 1, \ldots, N$; in the last case $p \in \mathscr{P}_{\boldsymbol{x}}^M$ means that the degree of p with respect each unknown x_i it is at most m_i, $i = 1, \ldots, N$.

We seek for explicit expansions

$$f(\boldsymbol{x}) = P_M[f](\boldsymbol{x}) + R_M[f](\boldsymbol{x}), \quad \boldsymbol{x} = (x_1, \ldots, x_N) \in \Omega, \tag{1}$$

where $f(\boldsymbol{x}) \in C^M(\Omega)$ and $P_M[f](\boldsymbol{x}) \in \mathscr{P}_{\boldsymbol{x}}^{K_M}$ is a polynomial which depends only on the values that $f(\boldsymbol{x})$ and some of its successive derivatives assume at the relevant boundary points of Ω (i.e. the vertices of the polygonal domain if $N > 1$); $R_M[f](\boldsymbol{x})$ is the exact remainder of the *boundary-type expansion* (1).

Polynomial expansions like (1) have found in the past decades several applications to the applied sciences (physical, engineering, etc.). For this reason we review classical and more recent univariate boundary-type expansions and

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investigate it in Section 2 in the following directions: interpolation to boundary data, bounds for the remainder and expansions in polynomial series, approximations by piecewise polynomials and splines. Moreover, in Section 3 we consider applications of the reviewed expansions to the numerical computation of the roots of nonlinear equations, to the construction of formulas for numerical differentiation and integration, to the approximation of solutions of special associated differential boundary value problems. Finally, in Section 4 we discuss briefly (merely because of reasons of limited space) some extensions of the expansions reviewed in Section 1 to rectangles or triangles in $\mathbb{R}^2$ as well as to cuboids or tetrahedrons in $\mathbb{R}^3$ and give some suggestions on its possible applications for future works: for example, to the production of new kernels of Sard's type or to the construction of *boundary-type cubature formulas*.

2. The univariate case

2.1. Existence of formulas

Since each interval $[a,b] \subset \mathbb{R}$ can be applied one to one onto the unit interval $[0,1]$ by means of the linear transformation of the variable $t \to x = t - a/(b-a)$, we can set $\Omega = [0,1]$ without loss of generality. In this section m always denotes a positive integer number. In the following we review all classical univariate boundary value formulas and some recent ones. All these formulas have the nice and important property of exactness on certain polynomial spaces; more precisely, from a simple analysis of the remainder terms it can be deduced that

$$P_m[f] \equiv f \quad \text{for each } f \in \mathscr{P}_x^{k_m} \tag{2}$$

for some $k_m \in \mathbb{N}$. If in addition there exists at least a $f \in \mathscr{P}_x^{k_m+1}$ for which

$$P_m[f] \neq f \tag{3}$$

then following a standard terminology we say that the algebraic degree of exactness of the operator $P_m[\cdot]$ is exactly k_m.

Two-point Hermite formula: As a particular case of the general Hermite interpolation formula [4,37,40,45,60], the two-point Hermite formula is the oldest univariate boundary-type formula (1878). Its formulation requires to fix two integer numbers $m, p, \; m \geqslant 2, \; 1 \leqslant p \leqslant m-1$; then

$$f(x) = P_m[f](x) + R_m[f](x), \quad x \in [0,1], \tag{4}$$

where $P_m[f](x)$ is the polynomial of degree not greater than $m-1$ defined by

$$P_m[f](x) = \sum_{j=0}^{p-1} S_{m,j}^p(x) f^{(j)}(0) + \sum_{j=0}^{m-p-1} (-1)^j S_{m,j}^{m-p}(1-x) f^{(j)}(1) \tag{5}$$

and

$$R_m[f](x) = \int_0^1 K_m(x,t) f^{(m)}(t)\,\mathrm{d}t. \tag{6}$$

The polynomials $S_{m,j}^p(x), \; 1 \leqslant p \leqslant m-1$, can be defined, for example, by (see [4, p. 66] for equivalent definitions)

$$S_{m,j}^p(x) = \frac{x^j}{j!}(1-x)^{m-p} \sum_{k=0}^{p-1-j} \binom{m-p+k-1}{k} x^k,$$

and the kernel function $K_m(x,t)$ is (see [4, p. 77] for equivalent definitions)

$$K_m(x,t) = \begin{cases} \displaystyle\sum_{j=0}^{p-1} \frac{(-t)^{m-j-1}}{(m-j-1)!} S_{m,j}^p(x), & t \leqslant x, \\ \displaystyle -\sum_{j=0}^{m-p-1} \frac{(1-t)^{m-j-1}}{(m-j-1)!} (-1)^j S_{m,j}^{m-p}(1-x), & x \leqslant t. \end{cases}$$

The algebraic degree of exactness of the operator $P_m[\cdot]$ is $m-1$.

Two-point Taylor formula: With this name it is usually indicated the particular case of the two-point Hermite formula which requires to use, at the endpoints of [0,1], the same number of derivatives; we consider it separately, since in many classical textbooks (see, for example [37, p. 37]) this expansion is presented without mentioning formula (4). In this case we have

$$f(x) = P_{2m}[f](x) + R_{2m}[f](x), \quad x \in [0, 1], \tag{7}$$

where $P_{2m}[f](x)$ is the polynomial of degree not greater than $2m-1$ defined by

$$P_{2m}[f](x) = \sum_{j=0}^{m-1} [Q_{m,j}(x) f^{(j)}(0) + (-1)^j Q_{m,j}(1-x) f^{(j)}(1)] \tag{8}$$

and

$$R_{2m}[f](x) = \int_0^1 K_{2m}(x,t) f^{(2m)}(t)\,\mathrm{d}t. \tag{9}$$

The polynomials $Q_{m,j}(x)$ can be defined, for example, by [55]

$$Q_{m,j}(x) = \frac{x^j}{j!}(1-x)^m \sum_{k=0}^{m-j-1} \binom{m+k-1}{k} x^k$$

and the kernel function $K_{2m}(x,t)$ is (see [4, p. 77] for equivalent definitions)

$$K_{2m}(x,t) = \begin{cases} \displaystyle\sum_{j=0}^{m-1} \frac{(-t)^{2m-j-1}}{(2m-j-1)!} Q_{m,j}(x), & t \leqslant x, \\ \displaystyle -\sum_{j=0}^{m-1} \frac{(1-t)^{2m-j-1}}{(2m-j-1)!} (-1)^j Q_{m,j}(1-x), & x \leqslant t. \end{cases}$$

The algebraic degree of exactness of the operator $P_{2m}[\cdot]$ is $2m-1$.

Lidstone expansion of first type: Lidstone [50] in 1929 and independently Poritsky [56] in 1932 introduced a generalization of the Taylor series that approximates a given function in the neighborhood of two points instead of one:

$$f(x) = P_{2m}[f](x) + R_{2m}[f](x), \quad x \in [0, 1], \tag{10}$$

where $P_{2m}[f](x)$ is the polynomial of degree not greater than $2m-1$ defined by

$$P_{2m}[f](x) = \sum_{k=0}^{m-1} [\Lambda_k(1-x) f^{(2k)}(0) + \Lambda_k(x) f^{(2k)}(1)] \tag{11}$$

and

$$R_{2m}[f](x) = \int_0^1 G_m(x,t) f^{(2m)}(t)\,\mathrm{d}t. \tag{12}$$

The polynomials $\Lambda_k(x)$, $k = 0, 1, \ldots,$ are called *Lidstone polynomials* and are defined, for example [4], by means of the recursive relations

$$\begin{cases} \Lambda_0(x) = x, \\ \Lambda_k''(x) = \Lambda_{k-1}(x), & k \geqslant 1, \\ \Lambda_k(0) = \Lambda_k(1) = 0, & k \geqslant 1, \end{cases}$$

and the kernel function is [7]

$$G_m(x,t) = -\begin{cases} \sum_{k=0}^{m-1} \dfrac{(1-t)^{2m-2k-1}}{(2m-2k-1)!}\Lambda_k(x), & t<x, \\ \sum_{k=0}^{m-1} \dfrac{t^{2m-2k-1}}{(2m-2k-1)!}\Lambda_k(1-x), & x \leqslant t. \end{cases} \tag{13}$$

The algebraic degree of exactness of the operator $P_{2m}[\cdot]$ is $2m-1$.

Bernoulli formula of first type: With this name we indicate a two-point formula that appears in classical textbooks [48, pp. 15–17; 46, p. 253] due to its several applications (Krylov [48] used it in studying the remainder term of some quadrature formulas of Newton–Cotes):

$$f(x) = P_m[f](x) + R_m[f](x), \quad x \in [0,1], \tag{14}$$

where $P_m[f](x)$ is the polynomial of degree not greater than m defined by

$$P_m[f](x) = \int_0^1 f(t)\,\mathrm{d}t + \sum_{j=1}^{m} \frac{B_j(x)}{j!}(f^{(j-1)}(1) - f^{(j-1)}(0)) \tag{15}$$

and

$$R_m[f](x) = -\frac{1}{m!}\int_0^1 B_m^*(x-t) f^{(m)}(t)\,\mathrm{d}t. \tag{16}$$

The *Bernoulli polynomials* $B_k(x),\ k=0,1,\ldots,$ are defined, for example [1, p. 804], [46, p. 231], by means of the recursive relations

$$\begin{cases} B_0(x) = 1, & \\ B_k'(x) = kB_{k-1}(x), & k \geqslant 1, \\ \int_0^1 B_k(x) = 0, & k \geqslant 1, \end{cases}$$

and $B_m^*(x)$ is the Bernoulli periodic function of order m, defined by

$$B_m^*(x) = B_m(x - [x]), \tag{17}$$

where $[x]$ denotes the largest integer not greater than x. The algebraic degree of exactness of the operator $P_m[\cdot]$ is m. So far as we know, formula (14) has not been yet studied from the point of view stated in Section 1.

Euler formula: With this name we indicate a two-point formula that appears in classical textbooks [46, p. 307]:

$$f(x) = P_{m+1}[f](x) + R_{m+1}[f](x), \quad x \in [0,1], \tag{18}$$

where $P_{m+1}[f](x)$ is the polynomial of degree not greater than m defined by

$$P_{m+1}[f](x) = \sum_{j=0}^{m} E_j(x)\frac{f^{(j)}(0) + f^{(j)}(1)}{2} \tag{19}$$

and

$$R_{m+1}[f](x) = \int_0^1 K_{m+1}(x,t) f^{(m+1)}(t)\,\mathrm{d}t. \tag{20}$$

The *Euler polynomials* $E_k(x),\ k=0,1,\ldots,$ are defined, for example [1, p. 804], [46, p. 309], by means of the generating function

$$\frac{2\mathrm{e}^{xt}}{\mathrm{e}^t + 1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \quad |t| < \pi,$$

and the kernel function $K_{m+1}(x,t)$ is

$$K_{m+1}(x,t)=\begin{cases}\dfrac{1}{2}\displaystyle\sum_{j=0}^{m}\frac{E_j(x)}{(m-j)!}(-t)^{(m-j)}, & t<x,\\[2ex] -\dfrac{1}{2}\displaystyle\sum_{j=0}^{m}\frac{E_j(x)}{(m-j)!}(1-t)^{(m-j)}, & x\leqslant t.\end{cases}$$

The algebraic degree of exactness of the operator $P_{m+1}[\cdot]$ is m. As in the case of formula (14) it is not known to be a study of formula (18) from the point of view stated in Section 1.

Two-point Abel–Gontscharoff formula: This formula is a particular case of the Abel–Gontscharoff formula, introduced in 1935 by Whittaker [65, p. 38] and subsequently by Gontscharoff [40, pp. 84–86]. In [4, p. 172] this formula is also referred as *two-point right focal interpolation formula*. Let $m,\alpha\in\mathbb{N}$, $m\geqslant 2$, $0\leqslant\alpha\leqslant m-1$; then

$$f(x)=P_m[f](x)+R_m[f](x),\quad x\in[0,1], \tag{21}$$

where $P_m[f](x)$ is the polynomial of degree $m-1$ defined by

$$P_m[f](x)=\sum_{i=0}^{\alpha}\frac{x^i}{i!}f^{(i)}(0)+\sum_{j=0}^{m-\alpha-2}\left(\sum_{i=0}^{j}\frac{(-1)^{j-i}x^{\alpha+1+i}}{(j-i)!(\alpha+1+i)!}\right)f^{(\alpha+1+j)}(1), \tag{22}$$

and

$$R_m[f](x)=\int_0^1 K_m(x,t)f^{(m)}(t)\,\mathrm{d}t. \tag{23}$$

The kernel function is [4, p. 177]

$$K_m(x,t)=\begin{cases}\displaystyle\sum_{i=0}^{\alpha}\frac{x^i(-t)^{m-i-1}}{i!(m-i-1)!}, & t\leqslant x,\\[2ex] -\displaystyle\sum_{i=\alpha+1}^{m}\frac{x^i(-t)^{m-i}}{i!(m-i-1)!}, & x\leqslant t.\end{cases} \tag{24}$$

The algebraic degree of exactness of the operator $P_m[\cdot]$ is $m-1$.

Modified Abel expansion: This formula has been introduced in [54] and corresponds to an interpolation problem considered in [59]:

$$f(x)=P_{2m}[f](x)+R_{2m}[f](x),\quad x\in[0,1], \tag{25}$$

where $P_{2m}[f](x)$ is the polynomial of degree not greater than $2m-1$ defined by

$$P_{2m}[f](x)=\sum_{i=0}^{m-1}\left[\mu_i(x)f^{(2i)}(1)+\mu'_{i+1}(x-1)f^{(2i+1)}(0)\right] \tag{26}$$

and

$$R_{2m}[f](x)=\int_0^1 K_m(x,t)f^{(2m)}(t)\,\mathrm{d}t. \tag{27}$$

The fundamental polynomials $\mu_k(x)$, $k=0,1,\ldots$, that are related to Bernoulli and Euler polynomials, can be defined, for example [54], by means of the recursive relations

$$\begin{cases}\mu_0(x)=1,\\ \mu''_k(x)=\mu_{k-1}(x), & k=1,2,\ldots,\\ \mu_k(1)=0,\ \mu'_k(0)=0, & k=1,2,\ldots\end{cases}$$

and the kernel functions can be defined by

$$K_m(x,t) = \begin{cases} \displaystyle\sum_{k=0}^{m-1} \frac{(-t)^{2m-2k-1}}{(2m-2k-1)!}\mu_k'(1-x), & t < x, \\ \displaystyle -\sum_{k=0}^{m-1} \frac{(1-t)^{2m-2k-1}}{(2m-2k-1)!}\mu_k(x), & x \leqslant t. \end{cases} \tag{28}$$

The algebraic degree of exactness of the operator $P_{2m}[\cdot]$ is $2m-1$.

(m, p) interpolation formula: This formula was introduced by Agarwal and Wong [4, p. 192]. Let $m, p \in \mathbb{N},\ m \geqslant 1,\ 0 \leqslant p \leqslant m-1,\ p$ fixed; then

$$f(x) = P_m[f](x) + R_m[f](x), \quad x \in [0,1], \tag{29}$$

where $P_m[f](x)$ is the polynomial of degree not greater than $m-1$ defined by

$$P_m[f](x) = \sum_{i=0}^{m-2} \frac{x^i}{i!} f^{(i)}(0) + \left(f^{(p)}(1) - \sum_{i=0}^{m-p-2} \frac{f^{(p+i)}(0)}{i!} \right) \frac{(m-p-1)!}{(m-1)!} x^{m-1} \tag{30}$$

and

$$R_m[f](x) = \int_0^1 K_m(x,t) f^{(m)}(t)\,dt. \tag{31}$$

The kernel function is defined by

$$K_m(x,t) = -\frac{1}{(m-1)!} \begin{cases} x^{m-1}(1-t)^{m-p-1} - (x-t)^{(m-1)}, & t \leqslant x, \\ x^{m-1}(1-t)^{m-p-1}, & x \leqslant t. \end{cases}$$

The algebraic degree of exactness of the operator $P_m[\cdot]$ is $m-1$. A formula symmetric to (29) with respect the axis $x = \frac{1}{2}$ is also possible [4, p. 193].

Bernoulli formula of second type: In [27] Costabile proposed the following boundary value formula:

$$f(x) = P_m[f](x) + R_m[f](x), \quad x \in [0,1], \tag{32}$$

where $P_m[f](x)$ is a polynomial of degree not greater than m defined by

$$P_m[f](x) = f(0) + \sum_{j=1}^{m-1} \frac{S_j(x)}{j!} \left(f^{(j)}(1) - f^{(j)}(0) \right), \tag{33}$$

where $S_j(x) = B_j(x) - B_j(0),\ \ j = 1, \ldots, m$ and

$$R_m[f](x) = \frac{1}{m!} \int_0^1 \left(B_m^*(x-t) + (-1)^{m+1} B_m(t) \right) f^{(m)}(t)\,dt. \tag{34}$$

The function $B_m^*(x)$ is the *periodic Bernoulli function* defined in (17). Formula (32) appears as a modification of previously established formula (14); for this reason in this context it is called *Bernoulli formula of second type*. The algebraic degree of exactness of the operator $P_m[\cdot]$ is m. A remarkable property of the expansion (32) is that when the boundary points coalesce, the expansion (32) converges towards the Taylor expansion of f; more precisely, if we denote by $P_m[f; a, h](x)$ the polynomial (33) for a generic interval $[a, a+h],\ \ a \in \mathbb{R},\ \ h > 0$, then

$$\lim_{h \to 0} P_m[f; a, h](x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(m)}(a)}{m!}(x-a)^m.$$

Note that this property is also satisfied by classical polynomial expansions like (4), (7) and (10) [51].

Lidstone formula of second type: In [33] it has been proposed the following expansion by means of boundary values:

$$f(x) = P_{2m}[f](x) + R_{2m}[f](x), \quad x \in [0, 1], \tag{35}$$

where $P_{2m}[f](x)$ is a polynomial of degree not greater than $2m-1$ defined by

$$P_{2m}[f](x) = f(0) + \sum_{k=1}^{m} (v_k(x) - v_k(0)) f^{(2k-1)}(1) - (v_k(1-x) - v_k(1)) f^{(2k-1)}(0) \tag{36}$$

and

$$R_{2m}[f](x) = \int_0^1 K_m(x,t) f^{(2m+1)}(t)\,\mathrm{d}t. \tag{37}$$

The polynomials $v_k(x)$, $k = 0, 1, \ldots,$ are defined by

$$\begin{cases} v_0(x) = 1, \\ v_k'(x) = \int_0^x v_{k-1}(t)\,\mathrm{d}t, & k \geqslant 1, \\ \int_0^1 v_{k-1}(x)\,\mathrm{d}x = 0, & k \geqslant 1, \end{cases}$$

and since [33]

$$v_k'(x) = \Lambda_{k-1}(x), \quad k \geqslant 1,$$

they are called *Lidstone polynomials of second type*. The kernel function can be defined by

$$K_m(x,t) = \begin{cases} -\sum\limits_{k=1}^{m} \dfrac{(1-t)^{2(m-k)+1}}{(2(m-k)+1)!}(v_k(x) - v_k(0)), & x \leqslant t, \\ \dfrac{t^{2m}}{(2m)!} + \sum\limits_{k=1}^{m} \dfrac{t^{2(m-k)+1}}{(2(m-k)+1)!}(v_k(1-x) - v_k(1)), & x \geqslant t. \end{cases} \tag{38}$$

The algebraic degree of exactness of the operator $P_{2m}[\cdot]$ is $2m-1$. A formula symmetric with respect the axis $x = \frac{1}{2}$ to (35) is also possible [33].

Other approximation formulas: Other approximation formulas exist, as particular case of *two-points Birkhoff conditions, two-points Abel–Gontscharoff–Hermite conditions* and *Abel–Gontscharoff–Lidstone conditions* [4]. For these formulas, however, we are not able to give explicit polynomial representations and therefore they are not considered in this work.

2.2. Interpolation to boundary data

We can view each of previously given polynomials as the unique solution of a particular interpolation problem. These problems are listed below.

Two point Hermite formula: The polynomial (5) satisfies the following interpolation condition:

$$\begin{cases} P_m^{(i)}[f](0) = f^{(i)}(0), & i = 0, \ldots, p-1, \\ P_m^{(i)}[f](1) = f^{(i)}(1), & i = 0, \ldots, m-p-1. \end{cases}$$

These conditions are a particular case of the general *Hermite interpolation conditions* [45].

Two point Taylor formula: The polynomial (8) satisfies the following interpolation condition:

$$\begin{cases} P_{2m}^{(i)}[f](0) = f^{(i)}(0), & i = 0, \ldots, m-1, \\ P_{2m}^{(i)}[f](1) = f^{(i)}(1), & i = 0, \ldots, m-1. \end{cases}$$

These conditions are called *two point Taylor interpolation conditions* [37, p. 37; 4, p. 63].

Lidstone formula of first type: The polynomial (11) satisfies the following interpolation condition:

$$\begin{cases} P_{2m}^{(2i)}[f](0) = f^{(2i)}(0), & i = 0, \ldots, m-1, \\ P_{2m}^{(2i)}[f](1) = f^{(2i)}(1), & i = 0, \ldots, m-1. \end{cases}$$

These conditions are called *Lidstone interpolation conditions*:

Bernoulli formula of first type: The polynomial (15) satisfies the following interpolation condition:

$$\begin{cases} \int_0^1 P_m[f](x)\,\mathrm{d}x = \int_0^1 f(x)\,\mathrm{d}x, \\ \Delta P_m^{(k-1)}[f](0) = \Delta f^{(k-1)}(0), & k = 1, \ldots, m-1, \end{cases}$$

where Δ is the *difference operator* [46], which acts on a generic function $\varphi(x)$ as follows:

$$\Delta\varphi(x) = \varphi(x+1) - \varphi(x). \tag{39}$$

We refer to these conditions as *Bernoulli interpolation conditions of first type*.

Euler formula: The polynomial (19) satisfies the following interpolation condition:

$$\mathscr{M} P_{m+1}^{(k)}[f](0) = \mathscr{M} f^{(k)}(0), \quad k = 0, \ldots, m,$$

where $\mathscr{M}$ is the *mean operator* [46], that acts on a generic function $\varphi(x)$ as follows:

$$\mathscr{M}\varphi(x) = \frac{\varphi(x) + \varphi(x+1)}{2}.$$

We refer to these conditions as *Euler interpolation conditions*.

Abel–Gontscharoff formula: The polynomial (22) satisfies the following interpolation condition:

$$\begin{cases} P_m^{(i)}[f](0) = f^{(i)}(0), & 0 \leqslant i \leqslant \alpha, \\ P_m^{(i)}[f](1) = f^{(i)}(1), & \alpha + 1 \leqslant i \leqslant n-1. \end{cases}$$

These conditions are called *two-point right focal conditions* [4, p. 172] and are a particular case of the general *Abel–Gontscharoff interpolation conditions* [4, p. 172; 37, p. 28].

Modified Abel expansion: The polynomial (26) satisfies the following interpolation conditions:

$$\begin{cases} P_{2m}^{(2i)}[f](0) = f^{(2i)}(0), \\ P_{2m}^{(2i+1)}[f](1) = f^{(2i+1)}(1), \end{cases} \quad i = 0, \ldots, m-1.$$

These conditions are called *modified Abel conditions* and are a particular case of the general *Abel–Gontscharoff interpolation conditions* [4, p. 172; 37, p. 28].

(m, p) *formula*: The polynomial (30) satisfies the following interpolation condition

$$\begin{cases} P_{m-1}^{(k)}[f](0) = f^{(k)}(0), & 0 \leqslant k \leqslant m-2, \\ P_{m-1}^{(p)}[f](1) = f^{(p)}(1), & 0 \leqslant p \leqslant m-1, \quad p \text{ fixed}. \end{cases}$$

These conditions are called (m, p) *interpolation conditions* [4, p. 192]. Symmetric (p, m) *interpolation conditions* can be also considered [4, p. 193].

Bernoulli formula of second type: The polynomial (33) satisfies the following interpolation condition:

$$\begin{cases} P_m[f](0) = f(0), \\ \Delta P_m^{(k-1)}[f](0) = \Delta f^{(k-1)}(0), & k = 1, \ldots, m-1, \end{cases}$$

where Δ is the *difference operator* defined in (39). Note that previous conditions imply that $P_m[f](x)$ interpolates the function f at $x = 1$ as well. We refer to these conditions as *Bernoulli interpolation conditions of second type*.

Lidstone formula of second type: The polynomial (36) satisfies the following interpolation condition:

$$\begin{cases} P_{2m}[f](0)=f(0), \\ P_{2m}^{(2k-1)}[f](0)=f^{(2k-1)}(0), \quad k=1,\ldots,m, \\ P_{2m}^{(2k-1)}[f](1)=f^{(2k-1)}(1), \quad k=1,\ldots,m. \end{cases} \tag{40}$$

We refer to these conditions as *Lidstone interpolation conditions of second type*. Symmetric interpolation conditions can be also considered [33].

2.3. Bounds for the remainder

Hermite formula: For the remainder $R_m[f](x)$ in the formula (4) there exists the following Cauchy representation:

$$R_m[f](x)=\frac{1}{m!}\omega_{m,p}(x)f^{(m)}(\xi), \quad \xi\in(0,1), \tag{41}$$

where

$$\omega_{m,p}(x)=x^p(x-1)^{m-p}. \tag{42}$$

Pointwise and uniform bounds of the error and its derivatives in terms of $M_m=\max_{0\leqslant x\leqslant 1}|f^{(m)}(x)|$ have been established in [6,69] and can be found in [4, Chapter 2.4]; in particular, the pointwise bounds are the following:

$$|R_m^{(k)}[f](x)|\leqslant \begin{cases} g_k(1-x)\dfrac{k!M_m}{m!}, & 0\leqslant x\leqslant \frac{1}{2}, \\ g_k(x)\dfrac{k!M_m}{m!}, & \frac{1}{2}\leqslant x\leqslant 1, \end{cases} \quad k=0,1,\ldots,m, \tag{43}$$

where

$$g_k(x)=\begin{cases} x^{m-\alpha-2}(1-x)^{\alpha+1-k}\left\{\binom{m-\alpha-1}{k}x+\left[\binom{m}{k}-\binom{m-\alpha-1}{k}\right](1-x)\right\}, & 0\leqslant k\leqslant\alpha, \\ x^{m-k-1}\left\{\binom{m-\alpha-1}{k-\alpha-1}x+\left[\binom{m}{k}-\binom{m-\alpha-1}{k-\alpha-1}\right](1-x)\right\}, & \alpha+1\leqslant k\leqslant m-1, \end{cases} \tag{44}$$

and $\alpha=\min\{p-1,m-p-1\}$.

Two point Taylor formula: The Cauchy representation of the remainder in (7) can be recovered from (41); after (43) the pointwise bounds of the error in term of $M_{2m}=\max_{0\leqslant x\leqslant 1}\left|f^{(2m)}(x)\right|$ determined by Ciarlet et al. [23]

$$|R_{2m}^{(k)}[f](x)|\leqslant\frac{x^{m-k}(1-x)^{m-k}M_{2m}}{k!(2m-2k)!}, \quad k=0,1,\ldots,m, \quad 0\leqslant x\leqslant 1, \tag{45}$$

are best possible for $k=0$ only; but for the cases $m=2$ and 3 the error on the derivatives $R_{2m}^{(k)}[f](x)$ later found by Birkhoff and Priver [11] are optimal.

Lidstone formula of first type: The error $R_{2m}[f](x)$ in the formula (10) has the following Cauchy's representation [4, p.18]:

$$R_{2m}[f](x)=E_{2m}(x)f^{(2m)}(\xi), \quad \xi\in(0,1). \tag{46}$$

Since $(-1)^mE_{2m}(x)>0$, the sign of $R_{2m}[f](x)$ depends on $f^{(2m)}(\xi)$ only. Agarwal and Wong [4,5] have obtained best possible pointwise as well as uniform bounds for the error (46) and it derivatives. For each $k=0,1,\ldots,m-1$ these bounds are the following [4, p. 19]:

$$\begin{aligned} |R_{2m}^{(2k)}[f](x)| &\leqslant(-1)^{m-k}E_{2m-2k}(x)M_{2m} \\ &\leqslant\frac{(-1)^{m-k}E_{2m-2k}}{2^{2m-2k}(2m-2k)!}M_{2m}, \quad 0\leqslant x\leqslant 1, \end{aligned} \tag{47}$$

for the even derivatives and

$$|R_{2m}^{(2k+1)}[f](x)| \leqslant (-1)^{m-k}[E_{2m-2k}(x) + (1-2x)E_{2m-2k-1}(x)]M_{2m}$$
$$\leqslant \frac{(-1)^{m-k+1}2(2^{2m-2k}-1)}{(2m-2k)!} B_{2m-2i} M_{2m}, \quad 0 \leqslant x \leqslant 1, \tag{48}$$

for the odd ones, where B_n and E_n, $n \geqslant 0$, denote, respectively, the Bernoulli and Euler numbers [1,46].

Bernoulli formula of first type: Since the kernel function in (16) change sign in [0, 1], the first mean value theorem for integrals [37, p. 7] cannot be applied in order to get a Cauchy representation of the error. From the analysis of (16) can be obtained the following uniform bound:

$$|R_m[f](x)| \leqslant \frac{1}{12(2\pi)^{m-2}} M_m, \quad 0 \leqslant x \leqslant 1, \quad m > 1.$$

Euler formula: A Cauchy representation of the error cannot be obtained directly from (20); the analysis of the error allows us to find the following pointwise bounds:

$$|R_{m+1}[f](x)| \leqslant \frac{1}{2} \sum_{j=0}^{m} \frac{|E_j(x)|}{(m-j+1)!} \left(x^{m-j+1} + (1-x)^{m-j+1}\right) M_{m+1}, \quad 0 \leqslant x \leqslant 1.$$

Two-point Abel–Gontscharoff formula: The Cauchy representation of the error in (21) it is known [4, p. 177]

$$R_{m-1}[f](x) = T_m(x) f^{(m)}(\xi), \quad \xi \in (0, 1), \tag{49}$$

where

$$T_m(x) = \frac{1}{m!} \sum_{i=\alpha+1}^{m} (-1)^{m-i} \binom{m}{i} x^i. \tag{50}$$

The bounds for the error and its derivatives have been established in [68] and can be found in [4, p. 178]:

$$|R_{m-1}^{(k)}[f](x)| \leqslant M_m \frac{(1-x)^{(m-k)}}{(m-k)!}, \quad 0 \leqslant x \leqslant 1, \quad 0 \leqslant k \leqslant m-1. \tag{51}$$

(m, p) *interpolation formula*: The Cauchy representation of the error in (29) is known [4, p. 195]:

$$R_{m-1}[f](x) = \frac{x^{m-1}}{(m-1)!} \left(\frac{x}{m} - \frac{1}{m-p}\right) f^{(m)}(\xi), \quad \xi \in (0, 1), \tag{52}$$

and best uniform bounds for the error and its derivatives have been also determined [4, p. 196]:

$$|R_{m-1}^{(k)}[f](x)| \leqslant \alpha_{m,k} M_m, \quad 0 \leqslant x \leqslant 1, \quad 0 \leqslant k \leqslant m-1, \tag{53}$$

where $M_m = \max_{0 \leqslant x \leqslant 1} |f^{(m)}(x)|$, and

$$\alpha_{m,k} = \frac{1}{(m-k)!} \times \begin{cases} \dfrac{p-k}{m-p}, & 0 \leqslant k \leqslant p-1, \\ \dfrac{(m-k-1)^{m-k-1}}{(m-p)^{m-k}}, & k = p, \\ \dfrac{k-p}{m-p}, & p+1 \leqslant k \leqslant m-1. \end{cases} \tag{54}$$

Modified Abel expansion: Since the kernel functions (28) fulfill the inequalities [54]

$$(-1)^n K_n(x, t) \geqslant 0, \quad 0 \leqslant x, t \leqslant 1, \quad n = 1, 2, \ldots$$

the Cauchy representation of the error (27) can be easily obtained by applying the first mean value theorem for integrals [37, p. 7]

$$R_{2m}[f](x) = \mu_m(x) f^{(2m)}(\xi), \quad \xi \in (0, 1), \tag{55}$$

where we used the established in [54] equality

$$\mu_n(x) = \int_0^1 K_n(x, t)\,\mathrm{d}t, \quad n = 1, 2, \ldots .$$

Then the pointwise bound for the remainder is the following:

$$|R_{2m}[f](x)| \leqslant |\mu_m(x)| M_{2m}, \quad 0 \leqslant x \leqslant 1 \tag{56}$$

and taking into account that [54]

$$0 \leqslant (-1)^n \mu_n(x) \leqslant C(\pi/2)^{-2n}, \quad n = 0, 1, \ldots,$$

where $C > 0$ is a constant independent of n, an uniform bound can be easily recovered from (56).

Bernoulli formula of second type: The Cauchy representation for the error in (32) cannot be obtained directly from (34); the uniform bound for the remainder is

$$|R_m[f](x)| \leqslant \frac{1}{6(2\pi)^{m-2}} M_m, \quad 0 \leqslant x \leqslant 1. \tag{57}$$

Lidstone formula of second type: A Cauchy representation of the remainder in (35) exists [33]:

$$R_{2m}[f](x) = \frac{Q_{2m+1}(x)}{(2m)!} f^{(2m+1)}(\xi), \quad \xi \in (0, 1), \tag{58}$$

where

$$Q_{2m+1}(x) = \frac{1}{(2m+1)(m+1)} \left(B_{2m+2}(x) - B_{2m+2} - 2^{2m+2} \left(B_{2m+2}\left(\frac{x}{2}\right) - B_{2m+2} \right) \right). \tag{59}$$

The uniform bound is

$$|R_{2m}[f](x)| \leqslant 4 \frac{(2^{m+2} - 1)}{(2m+2)!} \, |B_{2m+2}| \, M_{2m+1}, \quad 0 \leqslant x \leqslant 1. \tag{60}$$

Other two-point mixed interpolation: Optimal error bounds for the derivatives of two-point mixed interpolation have been obtained in [70] for several different interpolating polynomials.

2.4. A method for explicit construction of interpolating polynomial $P_M[f](x)$

A way to recover the expansions in Section 2.1, that is different from the well-known method of successive integrations by parts of the remainder terms of the formulas, it is provided by the general theory of finite interpolation [37, Chapter 2.2]. This theory allows us not only to prove the existence and uniqueness of the solution of each of interpolation problems listed in Section 2.1, but also to give for each solution a determinantal representation in terms of basis functions and linear functionals, since each of polynomials listed in Section 2.1 is the unique solution of the corresponding interpolation problem of Section 2.2. The uniqueness of the solution implies also the exactness of the representation in the polynomial space spanned by the basis functions, therefore, we can get an integral representation of the remainder by applying the Peano's kernel theorem [37, p. 70], [38, p. 94]. In addition, this way of doing allows us, after some calculation, to derive determinantal representations of the basis functions. For example, in [27] it is derived the following representation for the Bernoulli polynomials:

$$B_n(x) = \begin{cases} 1, & n = 0, \\ \dfrac{(-1)^n}{(n-1)!} \begin{vmatrix} 1 & x & x^2 & x^3 & \dots & x^{n-1} & x^n \\ 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \dots & \frac{1}{n} & \frac{1}{n+1} \\ 0 & 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & 0 & 2 & 3 & \dots & n-1 & n \\ 0 & 0 & 0 & \binom{3}{2} & \dots & \binom{n-1}{2} & \binom{n}{2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \binom{n-1}{n-2} & \binom{n}{n-2} \end{vmatrix}, & n \geqslant 1, \end{cases} \tag{61}$$

and from (61) all classical properties of Bernoulli polynomials can be recovered by means of elementary linear algebra tools [32]. Determinantal representations for the Lidstone polynomials are discussed in [36]. Similar representations for the Euler polynomials are objects of current investigation by the authors.

2.5. *Convergence of series*

The formula (1) suggests to consider the formal series

$$\sum_{M}^{\infty} P_M[f](x)$$

in order to characterize its convergence, i.e. with the aim of finding a class of functions f for which

$$f(x) = \sum_{M}^{\infty} P_M[f](x), \quad x \in [0, 1].$$

In the recent paper [52] details are given on the two-point Taylor expansion (7), in particular on the region of convergence and on representations in terms of Cauchy-type integrals of coefficients and remainders of this expansion. Some information on this type of expansion is also given in [66]. The case of Lidstone formula of first type (10) has been studied by several authors [13,14,17,56,58,59,64,67]. In particular Widder [67] proved that a necessary and sufficient condition for $f(x)$ can be represented by an absolutely convergent Lidstone series is that it should be the difference of two minimal completely convex functions on [0, 1]. In particular, this implies that f is the restriction of an entire function and satisfies

$$f^{(m)}(0) = \mathrm{o}(\pi^m), \quad m \to \infty. \tag{62}$$

Later Boas [14] gave necessary conditions and sufficient conditions, in terms of the growth of the function in the complex plane, for representation by absolutely or conditionally convergent Lidstone series. More recently Buckholtz and Shaw [17] proved that the condition (62) is also sufficient for the convergence of the Lidstone series. In [25] Buckholtz and Shaw's result [17] is generalized to other two-point expansions. A sufficient condition for $f(x)$ can be represented by an absolutely convergent Bernoulli series of first type is that f is the restriction of an entire function and satisfies [15]

$$f^{(m)}(0) = \mathrm{o}((2\pi)^m), \quad m \to \infty.$$

For the Bernoulli second type expansion similar condition hold [28]. Pethe and Sharma [54] proved that the modified Abel expansion (25) converges for functions in a subclass of completely convex functions. So far as we know, in other cases listed in Section 2.1 the problem of convergence of the formal series has not been globally studied.

2.6. Piecewise polynomial and spline interpolation

Although polynomials have attractive features, polynomial interpolation of a given function often has the drawback of producing approximations that may be wildly oscillatory. To overcome this difficulty, one can divide the interval of interest into small subintervals and in each subinterval consider a polynomial of relatively low degree and finally "piece together" these polynomials. This subject has been developed for a lot of previous cases. In particular Birkhoff, Ciarlet et al. [12,23,24] have provided an explicit representation of piecewise Hermite interpolants; Agarwal and Wong have considered piecewise Hermite [8] and Lidstone [7] interpolants; these representations are then used to obtain error bounds for the derivatives of piecewise interpolants in L_∞ and L_2 norm. So far as we know, for all other cases listed in Section 2.1 the piecewise polynomial interpolations have not been studied. Spline interpolation is a improvement over piecewise polynomial interpolation. It uses less information of the given function, yet furnishes smoother interpolates. In addition to the piecewise cubic interpolation, i.e. splines obtained gluing together two-point Taylor polynomials (8) of degree 3, this subject has been adapted by Agarwall and Wong [4] to the Lidstone conditions. So as far we know, nothing is known about the representation of the other piecewise interpolants.

3. Applications

3.1. Numerical solution of nonlinear equations

Under assumption of regularity on the function f in the equation

$$f(x) = 0, \quad x \in [a, b], \tag{63}$$

some iterative methods for the computation of its zero α in an interval (a, b), $a < b$, can be obtained by neglecting in (1) the remainder term and then by substituting (63) with the equation

$$P_M[f](x) = 0.$$

In particular, if $m = 2$ and $P_2[f](x)$ satisfies the interpolation conditions

$$P_2[f](0) = f(0), \quad P_2[f](1) = f(1),$$

algorithms that join the sure convergence of the bisection-like algorithms with superlinear convergence speed can be obtained by an enclosing interval procedure [10]. These procedures are been obtained in the cases of Bernoulli formula of second type (32) and Abel–Gontscharoff formula (21) (this formula coincides, for $m=2$, with the (2, 1) interpolation formula (29)). More precisely, for the Bernoulli formula we have the iterative formula [35]

$$x_{i+2} = x_{i+1} + \frac{2 f_{i+1}(x_{i+1} - x_i)}{-\sigma_{i+1} \pm \sqrt{\sigma_{i+1}^2 - 2\gamma_{i+1}(x_{i+1} - x_i) f_{i+1}}}, \tag{64}$$

where

$$f_k = f(x_k), \quad f'_k = f'(x_k), \quad \gamma_k = f'_k - f'_{k-1}$$

and

$$\sigma_k = f_k - f_{k-1} + \gamma_k/2 = x_{k+1} - x_k.$$

This method converges to the zero of f with order 2 for all $f \in C^3$ as proved in [35]. Furthermore, we should stress that the cost of (64) is equal to the cost of well-known Newton method for each iteration. For the Abel–Gontscharoff formula we have the iteration [34]

$$x_{i+1} = x_i - \frac{2 f_i}{f'_i \pm \sqrt{f_i'^2 - 4\sigma_i f_i}}, \tag{65}$$

where

$$f_i = f(x_i), \quad f_i' = f'(x_i), \quad \sigma_i = f_{i-1} - f_i - f_i'(x_{i-1} - x_i)/(x_{i-1} - x_i)^2.$$

The order of convergence of sequence (65) is equal to $1+\sqrt{2}$ as stated in [34]. This new approach to numerical solution of nonlinear equation suggests the following open problem:

Problem. *Let us consider the one parameter family $F[f,a,b,\lambda](x)$ of quadratic polynomials such that*

$$F[f,a,b,\lambda](a) = f(a), \quad F[f,a,b,\lambda](b) = f(b);$$

find the $\overline{\lambda}$ such that the order of the iterative method generated from the quadratic polynomial $F[f,a,b,\overline{\lambda}](x)$ by dichotomic procedure is the greatest possible.

Previous methods are recently drawn on by other authors [41,39].

3.2. Numerical differentiation

Differentiations of the formula (1) in some special cases provide useful formulas for numerical differentiation. In particular, since

$$\Lambda_k''(x) = \Lambda_{k-1}(x), \quad k \geqslant 1,$$

$2k$ differentiations of the Lidstone formula of first type (10) give

$$f^{(2k)}(x) = P_{2m-1}^{(2k)}[f](x) + R_{2m-1}^{(2k)}[f](x),$$

where

$$P_{2m-1}^{(2k)}[f](x) = \sum_{i=k}^{m-1}[\Lambda_{i-k}(1-x)f^{(2i)}(0) + \Lambda_{i-k}(x)f^{(2i)}(1)],$$

for $k=1$ we get

$$P_{2m-1}''[f](x) = \sum_{i=1}^{m-1}[\Lambda_{i-1}(1-x)f^{(2i)}(0) + \Lambda_{i-1}(x)f^{(2i)}(1)].$$

For the odd order derivatives it is useful the Lidstone formula of second type (35); in this case, in fact, by the property

$$v_k'(x) = \Lambda_{k-1}(x), \quad k \geqslant 1,$$

$2k+1$ differentiations of formula (35) give

$$f^{(2k+1)}(x) = P_{2m}^{(2k+1)}[f](x) + R_{2m}^{(2k+1)}[f](x), \quad x \in [0,1],$$

where

$$P_{2m}^{(2k+1)}[f](x) = \sum_{j=k+1}^{m}[\Lambda_{j-k-1}(x)f^{(2j-1)}(1) - \Lambda_{j-k-1}(1-x)f^{(2j-1)}(0)], \tag{66}$$

for $k=0$ we get

$$P_{2m}'[f](x) = \sum_{j=1}^{m}[\Lambda_{j-1}(x)f^{(2j-1)}(1) - \Lambda_{j-1}(1-x)f^{(2j-1)}(0)]. \tag{67}$$

Bernoulli formulas (14), (32) give similar formulas for numerical differentiation both for even and odd order; in fact, by differentiating k-times formula (14) we get

$$f^{(k)}(x) = P_n^{(k)}[f](x) + R_n^{(k)}[f](x),$$

where

$$P_n^{(k)}(x) = \sum_{i=k}^{n} \frac{B_{i-k}}{(2i-k)!}(x)(f^{(i-1)}(1) - f^{(i-1)}(0)),$$

for $k = 1$ we get

$$f'(x) = \sum_{k=1}^{n} \frac{B_{k-1}(x)}{(k-1)!}(f^{(k-1)}(1) - f^{(k-1)}(0)) - n\int_0^1 B_{n-1}^{*}(x-t) f^{(n)}(t)\,\mathrm{d}t.$$

3.3. Numerical integration

Boundary-type quadrature formulae, i.e. formulae for numerical integration with all their evaluation points lying on the boundary of the integration domains, are useful in several context [43,49,61]. A series of such special kind of formulae can be obtained by integrating each of formulas listed in Section 2.1. It must be pointed out that the integration of the expansions of Section 2.1 yield quadrature formulas that are exact for functions f belonging to the polynomial space generated by the corresponding basis functions. In the following we report some of these formulas.

Two point Taylor formulae: In [49] Lanczos has given a general discussion of a class of quadrature formulas in which the value of the integral over a finite range is expressed in terms of the values of the integrand and its derivatives at the endpoints. Because of the use of derivatives of the integrand he called them *quadrature by differentiation* formulas. Lanczos determined the weights A_k^m of the two point Taylor quadrature formula

$$\int_0^1 f(x)\,\mathrm{d}x \approx \frac{f(0)+f(1)}{2} + \sum_{k=1}^{m} A_k^m \{f^{(k)}(0) + (-1)^k f^{(k)}(1)\}$$

by requiring the mth order formula to give the exact answer for polynomials of degree $2m-1$ and obtained

$$A_k^m = \frac{(1+2m-k)!(1+m)!}{(1+k)!(m-k)!(2+2m)!}, \quad k = 0, 1, \ldots, m.$$

The Taylor two-point quadrature can be obtained by integrating formula (7).

Lidstone type formulae: By integrating formula (10) we get

$$\int_0^1 f(x)\,\mathrm{d}x = \frac{f(0)+f(1)}{2} + \sum_{k=1}^{m-1} \frac{2^{2k+2}-1}{(2k+1)!} B_{2k+2}(f^{(2k)}(1) + f^{(2k)}(0)) + f^{(2m)}(\xi)\int_0^1 E_{2m}(t)\,\mathrm{d}t, \quad \xi \in (0,1), \tag{68}$$

where we used the following equations:

$$\int_0^1 \Lambda_k(x)\,\mathrm{d}x = \int_0^1 \Lambda_k(1-x)\,\mathrm{d}x = \begin{cases} \dfrac{1}{2}, & k = 0, \\ \dfrac{2^{2k+2}-1}{(2k+1)!} B_{2k+2}, & k \geqslant 1. \end{cases}$$

Similarly, by integrating (35) we get

$$\int_0^1 f(x)\,\mathrm{d}x = f(0) + \sum_{j=1}^{n} \frac{2^{2j}}{(2j)!} B_{2j}\ (f^{(2j-1)}(0) + (1-2^{1-2j}) f^{(2j-1)}(1)) + \int_0^1 \int_0^1 f^{(2m+1)}(t) K_m(x,t)\,\mathrm{d}t\,\mathrm{d}x,$$

where $K_m(x,t)$ is defined in (38).

Bernoulli type formulae: By integrating the *Bernoulli formulas of second type* (32) we get

$$\int_0^1 f(x)\,\mathrm{d}x = \frac{f(0)+f(1)}{2} - \sum_{k=1}^{m} \frac{B_k}{k!} (f^{(k-1)}(1) - f^{(k-1)}(0)) + \int_0^1 R_m[f](x)\,\mathrm{d}x,$$

where $R_m[f](x)$ is defined in (34). This quadrature formula is classic [44] and is the basis for the well-known *Euler-summation formula*.

Euler type formulae: Finally, by integrating the Euler formulas (18) we have

$$\int_0^1 f(x)\,\mathrm{d}x = \frac{f(0)+f(1)}{2} + \sum_{i=1}^{m-1} \frac{E_{2i+1}}{(2i-1)!} (f^{(2i)}(0) + f^{(2i)}(1)) + \int_0^1 R_{m+1}[f](x)\,\mathrm{d}x, \tag{69}$$

where $R_{m+1}[f](x)$ is defined in (20). Since [4]

$$E_{2k}(x) = \Lambda_k(x) + \Lambda_k(1-x), \quad k = 0, 1, \ldots,$$

the quadrature formulae (69) coincide with the formula (68); the formulae (69) have been obtained, by means of techniques that differ from previous one, by Henrici [44, p. 459] and Bretti and Ricci [16].

Other formulae can be obtained by similar techniques from remaining expansions of Section 2.1. In particular, for the case of the *modified Abel expansion*, the quadrature weights in terms of Bernoulli and Euler numbers can be obtained by integrating the fundamental polynomials $\mu_k(x)$, $\mu'_{k+1}(x-1)$, $k=0,1,\ldots$, respectively making use of the relations with the Bernoulli and Euler polynomials stated in [54].

3.4. Boundary value problem

Many problems in applied mathematics require solutions of differential equations specified by conditions at the two endpoints of an interval; these problems, called *two-point boundary value problems*, are considerably more difficult to deal with than initial value problems, largely because of their global nature. To each interpolation problem listed in Section 2.2 there can be associated a corresponding special boundary value problem, in which the differential equation

$$x^{(N)}(t) = f(t, \mathbf{x}(t)), \tag{70}$$

where $x(t)$ is supposed to be a continuous, N times differentiable function in [0, 1], f continuous at least in the interior of the domain of interest and $\mathbf{x}(t) = (x(t), x'(t), \ldots, x^{(q)}(t))$, $0 \leqslant q \leqslant N-1$ but fixed, it is coupled with the boundary condition related to the interpolation problem. More precisely, the following two point boundary value problems can be formulated.

Hermite boundary value problem: The differential equation is

$$x^{(m)}(t) = f(t, \mathbf{x}(t)), \tag{71}$$

coupled with the boundary conditions

$$\begin{cases} x^{(i)}(0) = A_i, & 0 \leqslant i \leqslant p-1, \\ x^{(i)}(1) = B_i, & 0 \leqslant i \leqslant m-p-1. \end{cases} \tag{72}$$

Theorems of existence and uniqueness of the solution of previous problem are stated in [4], where also an iterative procedure can be founded for the computation of the approximated solution and several its applications.

Taylor boundary value problem: The differential equation is

$$x^{(2m)}(t) = f(t, \mathbf{x}(t)), \tag{73}$$

coupled with the boundary conditions

$$\begin{cases} x^{(i)}(0) = A_i, \\ x^{(i)}(1) = B_i. \end{cases} \quad 0 \leqslant i \leqslant m-1. \tag{74}$$

As a special case of the Hermite boundary value problem, we refer the reader to [4].

Lidstone boundary value problem: This problem consists of the general $2m$th order nonlinear differential equation

$$(-1)^m x^{(2m)}(t) = f(t, \mathbf{x}(t)), \tag{75}$$

coupled with the boundary conditions

$$\begin{cases} x^{(2i)}(0) = A_i, \\ x^{(2i)}(1) = B_i, \end{cases} \quad 0 \leqslant i \leqslant m-1. \tag{76}$$

In recent years this problem has attracted considerable attention (see [4] and several reference therein) particularly because of its special cases frequently occur in engineering and other branches of physical sciences. For instance, the deflection of a uniformly loaded rectangular plate supported over the entire surface by an elastic foundation and rigidly supported along the edges leads to this type of problem with $m = 2$ and $f = a_0(t)x + b(t)$.

Bernoulli boundary value problem I: The problem consists in coupling the mth order nonlinear differential equation

$$x^{(m)}(t) = f(t, \mathbf{x}(t)), \tag{77}$$

with the boundary conditions

$$\begin{cases} \int_0^1 x(t)\,\mathrm{d}t = A, \\ \Delta x^{(2i)}(0) = A_i, \quad i = 1, \ldots, m-1. \end{cases} \tag{78}$$

This problem is currently under investigation by the authors.

Euler boundary value problem: The Euler boundary value problem consists of the general differential equation

$$x^{(m)}(t) = f(t, \mathbf{x}(t)), \tag{79}$$

with the boundary condition

$$\mathscr{M} x^{(i)}(0) = A_i, \quad i = 0, \ldots, m-1. \tag{80}$$

This problem is currently under investigation by the authors.

Abel–Gontsharoff boundary value problem: The mth order differential equation

$$x^{(m)}(t) = f(t, \mathbf{x}(t)), \tag{81}$$

coupled with the Abel–Gontsharoff boundary conditions [3]

$$\begin{cases} x^{(i)}(0) = A_i, \quad 0 \leqslant i \leqslant \alpha, \\ x^{(i)}(1) = B_i, \quad \alpha + 1 \leqslant i \leqslant m-1, \end{cases} \tag{82}$$

has been a subject matter of several recent investigations (see [4] and several reference therein). It turns out that almost all the results similar to that for the Hermite boundary value problem (71), (72) can be stated for the Abel–Gontsharoff boundary value problem (81), (82).

Bernoulli boundary value problem II: This problem consists of the general mth order differential (77) together with the boundary conditions

$$\begin{cases} x(0) = A, \\ \Delta x^{(2i)}(0) = A_i, \quad i = 1, \ldots, m-1. \end{cases} \tag{83}$$

This problem, of some interest in engineering, is in advanced stage of investigation by the authors; results similar to that of the Lidstone boundary value problem (75), (76) have been proved.

Lidstone boundary value problem II: This problem (stated in [2] for $m=3$ and in [33] in the general case) consists of the general $(2m+1)$th order differential equation

$$(-1)^m x^{(2m+1)}(t) = f(x(t), \mathbf{x}(t)), \tag{84}$$

together with the boundary conditions

$$\begin{cases} x(0) = A, \\ x^{(2i-1)}(0) = A_i, \quad 1 \leqslant i \leqslant m, \\ x^{(2i-1)}(1) = B_i, \quad 1 \leqslant i \leqslant m. \end{cases} \tag{85}$$

The problem is currently under investigation.

4. The bivariate and trivariate cases

4.1. Rectangular domains

A powerful method to extend univariate formulas to bivariate (trivariate) ones on rectangles (cuboids) is the tensor-product interpolation [20, p. 46–50]. In [62] Stancu considered exactly this method in order to construct certain approximation formulas in the case of two variables, which, at the same time, allows us to obtain also expressions for their remainders. Among the formulas listed in Section 2.1, so as far as we know in literature there are at all extensions of the Taylor two-points formula (as a particular case of the two-points Hermite formula [9,12,22, 4, p. 266]), Lidstone formula of first type [4, p. 274] and Bernoulli formula of second type [28,29]. The tensorial product extension of a univariate formula with a certain degree of exactness to a bivariate (trivariate) formula on the rectangle (cuboid) preserves the property of exactness; in particular, it can be shown that the above quoted expansions have a certain degree of exactness. As a consequence, in analogy with the Sard theory of kernels [57], all these expansions can be used to find new kernels useful for the integral representation of linear functionals of special type that includes the truncated error for cubature formulas. The case of Bernoulli kernels was discussed in [29].

4.2. Triangular domain

The direct extension of the univariate expansions listed in Section 2.1 to bivariate (trivariate) ones on triangles in $\mathbb{R}^2$ (tetrahedrons in $\mathbb{R}^3$) is rather difficulty and not always possible. The Hermite case has been considered by several authors. Multi-point *natural extensions* of the univariate Hermite interpolant have been proposed in [47,53] and widely studied in [19]. The idea is based on the requirement that the multivariate extensions are related to its univariate analog on the class of *ridge functions* $f(\boldsymbol{x}) = g(\boldsymbol{\lambda} \cdot \boldsymbol{x})$, $\boldsymbol{\lambda}, \boldsymbol{x} \in \mathbb{R}^n$. In [42] the author develops an explicit representation of a finite-dimensional Hermite polynomial interpolant for the simplex in R^n which matches the function and certain derivative values at its vertices. The basis functions of this scheme are then used in the construction of a C^N blending function interpolant (for an appropriate N) for the simplex which matches an infinite set of data (function and derivative values given on all the faces). In [21] by using Bezier representations of polynomials on simplices, the authors describe an approach to the use of interpolation conditions at the vertices to determine a polynomial on the simplex. These interpolants are then used as parts for vertex spline in simplicial partitions. Previous results are extended to the multidimensional context in [22] . In [63] the author constructs a polynomial which interpolates given function values and derivatives at the vertices of a regular N-simplex. The interpolant is written as a linear combination of basis polynomials which are all generated from a basis polynomial defined on the standard simplex. The approximant interpolates quadratic polynomials exactly and has the lowest possible degree. Under these conditions, the interpolant is unique and the lowest possible degree turns out to be 3. Such interpolants are useful in optimization methods. The Lidstone formula of first type (10) and the Bernoulli formula of second type (32) have been extended, respectively, in [18,30,31] to new approximation formulas for smooth enough functions on the triangles or tetrahedrons by means of a new technique that preserve the degree of exactness. For this reason, they can be applied to provide new boundary type cubature formulas on the triangles or tetrahedrons. The Bernoulli quadrature formulas have been proposed in [30]:

these formulae can be organized in such a way that a natural structure of embedded pairs of quadrature formulas [26] appear; moreover, the weights of the cubature formulas do not depend on the order. The case of Lidstone quadrature formulas is in an advanced stage of study.

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