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The general type N solution of new massive gravity

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ABSTRACT

We find the most general algebraic type N solution with non-vanishing scalar curvature, which comprises all type N solutions of new massive gravity in three dimensions. We also give the special forms of this solution, which correspond to certain critical values of the topological mass. Finally, we show that at the special limit, the null Killing isometry of the spacetime is restored and the solution describes AdS pp-waves.

1. Introduction

Recently, Bergshoeff, Hohm and Townsend formulated a new dynamical theory of massive gravity in three dimensions [1]. The theory is now called *new massive gravity* (NMG) and is described by the Einstein–Hilbert (EH) action complemented with a particular higher-derivative correction term. The latter provides propagating degrees of freedom in the theory, thereby curing dynamically barren property of general relativity in three dimensions [2]. In contrast to topologically massive gravity (TMG), formulated a long ago by Deser, Jackiw and Templeton [3,4], the theory of NMG preserves parity. The associated wave equation contains fourth-order derivatives of metric perturbations, representing a physical massive graviton with two polarization states. In TMG, the addition of the higher-derivative Chern–Simons term to the EH action violates parity, but it also makes the theory dynamical with a single propagating massive mode. On the other hand, there exist some similarities between TMG and NMG theories. First of all, the linearized behavior of the latter in the Minkowski background is similar to that of TMG, resulting in a unitary theory of two propagating massive graviton modes as long as one uses the reverse sign for the EH term in the action [1,5]. Attempts to extend the NMG theory to all higher dimensions have revealed that only the three-dimensional model is unitary in the tree level [6,7].

It is unfortunate that unitarity of both TMG and NMG theories requires to reverse the usual sign of the EH term in the total action. For instance, due to this peculiarity the mass of the BTZ black holes [8] becomes negative that in turn makes unsatisfactory the quantum description in context of the AdS/CFT correspondence. In the case of TMG, significant progress on this route was achieved recently in [9]. It was shown that at a "chiral" point, determined by a certain critical value of the topological mass, the bulk gravitons disappear and the BTZ black holes have nonnegative masses. That is, one obtains a unitary chiral quantum theory of gravity with the usual sign of the EH term, which is a dual of two-dimensional conformal field theory (CFT$_2$) on the boundary. This remarkable result has renewed the interest in TMG with the hope of finding other stable vacua for a consistent formulation of quantum gravity in three dimensions [10–13]. In particular, the authors of [12] performed the Petrov–Segre type algebraic classification of exact solutions to TMG, showing that almost all existing in the literature homogeneous space solutions locally reduce to either type D, biaxially squashed AdS$_3$ solutions [14–16] or type N, AdS pp-waves solutions [10,17–20]. New Kundt type solutions of TMG were found in [13].

A similar wave of activity has also appeared in NMG [21–26]. It was found that NMG admits AdS$_3$ (BTZ), warped AdS$_3$ black holes [1,21] as well as AdS-wave solutions [22]. A new class of asymptotically AdS$_3$ black hole solutions to NMG with special values of the cosmological term has been discussed in [23,24]. Some special cases of Kundt spacetimes and homogeneous space solutions to NMG were also considered in [27,28], respectively. In the quantum context, contrary to the case of TMG, the reconciliation of incompatibility of bulk/boundary theories, in the sense of their unitarity, still remains unsatisfactory, though a number of critical relations between the AdS$_3$ radius and the mass scale do exist as well [23,29–31]. This fact motivates one to look for all possible

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exact solutions of NMG that could provide stable vacua for satisfactory quantum aspects of the theory.

In a recent paper [32], we began an exhaustive programme for studying exact solutions to NMG. In particular, we found a simple framework that provided mapping all known Petrov–Segre types D and N exact solutions of TMG into NMG. Meanwhile, it should be emphasized that TMG is a strongly constrained theory as it does not admit static solutions besides "trivial" Einstein solutions [33,34]. However, NMG is a much richer theory that admits solutions which are absent in TMG [23,32].

In this Letter, we continue the programme of [32] and find the most general solution that describes all algebraic type N spacetimes of NMG. In Section 2 we briefly recall the field equations of cosmological NMG in terms of a first-order differential operator (resembling a Dirac type operator) acting on the traceless Ricci tensor. In Section 3 we introduce a triad basis consisting of two null and one spacelike vectors and show that for type N geometries, the covariant derivatives of these vectors are determined only by three scalar functions. In Section 4 we derive the general form of the spacetime metric with a single unknown function of two variables, which obeys a second-order linear differential equation. In Section 5 we obtain the most general algebraic type N solution and discuss its some special limits of interest.

2. The field equations

The field equations of NMG were obtained in [1] by varying the action

$$S = \frac{1}{16\pi G} \int d^3x \sqrt{-g} \left(R - 2\lambda - \frac{1}{m^2} K \right), \tag{1}$$

with respect to the spacetime metric. Here $R = g^{\mu\nu} R_{\mu\nu}$ is the Ricci scalar, λ is a cosmological term, m is a mass parameter and

$$K = R_{\mu\nu} R^{\mu\nu} - \frac{3}{8} R^2. \tag{2}$$

In a recent paper [32], it was shown that if one defines a first-order differential operator as

$$\displaystyle{\not{D}} \Phi_{\mu\nu} = \frac{1}{2} \left(\epsilon_\mu{}^{\alpha\beta} \nabla_\beta \Phi_{\nu\alpha} + \epsilon_\nu{}^{\alpha\beta} \nabla_\beta \Phi_{\mu\alpha} \right), \tag{3}$$

where $\Phi_{\mu\nu}$ is a symmetric tensor, then its action on the traceless Ricci tensor

$$S_{\mu\nu} = R_{\mu\nu} - \frac{1}{3} g_{\mu\nu} R, \tag{4}$$

yields

$$\displaystyle{\not{D}} S_{\mu\nu} = -C_{\mu\nu}. \tag{5}$$

Here $C_{\mu\nu}$ is the symmetric, traceless and covariantly constant Cotton tensor defined as

$$C_{\mu\nu} = \epsilon_\mu{}^{\alpha\beta} \nabla_\alpha \left(R_{\nu\beta} - \frac{1}{4} g_{\nu\beta} R \right). \tag{6}$$

With the quantities $\displaystyle{\not{D}}$ and $S_{\mu\nu}$, the field equations of NMG with a cosmological term can be put in the form of the massive Klein–Gordon type equation with curvature-squared source term. Thus, one obtains the field equations of NMG in the form

$$(\displaystyle{\not{D}}^2 - m^2) S_{\mu\nu} = T_{\mu\nu}, \tag{7}$$

where the traceless source term is given by

$$T_{\mu\nu} = S_{\mu\rho} S_\nu^\rho - \frac{R}{12} S_{\mu\nu} - \frac{1}{3} g_{\mu\nu} S_{\alpha\beta} S^{\alpha\beta}. \tag{8}$$

This equation is also accompanied by the equation

$$S_{\mu\nu} S^{\mu\nu} + m^2 R - \frac{R^2}{24} = 6m^2 \lambda, \tag{9}$$

which is an analogue of the trace equation in [1]. For a source tensor given by the relation

$$T_{\mu\nu} = \kappa S_{\mu\nu}, \tag{10}$$

where κ is a function of the scalar curvature, which is fulfilled for algebraic types D and N spacetimes, instead of Eq. (7), we have

$$\displaystyle{\not{D}}^2 S_{\mu\nu} = \mu^2 S_{\mu\nu}, \tag{11}$$

with

$$\mu^2 = m^2 + \kappa. \tag{12}$$

We recall that in this description the field equations of cosmological TMG acquire the Dirac type form

$$\displaystyle{\not{D}} S_{\mu\nu} = \mu S_{\mu\nu}. \tag{13}$$

Here μ is the mass parameter of TMG. Further details of this description of NMG can be found in [32]. We note that in (6) and in what follows, the Levi-Civita tensor $\epsilon_{\mu\alpha\beta}$ is given by the relation $\epsilon_{\mu\alpha\beta} = \sqrt{-g}\, \varepsilon_{\mu\alpha\beta}$ and we use the convention $\varepsilon_{012} = 1$.

For algebraic types D and N spacetimes such a description turns out to be very powerful for finding exact solutions to NMG. This has been demonstrated in [32] by mapping all known algebraic types D and N exact solutions of TMG into NMG as well as presenting new examples of such solutions, which are only inherent in NMG. In the following, we further demonstrate the advantages of this description and find the most general type N solution to NMG.

3. The type N geometries

We begin with introducing a triad basis of real vectors, $\{l_\mu, n_\mu, m_\mu\}$ such that

$$l_\mu n^\mu = 1, \qquad m_\mu m^\mu = 1 \tag{14}$$

with all other contractions vanishing identically. That is, we have two null vectors l_μ and n_μ and one spacelike unit vector m_μ. The latter vector with

$$m_\mu = \epsilon_{\mu\nu\sigma} l^\nu n^\sigma \tag{15}$$

provides an orientation of the three-dimensional manifold. Clearly, we also have

$$l_\mu = \epsilon_{\mu\nu\sigma} m^\nu l^\sigma, \qquad n_\mu = \epsilon_{\mu\nu\sigma} n^\nu m^\sigma. \tag{16}$$

The spacetime metric written in terms of the basis vectors has the form

$$g_{\mu\nu} = 2l_{(\mu} n_{\nu)} + m_\mu m_\nu. \tag{17}$$

We now recall that for type N spacetimes, in the Petrov–Segre classification of three-dimensional spacetimes, the canonical form of the traceless Ricci tensor is given by

$$S_{\mu\nu} = l_\mu l_\nu, \tag{18}$$

where l_μ is a null vector (see for instance, [12,35,36]). With this expression, Eq. (9) yields

$$\lambda = -\nu^2 \left(1 + \frac{\nu^2}{4m^2} \right). \tag{19}$$

Here, for a convenience in the following, we have introduced $v^2 = -R/6$, which defines the case of a negative scalar curvature. Equivalently, one can also define it as $v = \sqrt{-\Lambda}$, where Λ is the usual cosmological constant. Meanwhile, using Eq. (10) we have

$$\mu^2 = m^2 + \frac{v^2}{2}, \tag{20}$$

instead of (12).

Next, from the contracted Bianchi identity, $S_{\mu\nu}{}^{;\nu} = 0$, where the semicolon stands for covariant differentiation, we find the relation

$$l_\nu{}^{;\nu} l_\mu + l^\nu l_{\mu;\nu} = 0. \tag{21}$$

This enables one to establish the general representation for the covariant derivative of the vector l_μ in the form

$$l_{\mu;\nu} = \alpha l_\mu l_\nu + \beta l_\mu m_\nu + \gamma (2m_\mu m_\nu - l_\mu n_\nu) + \sigma m_\mu l_\nu, \tag{22}$$

where the coefficients of the expansion are functions of spacetime and $\gamma = l_\mu{}^{;\mu}$. Using this expression, with Eqs. (15) and (16) in mind, one can easily calculate the action of the operator $\not{D}$ on the traceless Ricci tensor $S_{\mu\nu}$. We find that

$$\not{D} S_{\mu\nu} = (\sigma - 2\beta) l_\mu l_\nu - 4\gamma l_{(\mu} m_{\nu)}. \tag{23}$$

Comparing this equation with that given in (13), we see that for all type N solutions of TMG the function γ must be zero that, along with (21), yields

$$l_\mu{}^{;\mu} = 0, \qquad l^\nu l_{\mu;\nu} = 0. \tag{24}$$

That is, the null vector l_μ forms a congruence of expansion-free null geodesics. In other words, all algebraic type N solutions of TMG are Kundt spacetimes [13].

Next, we shall show that the above statement remains true for the case of NMG as well. For this purpose, it is convenient to begin by assuming that $\gamma \neq 0$. Then, the use of the Lorentz transformation of the basis vectors

$$l_\mu \rightarrow l_\mu, \qquad m_\mu \rightarrow m_\mu - f l_\mu, \qquad n_\mu \rightarrow n_\mu + f m_\mu - \frac{1}{2} f^2 l_\mu, \tag{25}$$

where f is a real function, enables one to set $\sigma = 2\beta$ in the new basis. As a consequence, we have

$$l_{\mu;\nu} = \alpha l_\mu l_\nu + \beta (l_\mu m_\nu + 2m_\mu l_\nu) + \gamma (2m_\mu m_\nu - l_\mu n_\nu). \tag{26}$$

The use of this expression, along with the orthogonality condition $l_\mu m^\mu = 0$, yields the equation

$$m_{\mu;\nu} = -2\beta n_\mu l_\nu - 2\gamma n_\mu m_\nu + l_\mu x_\nu, \tag{27}$$

where x_μ is a real vector. It is also straightforward to show that

$$\not{D} S_{\mu\nu} = -4\gamma l_{(\mu} m_{\nu)}. \tag{28}$$

With Eqs. (26), (27) and (28) we are able to calculate the explicit form of $\not{D}^2 S_{\mu\nu}$, which turns out to involve the term proportional to $(l^\mu \gamma_{;\mu} - \gamma^2)(g_{\mu\nu} - 3m_\mu m_\nu)$. Clearly, this term must vanish, as a consequence of the field equations in (11). Thus, we arrive at the equation

$$l^\mu \gamma_{;\mu} - \gamma^2 = 0. \tag{29}$$

We now use the fact that for any vector x_μ,

$$x_{\nu;\mu}{}^{;\nu} - (x_\nu{}^{;\nu})_{;\mu} = R_{\mu\nu} x^\nu = \left(S_{\mu\nu} + \frac{R}{3} g_{\mu\nu}\right) x^\nu, \tag{30}$$

where we have used Eq. (4). Replacing here x_ν by l_ν and then contracting the result with l^μ, we obtain that

$$l^\mu \gamma_{;\mu} + 3\gamma^2 = 0. \tag{31}$$

Combining now Eqs. (29) and (31), we see that $\gamma = 0$, that contradicts with our initial assumption $\gamma \neq 0$, i.e. we again arrive at equations in (24). Thus, *all algebraic type N solutions of NMG are Kundt spacetimes.*

Next, it is important to establish for these spacetimes the most suitable representation of the covariant derivatives of the basis vectors. We first note that

$$l^\mu \partial_\mu \beta = 0, \qquad l^\mu \partial_\mu \sigma = 0, \tag{32}$$

where, with (24) in mind, the first equation is obtained when using expression (22) in (30) for $x_\nu = l_\nu$ and contracting the result with the basis vector m^μ, whereas the second equation follows from the vanishing divergence of Eq. (23). Equations in (32) allow us to reduce Eq. (22) into the form

$$l_{\mu;\nu} = \alpha l_\mu l_\nu + \beta (l_\mu m_\nu - l_\nu m_\mu) \tag{33}$$

by means of the Lorentz transformation of the triad given by $l_\mu \rightarrow k^{1/2} l_\mu$, $n_\mu \rightarrow k^{-1/2} n_\mu$, $m_\mu \rightarrow m_\mu$, provided that $l^\mu \partial_\mu k = 0$. We see that Eq. (33) involves only two functions α and β. However, the canonical form of the traceless Ricci tensor now acquires an extra function k,

$$S_{\mu\nu} = k l_\mu l_\nu, \tag{34}$$

not violating the conditions in (24). Using the fact that expression (33) remains invariant under the Lorentz transformation (25), one can choose the vector m^μ to be commuting with the vector l^μ in the sense of their Lie bracket,

$$[l, m] = 0. \tag{35}$$

This in turn allows us to specify the covariant derivative of the vector m in the form

$$m_{\mu;\nu} = \tau l_\mu l_\nu + \beta (l_\mu n_\nu + l_\nu n_\mu) + \chi l_\mu m_\nu. \tag{36}$$

With Eqs. (33) and (36) one can easily write out the covariant derivative of m^μ as follows

$$n_{\mu;\nu} = -\alpha n_\mu l_\nu - \beta (n_\mu m_\nu + m_\mu n_\nu) - \tau m_\mu l_\nu - \chi m_\mu m_\nu. \tag{37}$$

We note that the Lie bracket in (35) is preserved with respect to transformation (25), provided that the function f obeys the condition $l^\mu \partial_\mu f = 0$. On the other hand, using Eqs. (33), (36) and (37) successively in (30), after some manipulations, we also find that $l^\mu \partial_\mu \chi = 0$. With these two conditions in mind, one can discard the function χ in (36) and (37) by means of the Lorentz transformation in (25). Thus, we find that *for the type N geometries, the covariant derivatives of the basis vectors are determined only by three scalar functions.*

4. The construction of the metric

It is straightforward to show that the associated determining equations for the scalar functions α, β and τ are obtained from Eq. (30) by an appropriate using expressions (33), (34), (36) and (37) in it. As a consequence, we have the set of simple equations

$$\partial_\rho \tau = -k - 4\beta \tau, \tag{38}$$

$$\partial_v \tau = \partial_\rho \alpha + 2\alpha \beta = n^\mu \partial_\mu \beta, \tag{39}$$

$$\partial_\rho \beta = -\partial_v \alpha = v^2 - \beta^2, \tag{40}$$

$$\partial_v \beta = 0, \tag{41}$$

which are easily solved. Here we have used the definitions $l = l^\mu \partial_\mu = \partial_v$ and $m = m^\mu \partial_\mu = \partial_\rho$. From Eq. (40), we immediately see that for $v^2 = \beta^2$, the function α is constant along the null vector l. Thus, the null vector determines a null Killing vector that can be seen from (33), discarding the function α by a Lorentz transformation. This case corresponds to a general AdS pp-waves solution [22] (see also below, Section 5).

For $v^2 \neq \beta^2$, the most general solutions to the above set of equations are given by

$$\alpha = (v^2 - \beta^2)[-v + \beta b(u) + c(u)], \qquad (42)$$

$$\tau = (v^2 - \beta^2)^2[vb(u) + g(u, \rho)], \qquad (43)$$

where $b(u)$ and $c(u)$ are arbitrary functions of the coordinate u, the function $g(u, \rho)$ obeys the equation

$$\partial_\rho g + \frac{k}{(v^2 - \beta^2)^2} = 0, \qquad (44)$$

and β is determined by Eqs. (40) and (41), which admit the following solutions

$$\beta = v \tanh(v\rho), \qquad (45)$$

and

$$\beta = v \coth(v\rho). \qquad (46)$$

Here we have used a coordinate transformation to drop a redundant function of u.

Next, using the Lie brackets

$$[n, l] = \alpha l, \qquad [n, m] = 2\beta n + \tau l, \qquad (47)$$

established by means of Eqs. (33), (36) and (37) as well as Eq. (39), we find the following representation for the null vector

$$n = n^\mu \partial_\mu = (v^2 - \beta^2)[A\partial_v + b(u)\partial_\rho + \partial_u], \qquad (48)$$

where A is given by

$$A = \frac{1}{2}[v^2 - a(u, \rho)] - v[\beta b(u) + c(u)], \qquad (49)$$

and the function $a(u, \rho)$ is determined by the equation

$$\partial_\rho a = 2(v^2 - \beta^2)g. \qquad (50)$$

Taking once again the derivative of this equation with respect to ρ and combining the result with Eqs. (38) and (44), we obtain the second-order linear inhomogeneous differential equation for $a(u, \rho)$

$$\partial_\rho^2 a + 2\beta \partial_\rho a = -\frac{2k}{v^2 - \beta^2}. \qquad (51)$$

The associated dual 1-forms

$$l_\mu \, dx^\mu = \frac{1}{v^2 - \beta^2} \, du, \qquad n_\mu \, dx^\mu = dv - A \, du,$$

$$m_\mu \, dx^\mu = d\rho - b(u) \, du, \qquad (52)$$

define the metric

$$ds^2 = \frac{2}{v^2 - \beta^2} \, du(dv - A \, du) + [d\rho - b(u) \, du]^2, \qquad (53)$$

that by means of the coordinate transformation

$$v \rightarrow v + \beta b(u) + c(u), \qquad (54)$$

can also be put in the form

$$ds^2 = d\rho^2 + \frac{2}{v^2 - \beta^2} \, du \, dv + \frac{1}{v^2 - \beta^2}[a(u, \rho) - v^2] \, du^2. \qquad (55)$$

In obtaining this expression, we have used the invariance of Eqs. (44) and (50) with respect to the transformation $a \rightarrow a + \beta g_1(u) + g_2(u)$, that removes two redundant functions in the metric. Thus, the most general metric for algebraic type N geometries is characterized by a single unknown function $a(u, \rho)$ governed by Eq. (51). Clearly, this metric does not admit a null Killing vector field.

The remaining step is to find the explicit form of this function. In doing so, as it follows from Eqs. (51), we first need to know the function k. For this purpose, let us calculate the action of the operator $\not{\partial}$ on the tensor in (34). We find that

$$\not{\partial} S_{\mu\nu} = -z S_{\mu\nu}, \qquad (56)$$

where

$$z = 3\beta + \partial_\rho \ln k. \qquad (57)$$

Comparing this equation with that in (13), we see that for $z = \pm\mu$, we arrive at TMG theory and the corresponding type N solutions recover those found in [13]. These solutions of TMG can be mapped into NMG using the prescription given in [32].

It is easy to see that the action of the operator $\not{\partial}$ on Eq. (56) yields

$$\not{\partial}^2 S_{\mu\nu} = (\partial_\rho z + z^2) S_{\mu\nu}, \qquad (58)$$

that, with the field equations of NMG given in (11), leads to the equation

$$\partial_\rho z = \mu^2 - z^2. \qquad (59)$$

The nontrivial solutions of this equation are given by

$$z = \mu \tanh[\mu\rho + h(u)], \qquad (60)$$

and

$$z = \mu \coth[\mu\rho + h(u)], \qquad (61)$$

where we keep an arbitrary function $h(u)$ as one cannot gauge out it simultaneously with that entering into the solutions for β (see Eqs. (45) and (46)).

Next, combining Eqs. (57) and (59) and taking into account Eq. (40), we find that

$$k = -\frac{1}{2}(v^2 - \beta^2)\left(\frac{v^2 - \beta^2}{\mu^2 - z^2}\right)^{1/2} F(u), \qquad (62)$$

where $F(u)$ is an arbitrary function.

5. The general solution

Substituting now the expression of k given above into Eq. (51), making use of solutions (45), (46) and (60), (61), we can solve it for the metric function $a(u, \rho)$.

We first consider the generic case $\mu^2 \neq v^2$ and begin with $\beta = v \tanh(v\rho)$. Then, for the function z given in Eqs. (60) and (61), the solutions of Eq. (51) are respectively given by

$$a = \frac{\cosh[\mu\rho + h(u)]}{\cosh(v\rho)} f(u) + \tanh(v\rho) f_1(u) + f_2(u), \qquad (63)$$

and

$$a = \frac{\sinh[\mu\rho + h(u)]}{\cosh(v\rho)} f(u) + \tanh(v\rho) f_1(u) + f_2(u). \qquad (64)$$

Since $h(u)$ is an arbitrary function, these two solutions can be "glued" together to give the single solution

$$a = \frac{1}{\cosh(\nu\rho)} \big[\cosh(\mu\rho)F_1(u) + \sinh(\mu\rho)F_2(u)$$
$$+ \cosh(\nu\rho)f_1(u) + \sinh(\nu\rho)f_2(u)\big], \tag{65}$$

which involves four arbitrary functions of u. We recall that this solution corresponds to the case of negative scalar curvature. However, making an analytical continuation $\nu \to i\nu$ or taking the limit $\nu \to 0$, we obtain the solutions corresponding to the positive or zero values of the scalar curvature, respectively. It is also important to note that, as it follows from Eq. (20), the quantity μ^2 can take on both negative and zero values. The associated solutions are also recovered by (65) when performing an analytical continuation $\mu \to i\mu$ or taking the limit $\mu \to 0$.

Similarly, for $\beta = \nu \coth(\nu\rho)$ and $\mu^2 \neq \nu^2$, we have the solution

$$a = \frac{1}{\sinh(\nu\rho)} \big[\cosh(\mu\rho)F_1(u) + \sinh(\mu\rho)F_2(u)$$
$$+ \cosh(\nu\rho)f_1(u) + \sinh(\nu\rho)f_2(u)\big]. \tag{66}$$

As in the previous case, one can make appropriate analytical continuations to include the case of positive or zero scalar curvature as well as the case of negative or zero μ^2.

It is interesting to note that the most general solution, comprising all these solutions, is given by

$$ds^2 = d\rho^2 + \frac{2\,du\,dv}{\nu^2 - \beta^2} + \left[Z(u,\rho) - \frac{\nu^2}{\nu^2 - \beta^2}\right]du^2, \tag{67}$$

where the metric function has the form

$$Z(u,\rho) = \frac{1}{\sqrt{\nu^2 - \beta^2}} \big[\cosh(\mu\rho)F_1(u) + \sinh(\mu\rho)F_2(u)$$
$$+ \cosh(\nu\rho)f_1(u) + \sinh(\nu\rho)f_2(u)\big] \tag{68}$$

and β is given as in either (45) or (46). We recall that ν is related to the cosmological term λ as given in (19) and the general metric does not admit a null Killing vector field. For $F_2(u) = f_1(u) = f_2(u) = 0$ and β in (45), this solution reduces to that obtained in [32]. We see that the general solution involves two extra functions $f_1(u)$ and $f_2(u)$. In fact, one of these functions can be discarded by means of coordinate transformations. In order to show this, it is convenient to write Eq. (68) in the following alternative form

$$Z(u,\rho) = \frac{\cosh(\mu\rho)F_1(u) + \sinh(\mu\rho)F_2(u)}{\sqrt{\nu^2 - \beta^2}}$$
$$+ \frac{h_1(u) + \beta h_2(u)}{\nu^2 - \beta^2}. \tag{69}$$

Here $h_1(u) \to f_1(u)$ and $h_2(u) \to f_2(u)$ for β given in (45), whereas $h_1(u) \to f_2(u)$ and $h_2(u) \to f_1(u)$ for β given in (46). Passing now to the new coordinates $v \to vG(u) + H(u)$, such that $\partial_u G = G(u)H(u)$, and $du \to du/G(u)$, it is straightforward to see that one can eliminate $h_1(u)$ by choosing $G(u)$ and $H(u)$. As a result, we have

$$Z(u,\rho) = \frac{\cosh(\mu\rho)F_1(u) + \sinh(\mu\rho)F_2(u)}{\sqrt{\nu^2 - \beta^2}}$$
$$+ \sinh(2\nu\rho)F_3(u). \tag{70}$$

Thus, the most general solution is characterized by three arbitrary functions. In the following, for some future purposes, we keep both functions $f_1(u)$ and $f_2(u)$. We proceed with the special forms of (67), which are of interest as well.

(i) $\mu^2 = \nu^2$ (or, as it follows from (20), $m^2 = \nu^2/2$). In this case, taking properly the limit of (68) or equivalently solving Eq. (51), with the explicit forms of β and z given above, we find that

$$Z(u,\rho) = \frac{1}{\sqrt{\nu^2 - \beta^2}} \big\{\cosh(\mu\rho)\big[\rho F_1(u) + f_1(u)\big]$$
$$+ \sinh(\mu\rho)\big[\rho F_2(u) + f_2(u)\big]\big\}. \tag{71}$$

(ii) $\mu^2 = 0$ (or, as it follows from (20), $m^2 = -\nu^2/2$). Then, from Eq. (68) it immediately follows that

$$Z(u,\rho) = \frac{1}{\sqrt{\nu^2 - \beta^2}} \big[F_1(u) + \rho F_2(u)$$
$$+ \cosh(\nu\rho)f_1(u) + \sinh(\nu\rho)f_2(u)\big]. \tag{72}$$

Again, one can discard the redundant function $f_1(u)$ in Eqs. (71) and (72) by means of coordinate transformations.

(iii) $\beta^2 \to \nu^2$. In this case, making the coordinate transformation $\nu \to (\nu^2 - \beta^2)\nu$, one can put the metric in (67) in the form

$$ds^2 = d\rho^2 + 2\,du\,dv - 4\beta\nu\,d\rho\,du$$
$$+ \big[Z(u,\rho) - (\nu^2 - \beta^2)\nu^2\big]du^2. \tag{73}$$

We see that in the limit $\beta^2 \to \nu^2$, the term proportional to ν^2 disappears. That is, the Killing isometry of the spacetime is restored and the vector ∂_ν becomes a null Killing vector. This in turn means that the resulting metric must describe AdS pp-waves. Indeed, redefining once again the coordinate v as $v \to e^{2\nu\rho}v$, we arrive at the metric

$$ds^2 = d\rho^2 + 2e^{2\nu\rho}\,du\,dv + \big[e^{\nu\rho}\cosh(\mu\rho)F_1(u)$$
$$+ e^{\nu\rho}\sinh(\mu\rho)F_2(u) + e^{2\nu\rho}f_1(u) + f_2(u)\big]du^2, \tag{74}$$

where, in the contrary with the general case (67), both functions $f_1(u)$ and $f_2(u)$ can be gauged out by means of coordinate transformations. This metric is nothing but the AdS pp-waves solution of NMG that was earlier found in [22].

(iv) Finally, for $F_1(u) = \pm F_2(u) = f(u)$, the solution in (67) recovers that obtained from the TMG case (see Refs. [13,32]). The same remains true for solution (71) as well.

Thus, we have the most general solution given in (67), which comprises all algebraic type N spacetimes of NMG.

6. Conclusion

A novel description of NMG in three dimensions, given in our previous work [32], turns out to be a very powerful tool for finding exact solutions to the theory. In this Letter, we have further demonstrated the advantages of this formalism. Describing three-dimensional algebraic type N spacetimes in terms of a triad basis (with two null vectors and one spacelike vector), we have shown that all such solutions of NMG are Kundt spacetimes. Using this property along with freedoms, provided by the Lorentz symmetries underlying the system, we have found that the covariant derivatives of the basis vectors are in general determined only by three scalar functions, which obey a chain of simple equations.

For algebraic type N geometries, we have obtained the general form of the spacetime metric with a single unknown function of two variables. Remarkably, this function is governed by a second-order linear inhomogeneous differential equation. Finally, solving the linear differential equation, we have found the most general algebraic type N solution with non-vanishing scalar curvature, which comprises all type N solutions to the theory of NMG. We have also considered some special cases of interest.

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