Despite the apparent power of tight closure techniques, the tight closure operation itself is quite difficult to handle in practice. For example, it is generally difficult to find the tight closure of an arbitrary ideal. Also, it is not known whether tight closure behaves well under localization. We would like to know whether it is true that $I^*W^{-1}R = (IW^{-1})^*$ where $I$ is an ideal of a ring $R$, $W$ is an arbitrary multiplicative system, and $I^*$ denotes the tight closure of $I$. It is not even known that if all of the ideals of $R$ are tightly closed, then all of the ideals of $R_p$ are tightly closed.

One obstacle to computing tight closure is the fact that one must check certain conditions for possibly infinitely many values of $q = p^e$. It would be helpful if there were a bound on the power of $p$ necessary to check whether an element is in the tight closure of an ideal. Not only would the existence of a bound make it feasible to calculate tight closure, but the existence of such a bound would solve the localization problem.

In this paper, we examine certain mapping properties of rings of characteristic $p$. The existence of these maps implies a bound on the power of $p$ necessary to test tight closure. The existence of such a bound is one of many conditions that implies that tight closure commutes with localization.

The main result of this paper is Theorem 4.5 which establishes that the mapping property holds for $F$-finite domains whose integral closures are regular. In particular, the mapping property holds for one-dimensional $F$-finite domains. It seems doubtful that these maps exist in higher dimensions in general, but the question remains quite open. Moreover, there is some evidence, particularly in the ring $K[[x, y, z]]/(x^3 + y^3 + z^3)$ when
the characteristic of $K$ is two, that a bound exists on the power of $p$ needed to test tight closure. Thus, it may be that there is a weaker property which holds in general and implies the existence of a bound.

The mapping property we will be discussing is the following: given a test element $c$, we call $R$ strongly bounded relative to $c$ if there exists an $R$-linear map $\theta: R^{1/q} \to R^{1/pq}$ taking $c^{1/q}$ to $c^{1/pq}$ for some $q$, and we call $q$ the bounding exponent. Note that the map $\theta$ and the exponent $q$ both depend on $c$.

First we will show that if a ring is strongly bounded relative to a test element, then tight closure commutes with localization in that ring. We will then show that in certain rings there are test elements such that the ring is strongly bounded relative to those test elements. We also discuss a related boundedness criterion for Frobenius closure.

1. BACKGROUND ON TIGHT CLOSURE

In this section we provide a brief introduction to the theory of tight closure. For more information, see [HH1, Hu1]. We are primarily interested in the case where tight closure is an operation performed on ideals in a commutative Noetherian ring of characteristic $p$. Tight closure is also defined for submodules of modules over a Noetherian ring. In addition there are several notions of tight closure in equal characteristic zero which involve reduction mod $p$. It is still unclear how to define tight closure in an effective way in mixed characteristic.

Throughout this paper, $R$ denotes a commutative Noetherian ring of characteristic $p$ and $q$ always denotes some prime power, $p^e$, for some non-negative integer $e$.

Notations and Conventions

The definition of the tight closure operation involves iterating the Frobenius endomorphism of a ring. We denote by $F$ or $F_R$ the Frobenius endomorphism of a ring $R$ of characteristic $p$, and we denote by $F^e$ the $e$th iteration of $F$, so that $F^e(r) = r^p$. If $R$ is a reduced ring of characteristic $p$, we write $R^{1/q}$ for the ring obtained by adjoining $q$th roots of all elements of $R$. The inclusion map $R \to R^{1/q}$ is isomorphic with $F^e: R \to R$. We write $R^*$ for $\bigcup_q R^{1/q}$. Note that $R^*$ is almost never Noetherian.

If $I \subseteq R$ and $q = p^e$, then $I^{[q]}$ denotes the ideal $(i^q; i \in I)R$, which is also the expansion $F(I)R$ of $I$ under the Frobenius map $F: R \to R$. Note that if $T$ denotes a set of generators for $R$, then $(t^{q^e}; t \in T)$ generates $I^{[q]}$.

In any commutative ring $R$, $R^0$ denotes the complement of the union of the set of minimal primes. Thus, if $R$ is a domain, $R^0 = R \setminus \{0\}$. 

Tight Closure

1.1 Definition. Let \( R \) be a ring of characteristic \( p \) and \( I \) an ideal of \( R \). We say that \( x \in I^* \), the tight closure of \( I \), if there exists \( c \in R^0 \) such that \( cx^q \in I^{(q)} \) for all \( q \gg 0 \). If \( I = I^* \), we say that \( I \) is tightly closed.

1.2 Discussion. Let \( R \) be a ring of characteristic \( p \) and \( N \subseteq M \) finitely generated \( R \)-modules. We can map a finitely generated free module \( G \) onto \( M \), say \( f: R^h \to M \). Let \( H = f^{-1}(N) \) and let \( v \in R^h \) map to \( u \in M \). Then \( u \in N_M^h \), the tight closure of \( N \) in \( M \), if and only if \( u \in H_M^h \). In the free case, the definition of tight closure is exactly the same as for ideals. We set \( (r_1, \ldots, r_h)^* = (r_1^*, \ldots, r_h^*) \) and let \( H^{(i)} \) be the \( R \)-span in \( M \) of the elements \( h^* \) for \( h \in N \). A more functorial approach to tight closure in modules is given in [HH1], for example.

We now outline some basic properties of tight closure.

1.3 Proposition. Let \( R \) be a Noetherian ring of characteristic \( p \), and let \( I, J \) be ideals of \( R \).

(a) \( I^* \) is an ideal of \( R \) containing \( I \).
(b) \( (I^*)^* = I^* \).
(c) If \( I \subseteq J \), then \( I^* \subseteq J^* \).
(d) If \( I \) has positive height or if \( R \) is reduced, then \( x \in I^* \) if and only if there exists \( c \in R^0 \) such that \( cx^q \in I^{(q)} \) for all \( q = p^r \).
(e) For any \( I \subseteq R \), \( I^* \) is the inverse image in \( R \) of \( (IR_{\text{red}})^* \).
(f) If \( I \) is tightly closed, then \( I: J \) is tightly closed for any ideal \( J \).
(g) \( I^* \subseteq \overline{I} \), where \( \overline{I} \) denotes the integral closure of \( I \).
(h) An element \( x \in R \) is in \( I^* \) if and only if the image of \( x \) in \( R/P \) is in the tight closure of \( (I + P)/P \) for every minimal prime \( P \) of \( R \).

Proof. See [HH1]. Parts (a)–(f) can be found in Proposition 4.1. Part (g) is Theorem 5.2, and (h) is Proposition 6.25(a).

1.4 Remark. Virtually all of the theory of tight closure reduces to the case where \( R \) is a reduced ring. If \( N \) denotes the nilradical of \( R \) and \( I \subseteq R \), then \( u \in I^* \) if and only if \( \overline{u} \in (I/N)^* \) where \( \overline{u} \) is the image of \( u \) in \( I/N \). Also note that Proposition 1.3(h) means that we can reduce to the domain case.

Test Elements

In many applications one would like to be able to choose the element \( c \) in the definition of tight closure independent of \( x \) or \( I \). It is very useful when a single choice of \( c \), a test element, can be used for all tight closure tests in a given ring.
The theory of test elements has turned out to be very important in tight closure theory. For example, the existence of test elements is used to prove that tight closure persists under homomorphisms, i.e., given $I \subseteq R$ and $z \in I^*$, then $\phi(z) \in (IS)^*$, where $\phi: R \to S$ is a homomorphism. Also, from a computational point of view, test elements make it easier to show that an element is not in the tight closure of a given ideal. If $c$ is a test element and $cu^q \notin I^{[q]}$ for one $q$, then $u \notin I^*$.

(1.5) Definition. An element $c \in R^\diamond$ is a test element for $R$ if for all ideals $I \subseteq R$ and all $u \in R$, if $u \in I^*$, then $cu^q \in I^{[q]}$ for all $q$ (equivalently $cu \in I$). We say that an element of $R$ is a completely stable test element if its image in every local ring of $R$ is a test element and its image in the completion of every local ring of $R$ is a test element.

Fortunately, completely stable test elements exist for the rings we will consider.

(1.6) Theorem. Let $R$ be a reduced excellent local ring. If $c \in R^\diamond$ has the property that $R_c$ is regular, then $c$ has a power which is a completely stable test element for $R$. In particular, completely stable test elements always exist for reduced excellent local rings.

Proof. See [HH3, Theorem 6.2].

Localization

We do not know, in general, how tight closure behaves under localization. In particular, we would like to know the answer to the following question: Given an ideal, $I$, of a ring, $R$, and an arbitrary multiplicative system $W$, is it true that $I^*W^{-1}R = (IW^{-1}R)^*$? The localization problem has been solved in a number of special cases. For example, localization at a maximal ideal behaves as desired.

(1.7) Proposition. Let $R$ be a Noetherian ring of characteristic $p$, and let $I$ be an ideal primary to a maximal ideal $m$. Then $(IR_m)^* = I^*(R_m)$. Hence, if $I$ is tightly closed, so is $IR_m$. Moreover, $I^*$ is the contraction of $(IR_m)^*$.

Proof. See Proposition 4.14 of [HH1].

Also, if $I$ is generated by parameters in a locally excellent equidimensional ring $R$, then $I^*W^{-1}R = (IW^{-1}R)^*$ for any multiplicative system $W$ of $R$ [AHH]. In addition, we also know that tight closure commutes with localization for $N \subseteq M$ modules such that $M/N$ has a finite phantom projective resolution [AHH].
2. LOCALIZATION

Before proving that the mapping property implies that tight closure commutes with localization, we need the following lemma.

(2.1) Lemma. Let $R$ be a reduced ring of characteristic $p > 0$ and $c \in R^\times$. Suppose that $R$ is strongly bounded relative to $c$ with bounding exponent $q$ for some $q$. Then $R$ is strongly bounded relative to $c$ with bounding exponent $q'$ for all $q' \geq q$.

Proof. Suppose there exists an $R$-linear map $\theta: R_1^{1/q} \to R_1^{1/pq}$ taking $c^{1/q}$ to $c^{1/pq}$. We can define a map $\tilde{\theta}: R_1^{1/pq} \to R_1^{1/pq}$ with the desired properties. Let $\tilde{\theta}(c^{1/pq}) = [\theta(c^{1/q})]^{1/p}$. First we check that $\tilde{\theta}(c^{1/pq}) = c^{1/p^2q}$. By construction

$$\tilde{\theta}(c^{1/pq}) = \left[\theta(c^{1/q})\right]^{1/p} = \left(c^{1/pq}\right)^{1/p} = c^{1/p^2q}.$$  

Next we check that $\tilde{\theta}$ is $R$-linear. Let $x^{1/p} \in R_1^{1/p}$. Then $\tilde{\theta}(x^{1/p}) = [\theta(x)]^{1/p}$. But $x \in R$ and $\theta$ is $R$-linear, so $\theta(x) = x$ and hence $\tilde{\theta}(x^{1/p}) = x^{1/p}$. Thus $\tilde{\theta}$ is $R_1^{1/p}$-linear and hence $R$-linear.

We can now prove that the existence of these maps implies that there is a bound on the power of $p$ needed to test tight closure. The existence of this bound also implies that tight closure commutes with localization.

(2.2) Proposition. Let $R$ be a reduced Noetherian ring of characteristic $p$ and let $c$ be a test element. Let $I \subseteq R$ be an ideal and $u \in R$.

(a) If $R$ is strongly bounded relative to $c$ with bounding exponent $q$, then $u \in I^*$ if and only if $cu^q \in I^{[q]}$.

(b) Suppose there exists $q$ and an $R$-linear map $\theta: R_1^{1/q} \to R_1^{1/pq}$ such that $\theta(c^{1/q}) = c^{1/pq}$. If $W$ is a multiplicative system in $R$, then $(W^{-1}I)^* = W^{-1}I^*$.

Proof. In the proof of (a), $\Rightarrow$ is clear. For the other direction, if $cu^q \in I^{[q]}$, taking $q$th roots gives $c^{1/q}u \in IR_1^{1/q}$. Apply the map $\theta: R_1^{1/q} \to R_1^{1/pq}$ to get $c^{1/pq}u \in IR_1^{1/pq}$. This implies that $cu^{pq} \in I^{[pq]}$. Repeating the argument using Lemma 2.1 shows that $cu^q \in I^{[q]}$, all $q \geq q$. Hence $u \in I^*$.

To prove (b), first note that it suffices to see that $(W^{-1}I)^* \subseteq W^{-1}I^*$. Pick $u \in (W^{-1}I)^*$. So $cu^q \in (W^{-1}I)^{[q]} = W^{-1}I^{[q]}$. We can pick $f \in W$ so that $fcu^q \in I^{[q]}$. This implies that $f^{\theta}cu^q \in I^{[q]}$. So $c(fu)^q \in I^{[q]}$. By part (a), $fu \in I^*$, and so $u \in W^{-1}I^*$.  

3. REDUCTIONS TO THE COMPLETE LOCAL CASE

Before proving the main result, we will discuss several reductions in the problem. The following two lemmas establish that the issue of giving a map $R^{1/q} \to R^{1/pq}$ is local and unaffected by completion.

(3.1) Lemma. (a) Let $R$ be a Noetherian ring. Let $M$ and $N$ be $R$-modules with $M$ finitely presented, $v \in M$ and $w \in N$. Suppose for all $P \in \text{Spec } R$ there exists a map $\phi_P : M_P \to N_P$ (R-linear or $R_P$-linear) with $\phi_P(v/1) = w/1$. Then there exists an $R$-linear map $\phi : M \to N$ such that $\phi(v) = w$.

(b) Let $R$ be a reduced $F$-finite Noetherian ring. Let $c$ be a stable test element for $R$. Suppose $R_p$ is strongly bounded relative to $c$ with bounding exponent $q$ for all $P \in \text{Spec } (R)$. Then $R$ is strongly bounded relative to $c$ with bounding exponent $q$.

Proof. (a) Let $J = \{\psi(v) : \psi \in \text{Hom}_R(M, N)\} \subseteq N$. If $w \notin J$, we can choose $P$ such that $P \supseteq J : w$. Then $w/1 \notin J_P$. Since $M$ is finitely presented, $\text{Hom}$ commutes with localization. So $J_P = \{(\phi(v)/1) : \phi \in \text{Hom}_R(M_P, N_P)\}$. Thus $w/1 \notin \{(\phi(v)/1) : \phi \in \text{Hom}_R(M_P, N_P)\}$.

(b) Since $R$ is $F$-finite, $R^{1/q}$ is module-finite over $R$. So $R^{1/q}$ is Noetherian and thus finitely presented as an $R$-module. The result now follows from (a).

(3.2) Lemma. Let $R$ be a local ring, $M$ and $N$ finitely generated $R$-modules, $v \in M$ and $w \in N$. Suppose there exists a map $\phi : M \to N$ such that $\phi(v) = w$ where $\tilde{v}$ and $\tilde{w}$ are the images of $v$ and $w$ in $\hat{R}$. Then there exists a map $\phi : M \to N$ such that $\phi(v) = w$.

In particular, if $R$ is a reduced local $F$-finite ring and $c$ is a stable test element for $R$, then $R$ is strongly bounded relative to $c$ if and only if $\hat{R}$ is strongly bounded relative to $\tilde{c}$ where $\tilde{c}$ is the image of $c$ in $\hat{R}$.

Proof. Let $J = \{\phi(v) : \phi \in \text{Hom}(M, N)\}$. If $w \notin J$, then there exists $t$ such that $w \notin J + m'N$. Since $R$ is Noetherian and $M$ and $N$ are finitely generated, we have

$$\text{Hom}_R(M, N) = \hat{R} \otimes_R \text{Hom}_R(M, N) \cong \text{Hom}_R(M, N).$$

If we think of $\phi \in \text{Hom}_R(M, N)$, then we can find $\theta \in \text{Hom}_R(M, N)$ such that $\phi = \theta$ in $\text{Hom}_R(M, N)$, is within $m'$ of $\phi$. Then $\phi(\tilde{v}) = \theta(\tilde{v}) \in m'N$, which implies that $\tilde{w} \in \theta(\tilde{v}) + m'N$. So $\tilde{w} \in J + m'N$, and hence $w \in J + m'N$, which is a contradiction.

To see the last statement in the lemma, we first show that $F^c(\hat{R}) \cong F^c(\hat{R})$. We know that $F^c(\hat{R}) \cong R$, so it follows that $F^c(\hat{R}) \cong \hat{R}$. Also, $F^c(\hat{R}) \cong \hat{R}$. Hence $F^c(\hat{R}) \cong F^c(\hat{R})$ since both are isomorphic to $\hat{R}$. 
Since \( R \) is reduced, we know that \( F^e(R) \cong R^{1/q} \), and hence \( F^e(R) \equiv R^{1/q} \). As \( R \) is \( F \)-finite, \( R \) is excellent and hence \( \hat{R} \) is reduced. Thus \( F^e(\hat{R}) \equiv \hat{R}^{1/q} \). Combining the previous results we see that \( \hat{R}^{1/q} \). Combining the previous results we see that \( \hat{R}^{1/q} \equiv \hat{R}^{1/q} \). Thus \( \text{Hom}_R(\hat{R}^{1/q}, \hat{R}^{1/pq}) \equiv \text{Hom}_R(\hat{R}^{1/q}, \hat{R}^{1/pq}) \). As \( R \) is \( F \)-finite, \( R^{1/q} \) is module-finite over \( R \), and the result now follows from the first part of the lemma.

### 4. STRONGLY BOUNDED RINGS

Our first example of strongly bounded rings is strongly \( F \)-regular rings.

**4.1 Definition.** A Noetherian domain \( R \) of characteristic \( p \) is strongly \( F \)-regular if \( R^{1/p} \) is a finite \( R \)-module and if for every non-zero \( d \in R \), there exists a \( q \) such that the map \( R \to R^{1/q} \) sending 1 to \( d^{1/q} \) splits as a map of \( R \)-modules.

Strong \( F \)-regularity is only defined for reduced rings \( R \) such that \( R^{1/p} \) is module-finite over \( R \). However, finitely generated algebras over a perfect field \( K \) and complete local rings \( R \) with a perfect residue class field \( K \) both satisfy this condition, so it is not particularly limiting.

**4.2 Proposition.** Let \( R \) be a strongly \( F \)-regular ring and let \( c \) be a test element. Then there exists an \( R \)-linear map \( R^{1/q} \to R^{1/pq} \) taking \( c^{1/q} \) to \( c^{1/pq} \) for some \( q \). In other words, a strongly \( F \)-regular ring is strongly bounded relative to every test element \( c \).

**Proof.** Since \( R \) is strongly \( F \)-regular, we know that the \( R \)-linear map \( R \to R^{1/q} \) that sends 1 to \( c^{1/q} \) splits as a map of \( R \)-modules. Let \( \theta \) be the splitting map. Then if we compose \( \theta \) with multiplication by \( c^{1/pq} \), we have

\[
\begin{align*}
R^{1/q} & \xrightarrow{\theta} R^{1/pq} \\
c^{1/q} & \to 1 \\
& \xrightarrow{c^{1/pq}}
\end{align*}
\]

and the composition is \( R \)-linear.

It is also quite easy to see that the desired maps exist for a one-dimensional \( F \)-finite domain \( R \) when the integral closure of \( R \) is contained in \( R^{1/q} \) for some \( q \).

**4.3 Proposition.** Let \( R \) be a one-dimensional \( F \)-finite domain and let \( S \) be the integral closure of \( R \) in its fraction field. Let \( c \) be a completely stable test element for \( R \). Suppose, for large \( q \), \( S \subseteq R^{1/q} \). Then there exists a \( q \) and a map \( \theta: R^{1/q} \to R^{1/pq} \) taking \( c^{1/q} \) to \( c^{1/pq} \).

**Proof.** We may pass to the local case by Lemma 3.1. By assumption we have that \( S \subseteq R^{1/q} \) for \( q \gg 0 \). Since normal implies regular in dimension
one and a one-dimensional regular domain is a principal ideal domain, $S$ is a PID. As $R$ is a domain, $R^{1/q}$ is torsion-free as an $S$-module. Finitely generated torsion-free modules over a PID are free, so $R^{1/q}$ is free as an $S$-module. We can give an $S$-linear map $R^{1/q} \to R^{1/pq}$ taking $c^{1/q}$ whenever we like if $c^{1/q}$ is part of an $S$-free basis for $R^{1/q}$. In the local case a free basis is just a minimal basis, so $c^{1/q}$ is part of an $S$-free basis if and only if $c^{1/q} \notin m_R^{1/q}$, and this is true if and only if $c \notin m_R^{[q]}$. Since $R$ is one-dimensional, $m_R$ contains some power of $m_R$. To see this, note that since $S$ is an integral extension of $R$ in its fraction field, we can pick $b \in R \setminus \{0\}$ such that $bS \subseteq R$. In fact, $bS \subseteq m_R$. The ideal $bS$ is a nonzero ideal of $R$ and since $R$ is one-dimensional, $m_R$ is the radical of $bS$. We can pick $q_0$ such that $m_R^{[q_0]} \subseteq bS \subseteq m_R$. Pick $q_1$ so large that $c \notin m_R^{[q_1]}$ and $q_1$ was chosen precisely so that $c \notin m_R^{[q_1]}$.  

Next we give an example of a simple case when the integral closure of $R$ is not contained in $R^{1/q}$, and we show that $R$ is strongly bounded.

(4.4) Example. Let $R = K + K(\lambda)x + K(\lambda)x^2 + K(\lambda)x^3 + \cdots$, where $\lambda$ is separable algebraic over $K$, a field of characteristic $p$. Then there exists an $R$-linear map $R^{1/q} \to R^{1/pq}$ taking $x^{1/q}$ to $x^{1/pq}$.

Proof. Let $S$ be the integral closure of $R$, so $S = K(\lambda) + K(\lambda)x + K(\lambda)x^2 + \cdots$, $R \subseteq S$ extends to injective $R$-linear maps $i^{1/q}: R^{1/q} \to S^{1/q}$ and $i^{1/pq}: R^{1/pq} \to S^{1/pq}$. Since $S$ is integrally closed, we can apply (4.3) to get an $S$-linear map $\theta: S^{1/q} \to S^{1/pq}$ taking $x^{1/q}$ to $x^{1/pq}$. We will construct $\theta$ explicitly later. So far we have the maps

$$R^{1/q} \xrightarrow{i^{1/q}} S^{1/q} \xrightarrow{\theta} S^{1/pq} \xrightarrow{i^{1/pq}} R^{1/pq}.$$  

Let $\phi = \theta \circ i^{1/q}$. It is enough to show that $\phi(R^{1/q}) \subseteq i^{1/pq}(R^{1/pq})$, for then the desired map is $\phi$ followed by restriction to $R^{1/pq}$. Let $1 = \mu_0, \mu_1, \ldots, \mu_h$ be a basis for $K^{1/q}$ over $K$. Since $K(\lambda)/K$ is separable and $K^{1/q}/K$ is purely inseparable, the two extensions are linearly disjoint. In other words, $1 = \mu_0, \mu_1, \ldots, \mu_h$ are linearly independent over $K(\lambda)$. So $1, \mu_1, \ldots, \mu_h$ is a basis for $K(\lambda)^{1/q}$ over $K(\lambda)$. Thus $\{\mu_i x^{1/q} \mid 0 \leq i \leq q\}$ is a free basis for $S^{1/q}$ over $S$. Define an $S$-linear map $S^{1/q} \to S^{1/pq}$ by

$$\theta(\mu_i x^{1/q}) = \begin{cases} x^{1/pq}, & i = 0 \text{ and } j = 1 \\ 0, & \text{otherwise.} \end{cases}$$
$R^{1/q}$ is generated over $R$ by $\{\mu_i x^{i/q}: 0 \leq i \leq h, 0 \leq t < q, 0 < s < q\}$. We can find $\phi(R^{1/q})$ by looking at what $\theta$ does to the generators of $R^{1/q}$.

Since $\mu_i x^{i/q} \in K(\lambda)^{1/q}$, we can express it in terms of the basis for $K(\lambda)^{1/q}$,

$$\mu_i x^{i/q} = \sum_{j=0}^{h} v_{ij} \mu_j x^{s/q}, \quad \text{where} \ v_{ij} \in K(\lambda).$$

Since $\theta(\mu_i x^{s/q}) = 0$ for $j \neq 0$, and $v_{ij} \in K(\lambda) \subseteq S$,

$$\theta(\mu_i x^{s/q}) = \begin{cases} v_{i0} x^{s/q}, & s = 1 \\ 0, & \text{otherwise}. \end{cases}$$

Recall that

$$S^{1/pq} = K(\lambda)^{1/pq} + K(\lambda)^{1/pq} x^{1/pq} + \cdots + K(\lambda)^{1/pq} x^{(p^s-1)/pq} + K(\lambda)^{1/pq} x + \cdots.$$ 

So $v_{i0} x^{s/pq} \in S^{1/pq}$ since $v_{i0} \in K(\lambda) \subseteq K(\lambda)^{1/pq}$. Then $\theta(R^{1/q}) \subseteq i^{1/pq}(R^{1/pq})$ as required.

We now prove the main result.

(4.5) Theorem. Let $R$ be an $F$-finite domain. Let $S$ be the integral closure of $R$ and suppose that $S$ is regular. For all $c \in R \setminus \{0\}$ such that $cS \subseteq R$, there exists a $q$ and a map $\theta: R^{1/q} \rightarrow R^{1/pq}$ taking $c^{1/q}$ to $c^{1/pq}$.

Note that this includes the case where $R$ is a one-dimensional $F$-finite domain. Since $S$ is one-dimensional and normal, it is regular.

Proof. We may reduce to the complete local case by Lemmas 3.1 and 3.2. A local ring of an $F$-finite ring is $F$-finite, its residue field is therefore $F$-finite, and so the completion is $F$-finite. So we are done by the following proposition.

(4.6) Proposition. Let $R$ be a complete local domain with residue field $K$, and let $S$ be the integral closure of $R$. Suppose that $K$ is $F$-finite and $S$ is regular. For any $c \in R \setminus \{0\}$ such that $cS \subseteq R$, there exists a $q$ and an $R$-linear map $R^{1/q} \rightarrow R^{1/pq}$ taking $c^{1/q}$ to $c^{1/pq}$.

Note that this includes the case where $R$ is a one-dimensional complete local $F$-finite domain. $S$ is regular as before. Also, for a complete local ring, $F$-finiteness is equivalent to the condition that the residue field be $F$-finite.

Proof. $R \subseteq S$ extends to an injective $R$-linear map $i^{1/q}: R^{1/q} \rightarrow S^{1/q}$. Let $i$ be the inclusion map. Then $i^{1/q}(c^{1/q}) = (i(c))^{1/q}$. 

Note that $S$ is local since it is module-finite over a complete local domain. Also, $S$ is $F$-finite since the complete ring $R$ is. As $S$ is regular, $S^{1/p}$ is flat over $S$, and since we are in the case where $S^{1/p}$ is module-finite over a local ring, $S^{1/p}$ is actually a free $S$-module. It follows that for all $q = p^i$, $S^{1/q}$ is also free over $S$. For large enough $q$, $c^{1/q}$ is part of a free $S$-basis for $S^{1/q}$. Since $S$ is regular, the Frobenius endomorphism is flat and so $S^{1/pq}$ is $S^{1/q}$-free. Hence there is an $S$-linear map $\theta : S^{1/q} \rightarrow S^{1/pq}$ taking $c^{1/q}$ to 1.

Multiplication by $c^{1/pq}$ gives a map $S^{1/pq} \rightarrow R^{1/pq}$. Composing $i^{1/q}$, $\theta$, and multiplication by $c^{1/pq}$ gives the required map.

\[
\begin{array}{cccc}
R^{1/q} & \xrightarrow{i^{1/q}} & S^{1/q} & \xrightarrow{\theta} & S^{1/pq} & \xrightarrow{c^{1/pq}} & R^{1/pq} \\
\end{array}
\]

Note that $cS \subseteq R$ implies $c^{1/pq}S^{1/pq} \subseteq R^{1/pq}$, so $c^{1/pq}\theta i^{1/q}$ is a map $R^{1/q} \rightarrow R^{1/pq}$ taking $c^{1/q}$ to $c^{1/pq}$. The map $i^{1/q}$ is $R$-linear, $\theta$ is $S$-linear, and multiplication by $c^{1/pq}$ is $S^{1/pq}$-linear, so the composition is at least $R$-linear. 

(4.7) Remark. Let $R$ be a domain and $S$ the integral closure of $R$ in its fraction field. Let $J = \{ j \in R : jS \subseteq R \}$, the conductor of $S$ into $R$. $J$ is an ideal of $R$, and if $R$ is one-dimensional, any non-zero, non-unital element of $R$ has a power in $J$. In particular, any test element for $R$ has a power in $J$. A power of a test element is still a test element, so we can always pick $c$ in Theorem 4.5 to be a test element. In other words, if $R$ is a one-dimensional $F$-finite domain and $c$ is a test element, then there exists an integer $N$ such that $R$ is strongly bounded relative to $c^N$.

Next we show that Proposition 4.6 holds for one-dimensional reduced rings.

(4.8) Proposition. Let $R$ be a one-dimensional complete local reduced ring, and let $S$ be the integral closure of $R$ in its total quotient ring. For any $c \in R \setminus \{0\}$ such that $cS \subseteq R$, there exists $q$ and an $R$-linear map $R^{1/q} \rightarrow R^{1/pq}$ taking $c^{1/q}$ to $c^{1/pq}$.

Proof. Let $p_1, \ldots, p_k$ be the minimal primes of $R$. Since $R$ is reduced, the integral closure of $R$ in its total quotient ring is just $\Pi_{i=1}^{k}(R/p_i)^{\gamma_i}$, where $(R/p_i)^{\gamma_i}$ denotes the normalization of $R/p_i$. Note that $c(R/p_i)^{\gamma_i} \subseteq R/p_i$ for all $i$. Since each $(R/p_i)^{\gamma_i}$ is a one-dimensional normal domain, there exists $q_i$ such that $c^{1/q_i}$ is part of a free $(R/p_i)^{1/q_i}$-basis. Let $q = \max(q_i)$. So we have maps $\theta_i : (R/p_i)^{1/q} \rightarrow (R/p_i)^{1/pq}$ taking $c^{1/q}$ to 1. We construct $\theta : \Pi(R/p_i)^{1/q} \rightarrow \Pi(R/p_i)^{1/pq}$ componentwise,

$$\theta((r_1, \ldots, r_k)) = (\theta_1(r_1), \ldots, \theta_k(r_k)).$$
Multiplication by $c^{1/pq}$ gives a map $S^{1/pq} = \prod (R/p_i)^{1/pq} \to R^{1/pq}$; thus we have the maps

$$
\begin{array}{cc}
\prod (R/p_i)^{1/q} & \cong \prod (R/p_i)^{1/q} \\
\uparrow & \theta \\
R^{1/q} & \cong (\prod R/p_i)^{1/pq}
\end{array}
$$

The composition of these maps takes $c^{1/q}$ to $c^{1/pq}$ and is $R$-linear.

We can also show that polynomial rings over strongly bounded rings are strongly bounded.

(4.9) Proposition. Let $R$ be a Noetherian ring of characteristic $p$, and let $c$ be a test element for both $R$ and $R[x_1, \ldots, x_n]$. If $R$ is strongly bounded relative to $c$, then $R[x_1, \ldots, x_n]$ is strongly bounded relative to $c$.

In particular, let $R$ be a local ring such that $R \to R$ has regular fibers, and let $d \in R^0$ be any element such that $(R_{reg})_d$ is regular. Then $d$ has a power, $d^N$, which is a completely stable test element for both $R$ and $R[x_1, \ldots, x_n]$. If $R$ is strongly bounded relative to $d^N$, then so is $R[x_1, \ldots, x_n]$.

Proof. Let $\phi$ be the $R$-linear map that makes $R$ strongly bounded relative to $c$ with bounding exponent $q$. Recall that $(R[x_1, \ldots, x_n])^{1/q} \cong R^{1/q}[x_1^{1/q}, \ldots, x_n^{1/q}]$. We can define a map

$$
\theta: (R[x_1, \ldots, x_n])^{1/q} \to (R[x_1, \ldots, x_n])^{1/pq}
$$

by $\theta(r^{1/q}) = \phi(r^{1/q})$ and $\theta(x_i^{1/q}) = x_i^{1/pq}$. So $\theta(c^{1/q}) = \phi(c^{1/q}) = c^{1/pq}$.

To see that $\theta$ is $R[x_1, \ldots, x_n]$-linear, just note that

$$
\theta(\sum r_i x_i) = \sum \theta(r_i) \theta(x_i) = \sum \phi(r_i) x_i = \sum r_i x_i.
$$

For the last statement, note that since $(R_{reg})_d$ is regular, some power of $d$, say $d^k$, is a test element for $R$ [HH3, Theorem 6.1]. The map $R \to R[x_1, \ldots, x_n]$ is smooth, so $R[x_1, \ldots, x_n]$ localized at the image of $d$ is also regular. Hence there is a power of the image of $d$, say $d^k$ which is a test element for $R[x_1, \ldots, x_n]$. So $d^N$ is a test element for both rings where $N = \max(k, k')$. 

We now show that a seemingly weaker condition than was necessary in Proposition 2.2 is sufficient to have tight closure commute with localization.

(4.10) Proposition. Let $R$ be a complete local domain of characteristic $p$. Suppose $R$ has the following property:

There exists a finite dimensional vector space, $V$, of $R$-forms that are test elements and $q_0$ such that there is an $R^{q_0}$-linear map $\theta: R \to R$ (*) taking $c^p \to d \in V \setminus \{0\}$ for all $c \in V \setminus \{0\}$. Note that $d$ depends on $c$. 

Then $W^{-1}I^* \equiv (W^{-1}I)^*$ where $I$ is an ideal of $R$ and $W$ is a multiplicative system in $R$.

**Proof.** It suffices to see that $(W^{-1}I)^* \subseteq W^{-1}I^*$. Pick $u \in (W^{-1}I)^*$. Set $q = q_0/p$. We have $cu^q \in (W^{-1}I)^{q_0} = W^{-1}I^{q_0}$, so we can pick $f \in W$ such that $f cu^q \in I^{q_0}$. This implies that $f^q cu^q \in I^{q_0}$. So $c(uf)^q \in I^{q_0}$. Taking $p$th powers gives $c^p(uf)^{q_0} \in I^{p q_0}$. Applying $\theta$ gives $d(uf)^{p q_0} \in I^{p q_0}$. So we get a different $d$ for each $q$. Such that $f u \in I^* R$. This implies that $f u \in I^* R$. Applying $\theta$ gives $df u \in I^* R$. So then $u \in W^{-1}I^*$ as desired.

We can also show that it is sufficient for the integral closure to have this property.

**4.11 Proposition.** Let $R$ be a domain and let $S$ be the integral closure of $R$ in its fraction field. Suppose $S$ has property $(\ast)$ from Proposition 4.10. Then $R$ has property $(\ast)$.

**Proof.** Pick $b \in R$ such that $bS \subseteq R$. Suppose $V$, $q_0$ and $\theta$ are the vector space and maps that make $S$ satisfy property $(\ast)$. Let $V_R = \{b^{q_0}v : v \in V\}$. Define $\bar{\theta} : V_R^p \to V$ by

$$\bar{\theta}(b^{q_0}v)^p = b^{p q_0} \theta(v^p) = b^{p q_0} d_{q_0}.$$

Then $\bar{\theta}(b^{q_0}v)^{p_0} = b^{q_0}d_{q_0}$ since $R^{q_0} \subseteq S^{q_0}$ and $\theta$ is $S^{q_0}$-linear. Thus $\bar{\theta}$ is $R^{q_0}$-linear.

5. A BOUNDEDNESS CRITERION FOR FROBENIUS CLOSURE

Frobenius closure is related to tight closure and sometimes it is easier to compute than tight closure. Also, in certain rings it is true that the Frobenius closure of an ideal is the same as the tight closure of that ideal (see [M], for example), so understanding Frobenius closure is particularly helpful in those situations. There is a mapping property similar to the
strongly bounded property which implies a bound on the power of \( p \) necessary to test Frobenius closure. We will denote the Frobenius closure of an ideal \( I \) by \( I^F \). Recall that \( I^F = \{ u \in R : u^q \in I^{[q]} \text{ for some } q \} \) and \( I^F \subseteq I^* \).

The mapping property we will be discussing is the following: there exists an \( R \)-linear map \( \theta : R^{1/pq} \to R^{1/q} \) taking 1 to 1 for some \( q \). Note that we can obtain maps for larger \( q \) by taking \( p \)th roots. If \( \theta : R^{1/pq} \to R^{1/q} \) is \( R \)-linear, then \( \theta^{1/p} : R^{1/p^2q} \to R^{1/pq} \) is \( R^{1/p} \)-linear and hence \( R \)-linear.

(5.1) Proposition. Let \( R \) be a reduced Noetherian ring of characteristic \( p \) and suppose that there exists an \( R \)-linear map \( \theta : R^{1/pq} \to R^{1/q} \) taking \( 1 \) to \( 1 \) for some \( q \). Let \( I \subseteq R \) be an ideal and \( u \in R \). Then \( u \in I^F \) if and only if \( u^q \in I^{[q]} \).

Proof. One direction follows directly from the definition of Frobenius closure. To see the other direction, suppose \( u \in I^F \). So \( u^q \in I^{[q]} \) for some \( q' \). If \( q' < q \), then by taking \( p \)th powers we see that \( u^q \in I^{[q]} \). If \( q' > q \), write \( q' = p^aq \). We have that \( u^{p^aq} \in I^{[p^aq]} \), and taking \( p^aq \)th roots gives \( u \in IR^{1/p^aq} \). Apply the map \( \theta : R^{1/p^aq} \to R^{1/p^aq} \) to get \( u \in IR^{1/p}R^{1/q} \). This implies that \( u^{p^{-1}q} \in I^{[p^{-1}q]} \). We repeat this argument to show that \( u^q \in I^{[q]} \) as desired. Hence if \( u \in I^F \), then \( u \in I^{[q]} \) and \( q \) is the desired bound.

REFERENCES


