Exact Controllability for Semilinear Wave Equations

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In this note, we prove the exact controllability for the semilinear wave equations in any space dimensions under the condition that the nonlinearity behaves like \( o(|\sqrt{\ln s}|) \) as \( s \to \infty \).

1 INTRODUCTION

In this note, we shall consider the exact controllability of the following semilinear wave equation,

\[
\begin{cases}
  y'' - \Delta y = f(y) + \chi_\omega(x)u(t, x) & \text{in } Q, \\
  y = 0 & \text{on } \Sigma, \\
  y(0) = y_0, \quad y'(0) = y_1 & \text{in } \Omega,
\end{cases}
\]  

(1.1)

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where $Q \triangleq (0, T) \times \Omega$, $\Sigma \triangleq (0, T) \times \Gamma$, $T > 0$, $\Omega \subset \mathbb{R}^n$ is a bounded domain with a $C^{1,1}$ boundary $\Gamma \triangleq \partial \Omega$, $\omega$ is a subdomain of $\Omega$, and $\chi_\omega$ denotes the characteristic function of the set $\omega$. In (1.1), $y(t, \cdot)$ is the state, $u(t, \cdot)$ is the control.

We assume the nonlinearity $f(\cdot)$ in (1.1) satisfies
\begin{equation}
\lim_{s \to \infty} \frac{f(s)}{\sqrt{\ln|s|}} = 0.
\end{equation}

The exact controllability problem of (1.1) can be formulated as follows: for any given $(y_0, y_1), (z_0, z_1) \in H^1_0(\Omega) \times L^2(\Omega)$, find (if possible) a control $u \in L^2((0, T) \times \omega)$ such that the weak solution $y(\cdot) \in C([0, T]; H^1_h(\Omega)) \cap C^1([0, T]; L^2(\Omega))$ of (1.1) satisfies
\begin{equation}
y(T) = z_0, \quad y'(T) = z_1.
\end{equation}

The study of exact (boundary and/or internal) controllability for the linear wave equation seems to be complete. Extensive related references can be found in [1–5, 7], and the rich works cited therein.

For the semilinear case, the situation is not so satisfactory. In this case, if the nonlinearity $f(\cdot)$ is globally Lipschitz continuous, one can find Zuazua’s controllability result [9] and its generalization by the author [8]; if the nonlinearity $f(\cdot)$ is allowed to grow superlinearly at infinity, to our best knowledge, there are only a few works [2, 10] giving positive and/or negative controllability results for (1.1) in one space dimension (i.e., the case $n = 1$).

In this note, we shall give a positive controllability result for (1.1) in any space dimension. Our main result (see Section 3) depends on a new explicit observability estimate for the wave equation with a potential in the $L^p$-classes (see Section 2).

2. EXPLICIT OBSERVABILITY ESTIMATE FOR THE LINEARIZED EQUATION

For any $S \in \mathbb{R}^n$ and $\varepsilon > 0$, we put $\mathcal{S}_\varepsilon(S) \triangleq \{ y \in \mathbb{R}^n | |y - x| < \varepsilon \text{ for some } x \in S \}$. Let us consider the equation
\begin{equation}
\begin{aligned}
&w'' - \Delta w = V(t, x)w & \quad & \text{in } Q, \\
&w = 0 & \quad & \text{on } \Sigma, \\
&w(0) = w_0, \quad w'(0) = w_1 & \quad & \text{in } \Omega.
\end{aligned}
\end{equation}

In (2.1), $V(\cdot)$ is a potential.
The following theorem will play a fundamental role in the sequel.

**Theorem 2.1.** Let \( \omega = \Omega \cap \mathcal{E}_c(\Gamma) \) for some \( \varepsilon_0 > 0 \) and \( T > T_0 \triangleq \text{diam}(\Omega \setminus \omega) \). Let \( V(\cdot) \in L^{1+\eta}(Q) \) (or \( V(\cdot) \in L^n(0, T; L^n(\Omega)) \)). Then the weak solution \( w(\cdot) \in C([0, T]; L^2(\Omega)) \cap C^1([0, T]; H^{-1}(\Omega)) \) of (2.1) satisfies

\[
|w_0|^2_{L^2(\Omega)} + |w_1|^2_{H^{-1}(\Omega)} \leq L(\ell) \int_0^T \int_\Omega |w|^2 \, dx \, dt,
\]

for some constant \( L = L(\ell) \) with \( \ell \triangleq |V(\cdot)|_{L^{1+\eta}(Q)} \) (or \( \ell \triangleq |V(\cdot)|_{L^n(0, T; L^n(\Omega))} \)). Furthermore, the constant \( L(\ell) \) has the explicit estimate

\[
L(\ell) = O(\exp(C\ell^2)) \quad \text{as } \ell \to \infty
\]

for some positive constant \( C = C(T, \Omega) \), independent of \( \ell \) and \( (w_0, w_1) \).

In order to give a proof of Theorem 2.1, let us introduce some notations. Fix any \( \eta \in (0, 1) \) and \( \mu > 0 \) and denote

\[
\begin{align*}
\varphi &= \varphi(t, x) \triangleq \eta^2 t^2 - |x|^2, \\
D_\mu &= \{ (t, x) \in \mathbb{R}^{1+n} \mid \varphi(x, t) > \mu \}.
\end{align*}
\]

We need the following known result (using our notations).

**Lemma 2.2 (Ruiz [6]).** Let \( K \) be a compact subset of \( D_\mu \). Then there is a \( \lambda_0 > 0 \) and a constant \( C = C(K, \mu) \) such that

\[
\lambda |e^{2\lambda \varphi} v|^2_{L^2(K)} \leq C |e^{2\lambda \varphi} (v'' - \Delta v)|_{H^{-1}(K)}, \quad \forall \lambda > \lambda_0 \text{ and } v \in C_0^\infty(K).
\]

Further, we need the following lemma, which is a simple generalization of a known result (see, for example, [8]).

**Lemma 2.3.** Let \( 0 \leq S_1 < S_2 < T_2 < T_1 \leq T \) and \( V(\cdot) \in L^{1+\eta}(Q) \) (or \( V(\cdot) \in L^n(0, T; L^n(\Omega)) \)) be given. Then there is a constant \( C > 0 \) such that (recall \( \ell \triangleq |V(\cdot)|_{L^{1+\eta}(Q)} \) (or \( \ell \triangleq |V(\cdot)|_{L^n(0, T; L^n(\Omega))} \))

\[
\int_{S_2}^{T_2} |w(t, \cdot)|^2_{H^{-1}(\Omega)} \, dt \leq C (1 + \ell) \int_{S_1}^{T_1} |w(t, \cdot)|^2_{L^2(\Omega)} \, dt,
\]

where \( w(\cdot) \) is the weak solution of (2.1).

**Proof.** Denote \( \phi(t) = (t - S_1)^2(T_1 - t)^2 \). Multiplying the first equation of (2.1) by \( \phi((-\Delta)^{-1}w) \), integrating it on \( (S_1, T_1) \times \Omega \), and proceeding exactly as in [8], we can obtain the desired result. \( \blacksquare \)
Finally, using the usual energy estimate and noting the time reversibility of (2.1), one can easily obtain the following lemma.

**Lemma 2.4.** It holds (recall $\mathcal{L} \triangleq |V(\cdot)|_{L^1(\Omega)}$ (or $\mathcal{L} \triangleq |V(\cdot)|_{L^2(0,T;L^2(\Omega))}$)

$$E(t) \leq E(s) e^{2T\tau}, \quad \forall t, s \in [0, T],$$

(2.7)

where

$$E(t) \triangleq \frac{1}{2} \left( |w(t, \cdot)|^2_{H^{-1}(\Omega)} + |w(t, \cdot)|^2_{L^2(\Omega)} \right)$$

(2.8)

with $w(\cdot)$ being the weak solution of (2.1).

Now, we can give a proof of Theorem 2.1.

**Proof of Theorem 2.1.** For simplicity, we assume that $0 \in \Omega$. We note that one can extend (2.5) to functions $v \in L^2_0(K) = H^1_0(H)$ such that $v'' - \Delta v \in H^{-1}(K)$. The proof is split into several steps.

**Step 1.** First of all, recall $T > T_0 \triangleq \text{diam}(\Omega \setminus \omega)$. Thus we can find a $\eta \in (0, 1)$ (close to 1), a $\varepsilon_1 \in (0, \varepsilon_0)$ (close to $\varepsilon_0$), and a $\mu > 0$ (close to 0) such that

$$\eta^2 T^2 > \text{diam}(\Omega \setminus \omega_1)^2 + \mu,$$

(2.9)

where $\omega_1 \triangleq \Omega \cap \mathcal{C}_\varepsilon(\Gamma) \subseteq \omega$. Then it is easy to see that one can find a small $\delta > 0$ such that

$$\eta^2 t^2 > \text{diam}(\Omega \setminus \omega_1)^2 + \mu, \quad \forall t \in [T - \delta, T].$$

(2.10)

By (2.10), we see that

$$K \triangleq (T - \delta, T) \times (\Omega \setminus \omega_1),$$

(2.11)

is a compact subset of $D_{\mu}$.

Next choose a function $\xi(\cdot) \in C^\infty(\overline\Omega; [0, 1])$ such that

$$\begin{cases}
\xi(x) = 1, & x \in \Omega \setminus \omega, \\
\xi(x) = 0, & x \in \omega_1.
\end{cases}$$

(2.12)

Denote

$$v = v(t, x) \triangleq \xi(x)w(t, x), \quad (t, x) \in Q,$$

(2.13)

where $w(\cdot)$ is the weak solution of (2.1). Then one sees that $V(\cdot)$ satisfies

$$\begin{cases}
v'' - \Delta v = V(t, x)v - w \Delta \xi - 2(\nabla w) \cdot (\nabla \xi) & \text{in } Q, \\
v = 0 & \text{in } (0, T) \times \omega_1.
\end{cases}$$

(2.14)
Step 2. Let us use Lemma 2.2. By Lemma 2.2 and (2.14), we conclude that
\[ \lambda |e^{2\lambda \xi}u|_{L^2(K)}^2 \leq C |e^{2\lambda \xi} [V(t, x) v - w \Delta \xi - 2(\nabla w) \cdot (\nabla \xi)] |_{H^{-1}(K)}, \]
\[ \forall \lambda > \lambda_0. \quad (2.15) \]

Note that by the Sobolev embedding theorem, and noting (2.12), we get (recall \( \mathcal{L} \triangleq |V(\cdot)|_{L^1(\Omega)} \) (or \( \mathcal{L} \triangleq |V(\cdot)|_{L^1(\Omega)} \))
\[ |e^{2\lambda \xi} [V(t, x) v - w \Delta \xi - 2(\nabla w) \cdot (\nabla \xi)] |_{H^{-1}(K)} \]
\[ = \sup_{0 \neq f \in H^1_0(K)} \frac{\int_K e^{2\lambda \xi} [V(t, x) v - w \Delta \xi - 2(\nabla w) \cdot (\nabla \xi)] f \, dx}{|f|_{H^1_0(K)}} \]
\[ \leq \sup_{0 \neq f \in H^1_0(K)} \frac{\int_K e^{2\lambda \xi} V(t, x) v f \, dx}{|f|_{H^1_0(K)}} \]
\[ + \sup_{0 \neq f \in H^1_0(K)} \frac{\int_K e^{2\lambda \xi} [-w \Delta \xi - 2(\nabla w) \cdot (\nabla \xi)] f \, dx}{|f|_{H^1_0(K)}} \]
\[ \leq \sup_{0 \neq f \in H^1_0(K)} \frac{\int_K e^{2\lambda \xi} V(t, x) v f \, dx}{|f|_{H^1_0(K)}} \]
\[ + \sup_{0 \neq f \in H^1_0(K)} \frac{\int_K e^{2\lambda \xi} w \Delta \xi \, dx}{|f|_{H^1_0(K)}} \]
\[ + \sup_{0 \neq f \in H^1_0(K)} \frac{\int_K w \sum_j (\hat{f}_j, e^{2\lambda \xi}) \, dx}{|f|_{H^1_0(K)}} \]
\[ \leq C[\mathcal{L} |e^{2\lambda \xi} v|_{L^2(K)} + e^{C\lambda} |w|_{L^2(0, T) \times \omega}]. \quad (2.16) \]

Combining (2.15)–(2.16), we have
\[ \lambda |e^{2\lambda \xi} u|_{L^2(K)}^2 \leq C_1 \left[ \mathcal{L}^2 |e^{2\lambda \xi} u|_{L^2(K)}^2 + e^{C_1 \lambda} |w|^2_{L^2(0, T) \times \omega} \right], \quad \forall \lambda > \lambda_0. \quad (2.17) \]

Denote
\[ \lambda_1 \triangleq C_1 \mathcal{L}^2 + 1. \quad (2.18) \]
Now, let us take
\[ \lambda \triangleq \max(\lambda_0, \lambda_1). \]
(2.19)

Then, by (2.17), we get
\[ |u|_{L^2(K)}^2 \leq C e^{CA} |w|_{L^2((0, T) \times \omega)}^2. \]
(2.20)

Thus, by (2.11)–(2.13), we arrive at
\[ |w|_{L^2((T - \delta, T) \times (\Omega \setminus \omega))}^2 \leq C e^{CA} |w|_{L^2((0, T) \times \omega)}^2. \]
(2.21)

Adding both sides of (2.21) by \(|w|_{L^2([0, T])}^2\), we end up with
\[ |w|_{L^2((T - \delta, T) \times \Omega)}^2 \leq C e^{CA} |w|_{L^2((0, T) \times \omega)}^2. \]
(2.22)

**Step 3.** Let us complete the proof of Theorem 2.1. By Lemma 2.3, we get
\[
2|w|_{L^2((T - \delta, T) \times \Omega)}^2 \geq |w|_{L^2((T - \delta/2, T - \delta/2 + \delta) \times \Omega)}^2 + |w|_{L^2((T - \delta, T) \times \Omega)}^2 \\
\geq |w|_{L^2((T - \delta/2, T - \delta/2 + \delta) \times \Omega)}^2 \\
+ \frac{C}{1 + \epsilon} \int_{T - \delta/2}^{T - \delta/2 + \delta} |w(t, \cdot)|_{H^{-1}(\Omega)}^2 \, dt \\
\geq \frac{C}{1 + \epsilon} \int_{T - \delta/2}^{T - \delta/2 + \delta} E(t) \, dt,
\]
(2.23)

where \(E(t)\) is defined by (2.8). Combining (2.22)–(2.23), we get
\[
\int_{T - \delta/2}^{T - \delta/2 + \delta} E(t) \, dt \leq C(1 + \epsilon) e^{CA} |w|_{L^2([0, T])}^2.
\]
(2.24)

However, by Lemma 2.4, we have
\[
\int_{T - \delta/2}^{T - \delta/2 + \delta} E(t) \, dt \geq C e^{-2T} E(0).
\]
(2.25)

Consequently, combining (2.24)–(2.25), we conclude that
\[
E(0) \leq C(1 + \epsilon) e^{CA} |w|_{L^2([0, T])}^2,
\]
(2.26)

which gives (2.2). Finally, by (2.18)–(2.19) and (2.26), we get (2.3) immediately. 

3. STATEMENT AND PROOF OF THE MAIN RESULT

Our main result can be stated as follows:

**THEOREM 3.1.** Let \( \omega = \Omega \cap \Theta_{e_0}(\Gamma) \) for some \( e_0 > 0 \) and \( T > T_0 \triangleq \text{diam} (\Omega \setminus \omega) \). Let \( f(\cdot) \in C^1(\mathbb{R}^d) \) satisfy (1.2). Then (1.1) is exactly controllable in \( H_0^1(\Omega) \times L^2(\Omega) \) at time \( T \) by using control \( u \in L^2((0, T) \times \omega) \).

**Proof.** Let us fix the initial and final date \((y_0, y_1), (z_0, z_1) \in H_0^1(\Omega) \times L^2(\Omega)\) and let us introduce the continuous function

\[
 h(s) \triangleq \begin{cases} 
 \left[ f(s) - f(0) \right] / s, & \text{if } s \neq 0; \\
 f'(0), & \text{if } s = 0. 
\end{cases} \tag{3.1}
\]

For any given \( z(\cdot) \in L^2(0, T; L^2(\Omega)) \), by Theorem 2.1, using HUM (see [4, 5]) and proceeding as in [10], we conclude that there exists a control \( u \in L^2((0, T) \times \omega) \) such that the solution \( y = y(\cdot ; z(\cdot)) \) of the equation

\[
\begin{align*}
 y'' - \Delta y &= h(z(\cdot))y + f(0) + \chi_\omega(x)u(t, x) \quad &\text{in } Q, \\
y &= 0 \quad &\text{on } \Sigma, \\
y(0) &= y_0, \quad y'(0) = y_1 \quad &\text{in } \Omega
\end{align*} \tag{3.2}
\]

satisfies

\[
y(T) = z_0, \quad y'(T) = z_1; \tag{3.3}
\]

furthermore, concerning the control \( u \), one has the estimate

\[
 |u|^2_{L^2((0, T) \times \omega)} \leq C \exp \left( C \int_{L^2(0, T; L^\infty(\Omega))} \right) \tag{3.4}
\]

for some constant \( C = C(T, \Omega, f(0), |y_0|_{H^1_0(\Omega)}, |y_1|_{L^2(\Omega)}, |z_0|_{H^1_0(\Omega)}, |z_1|_{L^2(\Omega)}) \). Thus, for any \( \epsilon \in (0, 4] \), we have

\[
 |u|^{2(1 + \epsilon)}_{L^2((0, T) \times \omega)} \leq C \exp \left( C \int_{L^2(0, T; L^\infty(\Omega))} \right). \tag{3.5}
\]

However, by calculus, we have

\[
\exp \left( C \int_{L^2(0, T; L^\infty(\Omega))} \right) = \sum_{j=0}^{\infty} \frac{C^j}{j!} \left( \int_{\Omega} \left| h(z(t, x)) \right|^n \, dx \right)^{2j/n}
\]

\[
= \sum_{j=0}^{n-1} \frac{C^j}{j!} \left( \int_{\Omega} \left| h(z(t, x)) \right|^n \, dx \right)^{2j/n}
\]

\[
+ \sum_{j=n}^{\infty} \left( \int_{\Omega} \left| h(z(t, x)) \right|^n \, dx \right)^{2j/n}
\]

\[
\text{for some constant } C = C(T, \Omega, f(0), |y_0|_{H^1_0(\Omega)}, |y_1|_{L^2(\Omega)}, |z_0|_{H^1_0(\Omega)}, |z_1|_{L^2(\Omega)}).
\]
\[\leq C + \text{ess sup}_{t \in (0, T)} \left( \int_{\Omega} |h(z(t, x))|^{n} \, dx \right)^{2} \]
\[+ \sum_{j=n}^{\infty} \frac{C^j}{j!} \text{ess sup}_{t \in (0, T)} \left( \int_{\Omega} |h(z(t, x))|^{n} \, dx \right)^{2j/n} \]
\[\leq C + 2 \sum_{j=n}^{\infty} \frac{C^j}{j!} \text{ess sup}_{t \in (0, T)} \left( \int_{\Omega} |h(z(t, x))|^{n} \, dx \right)^{2j/n}. \quad (3.6)\]

Note that for any \( j \geq n \), it holds
\[\int_{\Omega} |h(z(t, x))|^{n} \, dx \leq C \left( \int_{\Omega} |h(z(t, x))|^{2j} \, dx \right)^{n/2j}. \quad (3.7)\]

Thus, by (3.5)–(3.6), we have
\[
\begin{align*}
\exp(C|h(z(\cdot))|^{2}_{L^{\infty}(0, T; L^{n}(\Omega))}) \\
&\leq C \left[ 1 + \sum_{j=n}^{\infty} \frac{C^j}{j!} \text{ess sup}_{t \in (0, T)} \left( \int_{\Omega} |h(z(t, x))|^{2j} \, dx \right) \right] \\
&\leq C \left[ 1 + \text{ess sup}_{t \in (0, T)} \int_{\Omega} e^{Ch(z(t, x))L^{2}} \, dx \right]. \quad (3.8)
\end{align*}
\]

However, by our assumption (1.2), we have
\[e^{Ch(z(t, x))L^{2}} \leq C \left( 1 + |z(t, x)|^{2} \right). \quad (3.9)\]

Thus, by (3.5) and (3.8)–(3.9), we conclude that
\[|u|^{2}_{L^{2}(0, T; \Omega)} \leq C \left( 1 + |z|^{2}_{L^{2}(0, T; L^{2}(\Omega))} \right). \quad (3.10)\]

Thus
\[|u|^{2}_{L^{2}(0, T; \Omega)} \leq C \left( 1 + |z|^{2}_{L^{2}(0, T; L^{2}(\Omega))} \right). \quad (3.11)\]

Now, concerning (3.2), by means of the usual energy estimate, we end up with
\[|y|_{C(0, T; H^{1}(\Omega))} \leq C \left( 1 + |z|^{2}_{L^{2}(0, T; L^{2}(\Omega))} \right), \quad \forall \varepsilon \in (0, 4) \quad (3.12)\]
for some constant \( C = C(T, \Omega, f(0), |y_{0}|_{H^{1}(\Omega)}, |y_{1}|_{L^{2}(\Omega)}, |z_{0}|_{H^{1}(\Omega)}, |z_{1}|_{L^{2}(\Omega)}) \).

Consequently if we take \( \varepsilon = 4 \) in (3.12), the desired result follows from the fixed point technique. \( \blacksquare \)
REFERENCES