

Hilbert–Kunz Functions of Cubic Curves and Surfaces

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Received January 21, 1997

We determine the Hilbert–Kunz function of plane elliptic curves in odd characteristic, as well as over arbitrary fields the generalized Hilbert–Kunz functions of nodal cubic curves. Together with results of K. Pardue and P. Monsky, this completes the list of Hilbert–Kunz functions of plane cubics. Combining these results with the calculation of the (generalized) Hilbert–Kunz function of Cayley’s cubic surface, it follows that in each degree and over any field of positive characteristic there are curves resp. surfaces taking on the minimally possible Hilbert–Kunz multiplicity. © 1997 Academic Press

1. INTRODUCTION

Let $S = k[x_0, \dots, x_n]$ be the standard polynomial ring in $n + 1$ variables over a field k of prime characteristic p . Given a finite graded S -module M , the Hilbert–Kunz function of M is defined on powers of the characteristic, $q = p^n$, $n \in \mathbb{N}$, through

$$HK_M(q) := \dim_k M/m^{[q]}M,$$

where $m^{[q]} = (x_0^q, \dots, x_n^q)$ is the q th Frobenius power of the maximal homogeneous ideal $m = (x_0, \dots, x_n)$. If $I \subset S$ is a homogeneous ideal, and $R = S/I$ the homogeneous coordinate ring of the underlying projective scheme $X \subset \mathbb{P}_k^n$, the function $HK_R(q)$ is also called the Hilbert–Kunz function of X . Introduced by E. Kunz [6] in 1969, these functions were first

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studied in detail by P. Monsky [8], and he obtained the asymptotic formula

$$HK_M(q) = cq^m + O(q^{m-1})$$

with $c \geq 1$ some real number and m the Krull dimension of M . The number c is called the Hilbert-Kunz multiplicity of M . It is not known whether it is always rational. In general, it seems very difficult to determine these functions explicitly and a conceptual interpretation of the constant c is missing (see [5] for some surprising examples).

Here we exhibit the Hilbert-Kunz functions of plane elliptic curves in odd characteristic and of plane nodal cubics. Combining this work with results in [9, 10] completes the explicit determination of Hilbert-Kunz functions of plane cubic curves. The Hilbert-Kunz functions of reducible cubics were already determined by K. Pardue in [10], and he also predicted the following list for the irreducible ones on the basis of computer experiments. Note that the Hilbert-Kunz function is invariant under extensions of the coefficient field k , so that one may assume k algebraically closed.

THEOREM 1. *Let f be the equation of an irreducible cubic curve C in \mathbb{P}_k^2 over an algebraically closed field k , and let $HK_R(q)$ be the Hilbert-Kunz function of the homogeneous coordinate ring $R = S/f$. Clearly $HK_R(1) = 1$, whereas for $q > 1$ the values are as follows.*

(1) (K. Pardue [10]; also E. Kunz [7, Example 4.3], A. Conca [1]) *If C is a cuspidal cubic,*

$$HK_R(q) = \begin{cases} \frac{7}{3}q^2 & \text{for } p = 3, \\ \frac{7}{3}q^2 - \frac{4}{3} & \text{for } p \neq 3. \end{cases}$$

(2) (Theorem 3 below) *If C is a nodal cubic,*

$$HK_R(q) = \begin{cases} \frac{7}{3}q^2 - \frac{1}{3}q - 1 & \text{for } q \not\equiv 2 \pmod{3}, \\ \frac{7}{3}q^2 - \frac{1}{3}q - \frac{5}{3} & \text{for } q \equiv 2 \pmod{3}. \end{cases}$$

(3) (Theorem 4 below) *If C is an elliptic curve and $p \neq 2$,*

$$HK_R(q) = \frac{9}{4}q^2 - \frac{5}{4}.$$

(4) (P. Monsky [9]) *If C is an elliptic curve and $p = 2$, with $\delta = 1$ for $q = 2$ and $\delta = 0$ otherwise,*

$$HK_R(q) = \begin{cases} \frac{9}{4}q^2 - \delta & \text{if the } j\text{-invariant is } 0, \\ \frac{9}{4}q^2 - 1 - \delta & \text{if the } j\text{-invariant is not } 0. \end{cases}$$

At this stage, Hilbert–Kunz functions or multiplicities of plane curves of higher degree remain mysterious. However, a corollary of our work shows that for any $d \geq 2$ and for any field k of prime characteristic there exists a plane curve of degree d in \mathbb{P}_k^2 whose Hilbert–Kunz multiplicity is $\frac{3}{4}d$ —and this is the minimal possible value for such curves. In particular, the minimal Hilbert–Kunz multiplicity in each degree is rational and independent of the characteristic. We then determine explicitly the Hilbert–Kunz function of Cayley’s cubic surface in \mathbb{P}_k^3 , and the result allows us to conclude as well that for any $d \geq 2$ and for any field k of prime characteristic there exists a surface of degree d in \mathbb{P}_k^3 whose Hilbert–Kunz multiplicity is $\frac{2}{3}d$ —and this is again the minimal possible value, again rational and independent of the characteristic. By contrast, in higher dimensions the minimal Hilbert–Kunz multiplicity will depend upon the characteristic. For example, it follows from the algorithm given in [5] that the nonsingular quadric threefold in \mathbb{P}^4 has Hilbert–Kunz multiplicity $c = (29p^2 + 15)/(24p^2 + 12)$ for $p > 2$.

2. MINIMAL VALUES OF HILBERT–KUNZ FUNCTIONS

Let $I = (f)$ be a principal ideal generated by a homogeneous form f of degree $d > 0$ in S . The considerations in this section apply to the values of the *generalized* Hilbert–Kunz function of $R = S/I$, introduced by A. Conca in [1], and defined as

$$HK_{R,x}(q) = \dim_k S/(f, x_0^q, \dots, x_n^q),$$

where q is now *any* nonnegative integer, k any field. Unless k is of positive characteristic p , and q is a power of p , this dimension will generally depend upon the choice of the coordinate system $x = (x_0, \dots, x_n)$.

For each $q \in \mathbb{N}$ and each choice of coordinates x , set $x^{[q]} = (x_0^q, \dots, x_n^q)$ and consider the following graded S -modules of finite length,

$$\Theta = \frac{S}{x^{[q]}} = \bigoplus_i \Theta_i,$$

$$\theta = \frac{S}{f + x^{[q]}} = \bigoplus_i \theta_i,$$

$$\vartheta = \frac{(x^{[q]}:f)}{x^{[q]}} = \bigoplus_i \vartheta_i.$$

They are related by the exact sequence of graded S -modules

$$0 \rightarrow \vartheta(-d) \rightarrow \Theta(-d) \xrightarrow{f} \Theta \rightarrow \theta \rightarrow 0, \tag{1}$$

and $HK_{R,x}(q) = \dim_k \theta$. Evaluating dimensions yields universal bounds for the generalized Hilbert-Kunz function of $R = S/(f)$, when f varies over polynomials of degree d in $n + 1$ variables,

$$q^{n+1} = \sum_i \dim_k \Theta_i \geq HK_{R,x}(q) \geq \sum_i \max\{\dim_k \Theta_i - \dim_k \Theta_{i-d}, 0\}. \tag{2}$$

The upper bound, $HK_{R,x}(q) = q^{n+1}$, is achieved iff $f \in x^{[q]}$; for example, if $d > (n + 1)(q - 1)$, or if $q = p^n$ is a power of the characteristic, $d \geq q$ and $f = l^d$ for some linear form l . Here we are more concerned with the lower bound that is taken on if and only if f is of maximal rank at q , meaning that in each degree i the k -linear map $f|_{\Theta_{i-d}}$ is of maximal rank.

Whether a given polynomial f is of maximal rank at q can be decided by looking at the socle degree of the artinian ring θ ,

$$a(q) = \max\{i: \theta_i \neq 0\}, \tag{3}$$

and at the initial degree of ϑ ,

$$\iota(q) = \min\{i: \vartheta_i \neq 0\}. \tag{4}$$

Indeed, as the socle degree of Θ is $(n + 1)(q - 1)$, outside the range $d \leq i \leq (n + 1)(q - 1)$ source or target of $f|_{\Theta_{i-d}}$ is zero, whereas for a degree i inside that range the map is not surjective iff $i \leq a(q)$, not injective iff $i - d \geq \iota(q)$. Accordingly, all the k -linear maps induced by f are of maximal rank iff $a(q) < \iota(q) + d$. Moreover, the exact sequence (1) is self-dual, whence it suffices to know either $a(q)$ or $\iota(q)$:

LEMMA 1. For each $q \in \mathbb{N}$, and independent of f , one has

$$a(q) + \iota(q) = (n + 1)(q - 1). \tag{5}$$

Given q , all k -linear maps $f|_{\Theta_{i-d}}$ are of maximal rank iff

$$a(q) < \frac{(n+1)(q-1)+d}{2} < \iota(q) + d. \quad (6)$$

Moreover, each of the inequalities implies the other.

Proof. The ring $\Theta = S/x^{[q]}$ is a zero-dimensional complete intersection with its socle in degree $(n+1)(q-1)$. Thus for any finite graded Θ -module M , we have an isomorphism of graded Θ -modules

$$\mathrm{Hom}_k(M, k) \cong \mathrm{Hom}_{\Theta}(M, \omega_{\Theta}),$$

where $\omega_{\Theta} = \Theta((n+1)(q-1))$ is the canonical module of Θ , and $\mathrm{Hom}_k(M, k)$ is the Θ -module graded naturally through

$$(\mathrm{Hom}_k(M, k))_i = \mathrm{Hom}_k(M_{-i}, k).$$

As $\mathrm{Hom}_{\Theta}(\theta, \Theta) \cong (x^{[q]}:f)/x^{[q]} = \vartheta$, we get

$$\mathrm{Hom}_k(\theta, k) \cong \vartheta((n+1)(q-1)).$$

For the dimension of the finite dimensional k -vector space θ_i , this yields

$$\dim_k \theta_i = \dim_k \mathrm{Hom}_k(\theta_i, k) = \dim_k (\mathrm{Hom}_k(\theta, k))_{-i} = \dim_k \vartheta_{(n+1)(q-1)-i},$$

and the equality follows from the definition of $a(q)$ and $\iota(q)$. As f induces maps of maximal rank iff $a(q) < \iota(q) + d$, we can eliminate either one of the two invariants to obtain the last claim. ■

If $d > (n+1)(q-1)$, the information is already complete: f , inducing the zero map in (1), is trivially of maximal rank at q , and $HK_{R,x}(q) = q^{n+1}$. Also, if f is a polynomial of a single variable, $n = 0$, there are no secrets to discover. If $n > 0$, the (usual) Hilbert series

$$H_{\Theta}(t) = \sum_i (\dim_k \Theta_i) t^i = (1 + t + t^2 + \cdots + t^{q-1})^{n+1},$$

of the artinian k -algebra Θ is a *reciprocal* and *unimodal* polynomial of degree $l = (n+1)(q-1)$ in t , meaning that its coefficients, $\alpha_i = \dim_k \Theta_i$, satisfy

$$\begin{aligned} \alpha_i &= \alpha_{l-i} && \text{for every } i, \\ \alpha_i &< \alpha_{i+1} && \text{for } 0 \leq i < \left\lfloor \frac{l}{2} \right\rfloor. \end{aligned}$$

In particular, $\dim_k \Theta_i - \dim_k \Theta_{i-d} > 0$ iff $0 \leq i \leq m(q)$, where

$$m(q) = \left\lfloor \frac{(n + 1)(q - 1) + (d - 1)}{2} \right\rfloor;$$

as for $a(q)$, we suppress the dependence upon d from the notation. Thus the lower bound, $L(q)$, in inequality (2) evaluates to

$$\begin{aligned} L(q) &:= \sum_i \max\{\dim_k \Theta_i - \dim_k \Theta_{i-d}, 0\} \\ &= \sum_{m(q)-d+1}^{m(q)} \dim_k \Theta_i, \quad \text{as } H_\Theta(t) \text{ is unimodal and reciprocal} \\ &= \text{coefficient of } t^{m(q)} \text{ in } \frac{(1 - t^d)(1 - t^q)^{n+1}}{(1 - t)^{n+2}} \\ &= \frac{1}{2\pi\sqrt{-1}} \int_{|z|=\epsilon} \frac{(1 - z^d)(1 - z^q)^{n+1}}{(1 - z)^{n+2} z^{m(q)+1}} dz. \end{aligned}$$

As $m(q)$ is the largest integer smaller than $\frac{1}{2}((n + 1)(q - 1) + d)$, we get the following result.

THEOREM 2. *If $n > 0$, and if f is a homogeneous polynomial of degree $d \leq (n + 1)(q - 1)$ in $n + 1$ many variables, then the socle degree of the graded artinian k -algebra θ satisfies*

$$a(q) \geq m(q). \tag{7}$$

Furthermore, the following statements are equivalent:

- (i) The polynomial f is of maximal rank at q .
- (ii) The socle degree $a(q)$ is minimal, $a(q) = m(q)$.
- (iii) The initial degree $\iota(q)$ is maximal, $\iota(q) = (n + 1)(q - 1) - m(q)$.
- (iv) The Hilbert-Kunz function of f at q achieves the lower bound $L(q)$.

Proof. The first statement follows from the exact sequence (1), as

$$\dim_k \Theta_{m(q)} > \dim_k \Theta_{m(q)-d}.$$

As $a(q)$ is an integer and $m(q)$ is the largest integer smaller than $\frac{1}{2}((n + 1)(q - 1) + d)$, the just established lower bound for $a(q)$ implies the equivalences in view of Lemma 1. ■

EXAMPLE. For $d = 2, 3$ and $n = 2, 3$, we get the following table:

d	n	$m(q)$	Lower bound for $HK_{R,x}(q)$
2	2	$\left\lfloor \frac{3q}{2} \right\rfloor - 1$	$\left\lfloor \frac{3}{2}q^2 \right\rfloor = \begin{cases} \frac{3}{2}q^2 & \text{for } q \text{ even} \\ \frac{3}{2}q^2 - \frac{1}{2} & \text{for } q \text{ odd} \end{cases}$
	3	$2(q-1)$	$\frac{4}{3}q^3 - \frac{1}{3}q$
3	2	$\left\lfloor \frac{3q-1}{2} \right\rfloor$	$\left\lfloor \frac{9}{4}q^2 - \frac{5}{4} \right\rfloor = \begin{cases} \frac{9}{4}q^2 - 2 & \text{for } q \text{ even} \\ \frac{9}{4}q^2 - \frac{5}{4} & \text{for } q \text{ odd} \end{cases}$
	3	$2q-1$	$2q^3 - q$

For $d = 2$, it can be extracted from [1] that the quadric $x_0^2 - x_1x_2$ for $n = 2$, respectively the quadric $x_0x_1 - x_2x_3$ for $n = 3$, has generalized Hilbert–Kunz function that takes on the minimum value at each q .

Remark 1. P. Monsky noted that expressing the minimal possible value $L(q)$ of $HK_{R,x}(q)$ as a residue leads to an intriguing lower bound for Hilbert–Kunz multiplicities in terms of the integrals

$$\beta_{n+1} = \frac{1}{\pi} \int_{-\infty}^{+\infty} \left(\frac{\sin \alpha}{\alpha} \right)^{n+1} d\alpha = \frac{1}{2^n n!} \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i (n+1-2i)^n \binom{n+1}{i},$$

as

$$d\beta_{n+1} = \lim_{q \rightarrow \infty} \frac{1}{q^n 2\pi \sqrt{-1}} \int_{|z|=\epsilon} \frac{(1-z^d)(1-z^q)^{n+1}}{(1-z)^{n+2} z^{m(q)+1}} dz. \quad (8)$$

Thus, for a hypersurface of degree d in \mathbb{P}_k^n over a field k of positive characteristic, the Hilbert–Kunz multiplicity satisfies

$$c \geq d\beta_{n+1}. \quad (9)$$

A direct combinatorial proof is as follows. Expanding $(1-t^d)/(1-t)^{n+2}$ into its Taylor series at $t = 0$, one can write

$$\frac{1-t^d}{(1-t)^{n+2}} = \sum_{\nu=0}^{d-n-1} R(\nu)t^\nu + \sum_{\nu \geq 0} P(\nu)t^\nu,$$

where $R(\nu) \in \mathbb{Z}$, and $P(\tau) = (d/n!)\tau^n + O(\tau^{n-1})$ is the corresponding (Hilbert) polynomial, univariate over \mathbb{Q} of degree n with leading coeffi-

cient $d/n!$. Now use that $m(q)/q = (n + 1)/2 + O(1/q)$, that the coefficient of $t^{m(q)}$ in $(1/q^n)(1 - t^q)^{n+1} \sum_{\nu=0}^{d-n-1} R(\nu)t^\nu$ tends to zero with q , and that

$$(1 - t^q)^{n+1} \sum_{\nu \geq 0} P(\nu)t^\nu = \sum_{\nu \geq 0} \left(\sum_{i=0}^{\lfloor \nu/q \rfloor} (-1)^i P(\nu - iq) \binom{n+1}{i} \right) t^\nu,$$

to get

$$\begin{aligned} \lim_{q \rightarrow \infty} \frac{L(q)}{q^n} &= \lim_{q \rightarrow \infty} \sum_{i=0}^{\lfloor m(q)/q \rfloor} (-1)^i \frac{P(m(q) - iq)}{q^n} \binom{n+1}{i} \\ &= \lim_{q \rightarrow \infty} \sum_{i=0}^{\lfloor m(q)/q \rfloor} (-1)^i \frac{d}{n!} \left(\frac{m(q)}{q} - i \right)^n \binom{n+1}{i} \\ &= \frac{d}{2^n n!} \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i (n+1 - 2i)^n \binom{n+1}{i} \\ &= d\beta_{n+1}. \end{aligned}$$

That this combinatorial expression equals the indicated integral can now be checked in any table of integrals, e.g., [3, 3.836.5³, p 458]. It follows that the sequence $\{\beta_n\}$ of rational numbers decreases to zero. The first few values are

$$\beta_1 = 1, \quad \beta_2 = 1, \quad \beta_3 = \frac{3}{4}, \quad \beta_4 = \frac{2}{3}, \quad \beta_5 = \frac{115}{192}, \quad \beta_6 = \frac{11}{20}.$$

Remark 2. The same argument applies to $\liminf_{q \rightarrow \infty} HK_{R,x}(q)/q^n$. But a “generalized” Hilbert–Kunz multiplicity need not exist. For example, consider

$$R = k[x_0, x_1, x_2]/(x_0 + x_1 + x_2)$$

over a field k of positive characteristic $p > 2$. It is clear that

$$\frac{R}{(x_0^q, x_1^q, x_2^q)} \cong \frac{k[x, y]}{(x_0^q, x_1^q, (x_0 + x_1)^q)}$$

for any $q \in \mathbb{N}$. If q is a power of the characteristic, then

$$\frac{k[x_0, x_1]}{(x_0^q, x_1^q, (x_0 + x_1)^q)} = \frac{k[x_0, x_1]}{(x_0^q, x_1^q)},$$

and $HK_{R, (x_0, x_1, x_2)}(q) = q^2$. On the other hand, consider the sequence $\{2p^n\}$. For $q = 2p^n$, one has

$$\begin{aligned} J &= (x^q, y^q):(x+y)^q = (x^{2p^n}, y^{2p^n}):(x^{p^n} + y^{p^n})^2 \\ &= (x^{2p^n}, y^{2p^n}):x^{p^n}y^{p^n} = (x^{p^n}, y^{p^n}), \end{aligned}$$

and the exact sequence

$$0 \rightarrow k[x, y]/J \xrightarrow{(x+y)^q} k[x, y]/(x^q, y^q) \rightarrow k[x, y]/(x^q, y^q, (x+y)^q) \rightarrow 0$$

yields

$$HK_{R, (x, y, z)} = q^2 - \left(\frac{q}{2}\right)^2 = \frac{3}{4}q^2.$$

In this situation, there is thus no “generalized” Hilbert–Kunz multiplicity. P. Monsky pointed out that C. Han, in her thesis [4], determined *all* values of this generalized Hilbert–Kunz function.

For elliptic curves in odd characteristic, we will prove that the corresponding cubic polynomial is of maximal rank at any power q of the characteristic, whereas for the polynomial $x_0x_1x_2x_3(1/x_0 + 1/x_1 + 1/x_2 + 1/x_3)$, representing Cayley’s cubic surface, this will be even established at any $q \in \mathbb{N}$ over any field k . The proof is accomplished by showing that $a(q)$ equals the minimum value $m(q)$, and the respective Hilbert–Kunz function can then be read off from Table I.

None of this applies to elliptic curves in characteristic 2 (see [9] for details), nor does it hold for singular irreducible cubic curves in any characteristic. But in the latter case, the Hilbert–Kunz function can be determined completely from the rational parametrization of the curve as we show next.

3. SINGULAR IRREDUCIBLE CUBIC CURVES

The Hilbert–Kunz function of a cuspidal cubic is known from [10], see also [1], but our treatment here deals with the nodal and cuspidal case at the same time. Let k be an algebraically closed field—of any characteristic for now—and denote by C a singular irreducible plane cubic curve in \mathbb{P}_k^2 .

In suitable coordinates, C is given by a Weierstraß equation

$$f(x, y, z) = z(y^2 + a_1xy - a_2x^2) - x^3 = 0,$$

so that $o = [0, 0, 1] \in \mathbb{P}^2$ is its unique singular point. The curve has a node at o iff the tangential quadric $Q(x, y) = y^2 + a_1xy - a_2x^2$ has distinct roots iff $a_1^2 + 4a_2 \neq 0$, otherwise it is cuspidal.

The curve C is rational and a rational parametrization $\nu: \mathbb{P}^1 \rightarrow C \subset \mathbb{P}^2$ normalizes the curve, pulling back $\mathcal{O}_{\mathbb{P}^2}(1)$ along $C \hookrightarrow \mathbb{P}^2$ and then ν to $\mathcal{O}_{\mathbb{P}^1}(3)$. Algebraically, such a parametrization is given by the monomorphism of k -algebras

$$\alpha(x, y, z) = (sQ(s, t), tQ(s, t), s^3),$$

$$\begin{aligned} \alpha: R = \frac{k[x, y, z]}{f(x, y, z)} &\cong \bigoplus_{n \in \mathbb{Z}} H^0(C, \mathcal{O}_C(n)) \hookrightarrow \bigoplus_{n \in \mathbb{Z}} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3n)) \\ &\cong k[s, t]^{(3)} =: \tilde{R}, \end{aligned}$$

where $\tilde{R} = k[s, t]^{(3)}$ is the Veronese subring of the polynomial ring $k[s, t]$ spanned by all homogeneous polynomials whose degree is divisible by 3. Notice that \tilde{R}_n consists of all homogeneous polynomials of degree $3n$.

The cokernel of α can be identified as follows. A section $p(s, t) \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3n))$ comes via α from a section in $H^0(C, \mathcal{O}_C(n))$ iff $p(s, t)$ takes on the same value at the two points $Q(s, t) = 0$. Explicitly, write $p(s, t) = e_1(s) + e_2(s)t + e(s, t)Q(s, t)$ with uniquely determined polynomials $e_1, e_2 \in k[s]$ and $e \in k[s, t]$. The component $e_2(s)t$ represents the class of $p(s, t)$ in $k[s, t]/(k[s] + Qk[s, t])$ and $(k[s, t]/(k[s] + Qk[s, t]))^{(3)}$ is the cokernel of α . If $p(s, t) \in k[s, t]^{(3)}$, then $e_2(s) = b(s^3)s^2$ for some unique univariate polynomial b that is necessarily of degree $\frac{1}{3}\deg p - 1$.

LEMMA 2. (i) *The map*

$$\beta: \tilde{R} = k[s, t]^{(3)} \rightarrow k[z](-1),$$

associating to $p(s, t)$ the polynomial $b(z)$, is a degree preserving epimorphism of R -modules, the R -module structure on \tilde{R} given by α , the one on $k[z]$ by the natural projection $R = k[x, y, z]/f \rightarrow k[z]$.

(ii) *The sequence of graded R -modules*

$$0 \rightarrow R \xrightarrow{\alpha} \tilde{R} = k[s, t]^{(3)} \xrightarrow{\beta} k[z](-1) \rightarrow 0 \tag{10}$$

is exact.

Proof. If $p(s, t) = e_1(s) + b(s^3)s^2t + e(s, t)Q(s, t)$ is the unique representation of $p(s, t) \in \tilde{R}$, then

$$\alpha(x)p(s, t) = 0 + 0 \cdot s^2t + (sp(s, t))Q(s, t),$$

$$\alpha(y)p(s, t) = 0 + 0 \cdot s^2t + (tp(s, t))Q(s, t),$$

$$\alpha(z)p(s, t) = s^3e_1(s) + (s^3b(s^3))s^2t + (s^3e(s, t))Q(s, t),$$

In this diagram, E, F, G , and then also D , are finite dimensional and one has

$$HK_{R,(x,y,z)}(q) = \dim_k E = \dim_k F - \dim_k G + \dim_k D. \quad (11)$$

The dimension of $G \cong (k[z]/z^q)(-1)$ equals q , and the next lemma determines the dimension of F , that is, the value of the generalized Hilbert-Kunz function for \tilde{R} with respect to (x, y, z) at q .

LEMMA 3. (i) *Set $P = k[s, t]$, the polynomial ring in two variables with its natural grading. For any $q \in \mathbb{N}$, the P -module $M = k[s, t]/(\alpha(x)^q, \alpha(y)^q, \alpha(z)^q)$ has minimal graded resolution*

$$0 \rightarrow P(-4q) \oplus P(-5q) \xrightarrow{\begin{pmatrix} t^q & -s^q & 0 \\ s^{2q} & 0 & -Q(s,t)^q \end{pmatrix}} P(-3q)^{\oplus 3} \xrightarrow{\begin{pmatrix} s^q Q(s,t)^q \\ t^q Q(s,t)^q \\ s^{3q} \end{pmatrix}} P \rightarrow M \rightarrow 0.$$

(ii) *The middle column in the diagram above is obtained from that resolution by applying the functor $()^{(3)}$, and in particular $F = M^{(3)}$.*

(iii) *The dimension of F is given by*

$$HK_{\tilde{R},(x,y,z)}(q) = \dim_k F = \begin{cases} \frac{7}{3}q^2 - \frac{1}{3} & \text{if } q \not\equiv 0 \pmod{3}, \\ \frac{7}{3}q^2 & \text{if } q \equiv 0 \pmod{3}. \end{cases}$$

Proof. (i) As s does not divide $Q(s, t)$, the module M is artinian, and the result follows from the Hilbert-Burch theorem: the (signed) (2×2) -minors of the leftmost matrix are respectively $s^q Q(s, t)^q = \alpha(x)^q$, $t^q Q(s, t)^q = \alpha(y)^q$, $s^{3q} = \alpha(z)^q$.

Part (ii) is clear and (iii) follows then easily from

$$\dim_k F_i = \dim_k M_{3i} = \dim_k P_{3i} - 3 \dim_k P_{3i-3q} + \dim_k P_{3i-4q} + \dim_k P_{3i-5q}$$

and $\dim_k P_j = \max\{0, j + 1\}$ for $j \in \mathbb{Z}$. ■

In Eq. (11) for $HK_{R,(x,y,z)}(q)$, it remains to determine the dimension of D . To this end, we exhibit the map $\bar{\beta}$ explicitly. As multiplication by z^q is injective on $k[z]$, one has $C \cong k[z]^{\oplus 2}(-1 - q)$. Furthermore, as $B \cong (k[s, t](-4q) \oplus k[s, t](-5q))^{(3)}$ by Lemma 3, a homogeneous element in B is represented by a pair $(p_1(s, t), p_2(s, t))$ of homogeneous polynomials satisfying

$$\deg p_1 = \deg p_2 + q \equiv -4q \pmod{3},$$

and such a pair is mapped to

$$p_1(s, t)(t^q, -s^q, 0) + p_2(s, t)(s^{2q}, 0, -Q(s, t)^q)$$

in $\tilde{R}^{\oplus 3}(-q)$. Thus

$$\bar{\beta}(p_1, p_2) = (\beta(t^q p_1 + s^{2q} p_2), \beta(-s^q p_1)) \in k[z]^{\oplus 2}(-1 - q) \cong C.$$

As the field k is algebraically closed, the quadric Q factors, $Q(s, t) = (t - us)(t - vs)$; $u, v \in k$; and the unique representation of $t^q \bmod Q(s, t)$ is

$$t^q = \tau_1 s^q + \tau_2 s^{q-1} t + \tau(s, t) Q(s, t),$$

where

$$\tau_1 = -uv \sum_{i=0}^{q-2} u^{q-2-i} v^i, \quad \tau_2 = \sum_{i=0}^{q-1} u^{q-1-i} v^i.$$

Writing now

$$p_1 = b_1(s) + b_2(s)t + b_3(s, t)Q(s, t),$$

$$p_2 = c_1(s) + c_2(s)t + c_3(s, t)Q(s, t),$$

for suitable polynomials b_i, c_i , it follows that

$$\bar{\beta}(p_1, p_2) = (\tau_2 b_1 s^{q-3} + (\tau_1 - a_1 \tau_2) b_2 s^{q-2} + c_2 s^{2q-2}, -b_2 s^{q-2}) \Big|_{s^3=z}.$$

So the image of $\bar{\beta}$ in $C \cong k[z]^{\oplus 2}(-1 - q)$ is generated as a $k[z]$ -module by the three pairs

$$(\tau_2, 0)z^{(q-3+\epsilon)/3}, \quad (\tau_1 - a_1 \tau_2, -1)z^{(q-2+\eta)/3}, \quad (1, 0)z^{(2q-2+\zeta)/3},$$

where $\epsilon, \eta, \zeta \in \{0, 1, 2\}$ are such that the exponents become integers. Accordingly, the dimension of D is equal to $(2q - 5 + \epsilon + \eta)/3$ if $\tau_2 \neq 0$, whereas it equals $(3q - 4 + \eta + \zeta)/3$ if $\tau_2 = 0$. Thus, the decisive factor is whether or not τ_2 vanishes.

LEMMA 4. (i) *If q is a power of the characteristic of k , then C is nodal over k iff $\tau_2 \neq 0$.*

(ii) *If $\tau_2 \neq 0$, then for any q*

$$\dim_k D = \begin{cases} 2\lfloor q/3 \rfloor & \text{for } q \not\equiv 0 \pmod{3}, \\ \frac{2}{3}q - 1 & \text{for } q \equiv 0 \pmod{3}. \end{cases}$$

Proof. (i) If q is a power of the characteristic of k , then $\tau_2 = (u - v)^{q-1}$, whence $\tau_2 \neq 0$ iff $u \neq v$ iff C is nodal. Part (ii) just evaluates the formula for $\dim_k D$ found above in terms of $q \bmod 3$. ■

Putting everything together yields the Hilbert-Kunz function in the nodal case.

THEOREM 3. *Let C be a nodal cubic over a field k of prime characteristic p . For a power q of p , the Hilbert-Kunz function at q is*

$$HK_C(q) = \begin{cases} \frac{7}{3}q^2 - \frac{1}{3}q - 1 & \text{for } q \not\equiv 2 \pmod{3}, \\ \frac{7}{3}q^2 - \frac{1}{3}q - \frac{5}{3} & \text{for } q \equiv 2 \pmod{3}. \end{cases}$$

If C is a cuspidal cubic, then $\tau_2 = 0$ for any q and we get immediately the generalized Hilbert-Kunz function—in accordance with [8] and [1],

$$HK_{C,(x,y,z)}(q) = \begin{cases} \frac{7}{3}q^2 & \text{for } q \equiv 0 \pmod{3}, \\ \frac{7}{3}q^2 - \frac{4}{3} & \text{for } q \not\equiv 0 \pmod{3}. \end{cases}$$

Note however that, if q is not a power of the characteristic, this last result will in general depend upon the choice of the coordinate system made relative to the given Weierstraß form. The case of the generalized Hilbert-Kunz function for a nodal cubic can be extracted as well—and the dependence upon the coordinate system becomes apparent: If the distinct roots u, v satisfy $u^q - v^q = 0$ for some q , the generalized Hilbert-Kunz function “jumps up,” it takes on the value from the cuspidal case. For any given q , we can avoid this situation by replacing y with $y + \alpha x$, for a general $\alpha \in k$. The curve C is then still in Weierstraß form, and with respect to $(x, y + \alpha x, z)$, the generalized Hilbert-Kunz function takes on the value predicted by Theorem 3. Unless the algebraically closed field k is an algebraic closure of a finite field, one can even find an $\alpha \in k$ that works for all q simultaneously.

4. ELLIPTIC CURVES IN ODD CHARACTERISTIC

In this section, we prove the announced result for elliptic curves in odd characteristic and deduce that the Hilbert-Kunz multiplicity of a generic plane curve equals $\frac{3}{4}d$ when $d \geq 2$.

THEOREM 4. *Let $f(x, y, z) \in S = k[x, y, z]$ be a cubic polynomial defining a plane elliptic curve over a field k of odd characteristic p . For any $n \in \mathbb{N}$ and $q = p^n$, the socle degree $a(q)$ of $\theta = S/(f + m^{[q]})$ is minimal,*

$$a(q) = \frac{3}{2}q - \frac{1}{2},$$

and the Hilbert–Kunz function of $R = S/(f)$ at q is given by

$$HK_R(q) = \frac{9}{4}q^2 - \frac{5}{4}.$$

In Subsection 4.1, we recall a classical result about determinants of Hankel matrices whose entries are Legendre polynomials, and in Subsection 4.2, we use it to determine the invariant $a(q)$ and to establish Theorem 4. But first we state a corollary and make a remark.

COROLLARY 1. *For any field k of prime characteristic p and any integer $d \geq 2$, there is a curve $C \subset \mathbb{P}_k^2$ of degree d whose Hilbert–Kunz multiplicity achieves the minimum $\frac{3}{4}d$.*

Proof. As shown in [10], the Hilbert–Kunz multiplicity of the quadric $g = x^2 - yz$ equals $\frac{3}{2}$. For elliptic curves in any prime characteristic, Theorem 1 shows that their Hilbert–Kunz multiplicities are minimal, equal to $\frac{9}{4}$. As any integer $d \geq 2$ can be written $d = 2u + 3v$ for some $u, v \in \mathbb{N}$, additivity of the Hilbert–Kunz multiplicity, see [8], implies that the curve of degree d , defined by $h = g^u f^v$, f a nonsingular cubic, will achieve the minimum. ■

Remark 3. Semi-continuity of the Hilbert–Kunz multiplicity yields that the Hilbert–Kunz multiplicity of a generic plane curve of degree $d \geq 2$ equals $\frac{3}{4}d$. Clearly $c = 1$ if the degree $d = 1$. So the Hilbert–Kunz multiplicity of a generic curve is rational and independent of the (positive) characteristic.

4.1. Hankel Determinants of Legendre Polynomials

The Hankel matrices associated to a sequence $a = \{a_i\}$ are

$$H_k^{(n)}(a) = \begin{pmatrix} a_n & a_{n+1} & \cdots & a_{n+k-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+k} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+k-1} & \cdots & \cdots & a_{n+2(k-1)} \end{pmatrix},$$

with corresponding Hankel determinants

$$D_k^{(n)}(a) = \det H_k^{(n)}(a).$$

The generating function for the Legendre polynomials, $\{P_n(t)\}_{n \in \mathbb{N}}$, is

$$F(t, x) = \frac{1}{\sqrt{1 - 2tx + x^2}} = \sum P_n(t)x^n \in \mathbb{Z}\left[\frac{1}{2}, t\right][[x]].$$

For each k , consider the determinant of the following Hankel matrix whose entries are Legendre polynomials,

$$D_k^{(0)}(P_i(t)) = \det \begin{pmatrix} P_0(t) & P_1(t) & \cdots & P_{k-1}(t) \\ P_1(t) & P_2(t) & \cdots & P_k(t) \\ \cdots & \cdots & \cdots & \cdots \\ P_{k-1}(t) & \cdots & \cdots & P_{2k-2}(t) \end{pmatrix}.$$

In [2], J. Geronimus gave the following beautiful formula.

THEOREM 5. $D_k^{(0)}(P_i(t)) = 2^{-(k-1)^2} (t^2 - 1)^{(k-1)k/2}.$

P. Monsky communicated a direct proof to us that we now present.

LEMMA 5. *If $t \in \mathbb{R}$ and $t > 1$, then*

$$P_n(t) = \frac{1}{\pi} \int_0^\pi [t + \sqrt{t^2 - 1} \cos \alpha]^n d\alpha.$$

Proof. If x is small,

$$\begin{aligned} (1 - 2tx + x^2)^{-1/2} &= \left[(1 - tx)^2 - (x\sqrt{t^2 - 1})^2 \right]^{-1/2} \\ &= \frac{1}{\pi} \int_0^\pi \frac{d\alpha}{(1 - tx) - x\sqrt{t^2 - 1} \cos \alpha} \\ &= \frac{1}{\pi} \int_0^\pi \frac{d\alpha}{1 - x[t + \sqrt{t^2 - 1} \cos \alpha]}. \end{aligned}$$

The expansion of the integrand into a power series of x yields the lemma. ■

Proof of Theorem 5 (P. Monsky). As both sides of Geronimus' formula are polynomials in t , it suffices to prove it for $t \in \mathbb{R}$ and $t > 1$.

Let V be the vector space of real valued continuous functions on $[0, \pi]$. Define a symmetric bilinear form (\cdot, \cdot) on V through

$$(f, g) = \frac{1}{\pi} \int_0^\pi f(\alpha)g(\alpha) d\alpha.$$

If $h_1, \dots, h_s \in V$, let $\Delta(h_1, \dots, h_s)$ be the determinant of the matrix $((h_i, h_j))$. Set $g_m = [t + \sqrt{t^2 - 1} \cos \alpha]^m$. By the preceding lemma, $(g_i, g_j) = P_{i+j}$. So the required Hankel determinant is $\Delta(g_0, \dots, g_{k-1})$.

Let $V_m \subset V$ be the subspace spanned by $\{1, \cos \alpha, \dots, (\cos \alpha)^m\}$. Then $f_m = (t^2 - 1)^{m/2} \cos(m\alpha) \in V_m$, and it is a linear combination of g_0, \dots, g_m . Furthermore, modulo V_{m-1} , $\cos(m\alpha) \equiv 2^{m-1}(\cos \alpha)^m$, and consequently $f_m \equiv 2^{m-1}(t^2 - 1)^{m/2}(\cos \alpha)^m \equiv 2^{m-1}g_m$. We conclude that

$$\begin{aligned} \Delta(f_0, \dots, f_{k-1}) &= \left(\prod_{m=1}^{k-1} 2^{m-1} \right)^2 \Delta(g_0, \dots, g_{k-1}) \\ &= 2^{(k-2)(k-1)} \Delta(g_0, \dots, g_{k-1}). \end{aligned}$$

But using the orthogonality of the f_i one finds that

$$\Delta(f_0, \dots, f_{k-1}) = 2^{-(k-1)}(t^2 - 1)^{(k-1)k/2}$$

and Theorem 5 follows. ■

Now consider

$$G(t, x) = \sqrt{1 - 2tx + x^2} = \sum \widetilde{P_n(t)} x^n.$$

As $F(t, x)G(t, x) = 1$, Geronimus' formula yields also the following corollary.

COROLLARY 2. $D_k^{(2)}(\widetilde{P_i(t)}) = (-1)^k D_{k+1}^{(0)}(P_i(t)) = (-2)^{-k^2} (t^2 - 1)^{k(k+1)/2}$.

Remark 4. The coefficients of Legendre polynomials are rational numbers whose denominators are powers of 2. Thus Geronimus' identity and the above corollary hold over any ring in which 2 is a unit, in particular over a field of odd characteristic.

4.2. The Invariant $a(q)$

We first prove the following theorem showing that in odd characteristic there are no nontrivial syzygies of low degree between the equation of an elliptic curve and Frobenius powers of the variables.

THEOREM 6. *Let k be a field of odd characteristic p , and let $f \in k[x, y]$ be a cubic polynomial defining an elliptic curve in \mathbb{A}_k^2 . For any $q = p^n$, with $n \in \mathbb{N}$, if $f|ux^q + vy^q + w$ for $u, v, w \in k[x, y]$ of degree at most $\frac{1}{2}(q - 1)$, then f divides each of u, v, w .*

Proof. We give the proof for $q \equiv 1 \pmod{4}$. The argument in the other case, $q \equiv 3 \pmod{4}$, is analogous and left to the reader. Without loss of generality assume that k is algebraically closed. Since the result is invariant under the action of $GL(2, k)$, and the characteristic of k is odd, we can put the cubic into the form $f = y^2 - x(1 - 2tx + x^2)$ with $t^2 \neq 1$.

If $f|ux^q + vy^q + w$ for some u, v, w of degree at most $\frac{1}{2}(q - 1)$, then

$$ux^q + vy^q + w = fh, \tag{12}$$

where h is a polynomial in x, y . Set

$$l = \frac{q - 1}{2},$$

$$g = x(1 - 2tx + x^2).$$

We can then write

$$u = \sum_{i=0}^l a_i y^i = A_0 + yA_1 + fu_1,$$

with $a_i \in k[x]$ of degree at most $l - i$, $u_1 \in k[x, y]$ and

$$A_0 = \sum_{j=0}^{l/2} a_{2j} g^j, \quad A_1 = \sum_{j=0}^{l/2-1} a_{2j+1} g^j,$$

polynomials in x . Similarly, we write

$$v = B_0 + yB_1 + fv_1$$

$$w = C_0 + yC_1 + fw_1$$

$$y^q = y^{2l+1} = yg^l + f\gamma$$

for polynomials $v_1, w_1, \gamma \in k[x, y]$; $B_0, B_1, C_0, C_1 \in k[x]$. Equation (12) then becomes

$$(x^q A_0 + C_0 + g^{l+1} B_1) + y(x^q A_1 + C_1 + g^l B_0) = fh_1.$$

Viewing both sides as polynomials in y , we get $h_1 = 0$ and

$$x^q A_0 + C_0 + g^{l+1} B_1 = 0, \tag{13}$$

$$x^q A_1 + C_1 + g^l B_0 = 0. \tag{14}$$

As $\deg C_0 \leq \deg x^q A_0 \leq \frac{7}{2}l + 1$, it follows that

$$\deg B_1 \leq \left(\frac{7}{2}l + 1\right) - 3(l + 1) = \frac{l}{2} - 2,$$

and similarly $\deg B_0 \leq l/2 - 1$. Thus we can write

$$B_1 = \alpha_{l/2-2}x^{l/2-2} + \cdots + \alpha_0,$$

$$B_0 = \beta_{l/2-1}x^{l/2-1} + \cdots + \beta_0,$$

for tuples

$$\alpha = (\alpha_i) \in k^{l/2-1}, \quad \beta = (\beta_i) \in k^{l/2}.$$

Since $\deg C_0 \leq \frac{3}{2}l$ and $\text{ord}(x^q A_0) \geq q = 2l + 1$, the intermediate powers of x in $g^{l+1}B_1$ have zero coefficients, whence we get a linear system of equations for α , say $E\alpha = 0$, where $E: k^{l/2-1} \rightarrow k^{l/2}$ is represented by the matrix

$$E = \begin{pmatrix} e_2 & e_3 & \cdots & e_{l/2} \\ e_3 & e_4 & \cdots & e_{l/2+1} \\ e_4 & \cdots & \cdots & e_{l/2+2} \\ \cdots & \cdots & \cdots & \cdots \\ e_{l/2+1} & \cdots & \cdots & e_{l-1} \end{pmatrix},$$

whose entries e_i are the coefficients in the expansion

$$(1 - 2tx + x^2)^{(q+1)/2} = x^{q+1} + e_q x^q + \cdots + e_0.$$

Analogously, the corresponding powers of x in $g^l B_0$ yield a system of equations for β , say $H\beta = 0$, where $H: k^{l/2} \rightarrow k^{l/2+2}$ is represented by the matrix

$$H = \begin{pmatrix} h_0 & h_1 & \cdots & h_{l/2-1} \\ h_1 & h_2 & \cdots & h_{l/2} \\ \cdots & \cdots & \cdots & \cdots \\ h_{l/2+1} & \cdots & \cdots & h_l \end{pmatrix},$$

whose entries h_i are the coefficients in the expansion

$$(1 - 2tx + x^2)^{(q-1)/2} = x^{q-1} + h_{q-2}x^{q-2} + \cdots + h_0.$$

As q is a power of the characteristic p , one has

$$\begin{aligned} (1 - 2tx + x^2)^{(q-1)/2} &\equiv \frac{(1 - (2tx)^q + x^{2q})^{1/2}}{\sqrt{1 - 2tx + x^2}} \\ &\equiv (1 - (2tx)^q + x^{2q})^{1/2} \sum P_n(t)x^n \pmod{p}, \end{aligned}$$

whence $h_i \equiv P_i(t) \pmod p$ and, analogously, $e_i \equiv \widetilde{P_i(t)} \pmod p$ for $i < q$, where $P_i(t)$ and $\widetilde{P_i(t)}$ are as in Subsection 4.1. As $t^2 \neq 1$ by assumption, Geronimus' Theorem and its corollary imply

$$\text{rank } E = \frac{l}{2} - 1, \quad \text{rank } H = \frac{l}{2},$$

whence each α_i or β_j equals zero, thus $B_0 = B_1 = 0$, so that $v = fv_1$. As $\deg C_i < \text{ord}(x^q A_i)$, for $i = 0, 1$, it follows further from Eqs. (13) and (14) that $C_i = A_i = 0$ and the theorem follows. ■

Now we can finish the *Proof of Theorem 4*.

As $d = 3 < 3(q - 1)$, for any power $q = p^n$, $n \in \mathbb{N}$, of an odd prime p , Theorem 2 yields the lower bound $a(q) \geq \frac{3}{2}q - \frac{1}{2}$, and the upper bound $\iota(q) \leq \frac{3}{2}q - \frac{5}{2}$. It remains thus to show $\vartheta_{(3/2)q-7/2} = 0$, or, equivalently, if $f|ux^q + vy^q + wz^q$ for $u, v, w \in k[x, y, z]_{(q-1)/2}$, then $f|u, v, w$. As it suffices to verify the above statement in the affine part ($z = 1$) of \mathbb{P}_k^2 , the result in Theorem 6 finishes the proof. ■

5. CAYLEY'S CUBIC SURFACE

Let $S = k[x, y, z, w]$ be the polynomial ring in four variables over an arbitrary field k and let $f = xyz + xyw + xzw + yzw$ be the Cayley cubic. We consider the generalized Hilbert-Kunz function of $R = S/f$, given at $q \in \mathbb{N}$ through

$$HK_{R,(x,y,z,w)}(q) = \dim_k S/(f, x^q, y^q, z^q, w^q).$$

THEOREM 7. *The socle degree of the artinian ring $\theta = S/(f, x^q, y^q, z^q, w^q)$ is*

$$a(q) = \begin{cases} 0 & \text{if } q = 1, \\ 2q - 1 & \text{if } q > 1, \end{cases}$$

and the value of the generalized Hilbert-Kunz function of Cayley's cubic at $q \in \mathbb{N}$ is

$$HK_{R,(x,y,z,w)}(q) = 2q^3 - q.$$

Proof. If $q = 1$, then $\theta \cong k$ and $a(1) = 0$. Now assume $q > 1$. Since $d = 3 < 4(q - 1)$, Theorem 2 yields the lower bound $a(q) \geq 2q - 1$. Thus it remains to show $\theta_{2q} = 0$, i.e., that any monomial $x^i y^j z^k w^l \in \theta_{2q}$ is equivalent to 0.

Case 1. $\max\{i, j, k, l\} + \min\{i, j, k, l\} \geq q$.

We argue by descending induction on $\max\{i, j, k, l\}$. Due to symmetry, we may assume $i \geq j \geq k \geq l$. If $i \geq q$, we are done; and as $i + l \geq q$, we may assume $l \neq 0$. Then in θ_{2q} ,

$$x^i y^j z^k w^l = -x^{i+1}(y^{j-1} z^k w^l + y^j z^{k-1} w^l + y^j z^k w^{l-1}),$$

and the induction applies.

Case 2. $\max\{i, j, k, l\} + \min\{i, j, k, l\} < q$.

We argue by descending induction on $\min\{i, j, k, l\}$. Again, we may assume $i \geq j \geq k \geq l$. Suppose $k = l$ or $l + 1$. Since $i + l \leq q - 1$, $j + k \leq i + k \leq q$. So $i + j + k + l \leq 2q - 1$, a contradiction. We conclude that i, j , and k are all at least $l + 2$. Then in θ_{2q} ,

$$x^i y^j z^k w^l = -(x^{i-1} y^j z^k + x^i y^{j-1} z^k + x^i y^j z^{k-1}) w^{l+1},$$

and induction applies.

The claimed result for the generalized Hilbert–Kunz function follows now from Theorem 2, as $d = 3 < 4(q - 1)$ whenever $q > 1$, and its validity for $q = 0, 1$ is clear. ■

COROLLARY 3. *For any field k of prime characteristic p and any integer $d \geq 2$, there is a surface $X \subset \mathbb{P}_k^3$ of degree d whose Hilbert–Kunz multiplicity achieves $\frac{2}{3}d$, the minimum possible for such surfaces.*

Proof. The Hilbert–Kunz multiplicity of the quadric surface $g = xy - zw$ equals $\frac{4}{3}$ by [1]. As just established, the Hilbert–Kunz multiplicity of the Cayley cubic f is equal to 2. Since $d = 2u + 3v$ for some $u, v \in \mathbb{N}$, additivity of the Hilbert–Kunz multiplicity implies that the surface defined by $g^u f^v$ has Hilbert–Kunz multiplicity equal to $\frac{2}{3}d$. ■

Remark 5. For any field k of positive characteristic, by virtue of the above corollary and semi-continuity, a generic surface in \mathbb{P}_k^3 of degree $d \geq 2$ achieves the minimal Hilbert–Kunz multiplicity $\frac{2}{3}d$. Also, $c = 1$ if $d = 1$. So the Hilbert–Kunz multiplicity of a generic surface is rational and independent of the characteristic. Note that the Hilbert–Kunz multiplicity of Cayley’s cubic is minimal although this surface is singular—in contrast to the case of cubic curves.

Remark 6. As pointed out at the end of the Introduction, in higher dimensions—at least in low degrees—the generic Hilbert–Kunz multiplicity will depend upon the characteristic.

ACKNOWLEDGMENTS

The authors thank P. Monsky, whose influence on this paper should be clear to every reader. We are especially grateful for his permission to include the direct proof of Geronimus' theorem and for modifications that led to shorter proofs. We also thank K. Pardue, who got us interested in the subject, and A. Conca for useful conversations.

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