European Journal of Combinatorics 31 (2010) 1171–1180

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Reversing a polyhedral surface by origami-deformation

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ARTICLE INFO

Article history:
Available online 13 October 2009

ABSTRACT

We introduce a new variety of flexatube, a *rhombotube*. It is obtained from a cardboard rhombohedron consisting of six rhombi with interior angles 60° and 120°, by removing a pair of opposite faces, and then subdividing the remaining four faces by pairs of diagonals. It is reversible, that is, it can be turned inside out by a series of folds, using edges and diagonals of the rhombi. To turn a rhombotube inside out is quite a challenging puzzle. We also consider the reversibility of general polyhedral surfaces. We show that if an orientable polyhedral surface with boundary is reversible, then its genus is 0, and for every interior vertex, the sum of face angles at the vertex is at least 2π. After defining the tube-attachment operation, we show that every polyhedral surface obtained from a rectangular tube by applying tube-attachment operations one after another, can be subdivided so that it becomes reversible.

1. Introduction

Deformation of geometric objects in a space has been studied by many researchers with great interest.

A deformation of a polygonal arc or polygonal cycle in the plane is a continuous motion of the arc or cycle such that during the motion, each edge remains a line segment of fixed length. The *carpenter's rule problem* asks whether every polygonal arc in the plane can be deformed, with *avoiding self-intersections*, into a polygonal arc lying on a straight line. Connelly et al. [3] proved, among other things, that this is always possible.

Since a state (locations of the vertices) of a polygonal cycle with n vertices can be represented by a point in $2n$-space, all states obtained by deforming the polygonal cycle (with allowing self-intersections) determine a subset of $2n$-space. The 'space of shapes' (the *configuration space*) of the

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doi:10.1016/j.ejc.2009.09.007

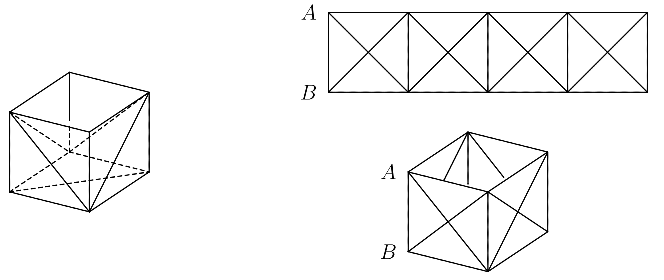

Fig. 1. A folding container (left) and a flexatube (right).

polygonal cycle is then obtained as the quotient space of this subset under the relation corresponding to 'congruence'. Havel [8] proved that the configuration space of an equilateral pentagon (that is, 5-vertex-polygonal cycle with equal edge-lengths) in the plane is a connected orientable closed 2-dimensional manifold of genus 4. Maehara [9] classified the configuration spaces for pentagons with edges of all different lengths in the plane.

A polyhedral surface M is a 2-dimensional manifold in R^3 obtained by attaching cardboard polygons along their edges. The cardboard polygons are supposed to be very thin, and the thickness is regarded to be 0. Each polygon of M is called simply a face of M. A subdivision of M is a polyhedral surface obtained by subdividing faces of M into small polygons.

Definition. An *origami-deformation* of a polyhedral surface $M \subset R^3$ is a continuous motion $f_t : M \to R^3$ $(0 \leq t \leq 1)$ of M such that (1) f_0 is the inclusion map, (2) for each face of M, the induced motion of the face is a rigid motion, (3) two faces may touch or overlap during the motion, but they never go through each other, and (4) the motion is not a rigid motion of the whole M.

Note that since two faces may overlap, $f_t : M \to R^3$ is not always an embedding for every $t \in [0, 1]$. However, since no two faces go through each other during the motion, it follows that if $f_1 : M \to R^3$ is an embedding, then M can be changed to $f_1(M)$ through 'topological embeddings', that is, the inclusion map f_0 and the embedding f_1 are isotopic.

A polyhedral surface that admits an origami-deformation is called *flexible*, otherwise, it is called *rigid*.

Cauchy proved in 1813 that every closed convex polyhedral surface in R^3 is rigid, and Gluck [6] proved that almost all closed polyhedral surfaces of genus 0 in R^3 with all triangular faces are rigid. However, Connelly [1,2] found a flexible closed polyhedral surface of genus 0 in R^3 with all triangular faces.

For polyhedral surfaces with a boundary, there are also interesting problems. If a polyhedral surface with a boundary can be deformed so that all dihedral angles become π, then the surface is called *developable*. A *face-cycle* of a polyhedral surface is a cyclic sequence of (at least three) distinct faces in which each pair of consecutive faces shares a common edge. Maehara [10] proved that the surface of a convex polyhedron cannot have a developable face-cycle.

Surfaces that are *flattenable* or *reversible* would also be interesting objects.

Example 1 (*Folding Container*). From a cubical box, remove a face and triangulate the remaining five faces as in Fig. 1 left. This polyhedral surface can be folded flat into a square.

Example 2 (*Flexatube*). From a cubical box, remove a pair of opposite faces, and triangulate the remaining four faces by pairs of diagonals, see Fig. 1 right. The resulting polyhedral tube consisting of 16 triangles is called a *flexatube* [5,11,13]. This tube is reversible!

What is meant by reversible? Paint the outside of a flexatube with red and the inside with blue. Then 'to reverse the flexatube' means to deform the flexatube so that its outside becomes blue.

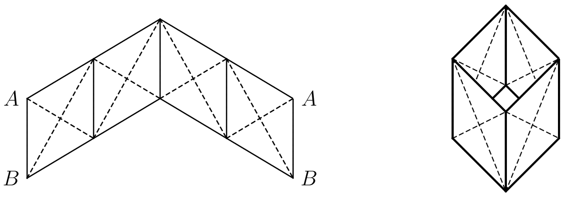

Fig. 2. A rhombotube.

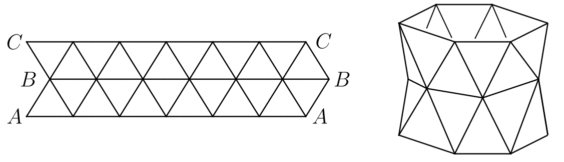

Fig. 3. Nakamura's reversible tube.

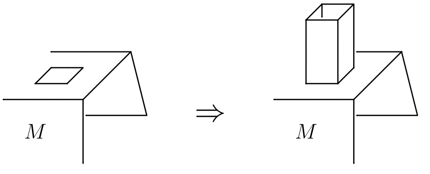

Fig. 4. A tube-attachment.

To reverse a flexatube is actually possible, though it is not easy. A flexatube is a variation of *flexagons* that were originally discovered in 1939 by Arthur H. Stone, see [11, p. 14].

Stimulated by the flexatube, I sought other intriguing variations of flexatube, and I devised one in 2006. Let us call it *Rhombotube*.

Example 3 (*Rhombotube*). From a hollow rhombohedron whose six faces are rhombi with interior angles 60° and 120°, remove a pair of opposite faces, and triangulate the remaining 4 faces by pairs of diagonals. The resulting polyhedral tube consisting of 16 triangles is the rhombotube. Fig. 2 shows how to make a paper model of rhombotube. This tube is also reversible.

To reverse a rhombotube is a challenging puzzle. A solution I found is exquisite and complicated, see Appendix. The condition that each rhombus has 60° angle seems to be essential to the solution.

When I talked on Rhombotube at KyotoCGGT 2007, Professor G. Nakamura (Research Institute of Educational Department, Tokai University, Japan) informed me that he also devised a reversible tube a long time ago.

Example 4 (*Nakamura's Reversible Tube*). In 1970, Gisaku Nakamura devised the following reversible tube (Fig. 3). To reverse it is not easy. This is also a beautiful and nice puzzle.

Motivated by these reversible tubes, we also consider the reversibility for general polyhedral surfaces. We prove that if an orientable polyhedral surface M (with boundary) is reversible, then M has genus 0, and for any interior vertex p of M, the sum of face-angles at p is greater than or equal to 2π (Theorems 1 and 2).

A *tube-attachment* operation is defined in the following way:

From a face of polyhedral surface M, cut out a rectangle, and attach a rectangular tube at the rectangular hole as shown in Fig. 4. (If necessary, we subdivide the face with rectangular hole to make it the union of polygons.)

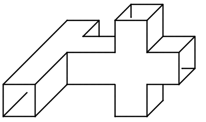

Fig. 5. By a subdivision, this surface becomes reversible.

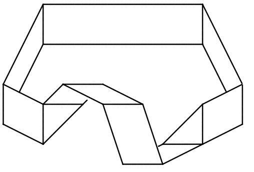

Fig. 6. Not *s*-reversible.

We prove that every polyhedral surface obtained from a polygonal tube by applying tube-attachment operations one after another can be subdivided so that it becomes reversible (Theorems 3 and 4).

Example 5. The surface shown in Fig. 5 can be subdivided so that it becomes reversible, since the surface can be obtained from a rectangular tube by applying tube-attachment operations one after another.

2. Reversibility and *s*-reversibility

We state here a precise definition of the reversibility for a general polyhedral surface. A polyhedral surface M (with boundary, not necessarily orientable) is called *reversible* if there is an origami-deformation $f_t : M \to R^3$ $(0 \le t \le 1)$ such that $f_1(M)$ is a mirror image of M with respect to a plane, and the correspondence

$$M \ni x \mapsto f_1(x) \in f_1(M)$$

is the reflection map.

If a subdivision M' of M is reversible, then M is called *subdivision-reversible* (shortly *s-reversible*).

In this sense, a flexatube and a rhombotube are reversible, and the surface obtained from a cubical box by removing a pair of opposite faces is *s*-reversible, since a flexatube is its subdivision. It is also obvious that a surface that is a part of an *s*-reversible surface is also *s*-reversible.

Lemma 1. *If a polyhedral surface M contains a link (α, β) with nonzero linking number, then M is not s-reversible.*

Proof. Suppose that there is an origami-deformation $f_t : M \to R^3$ $(0 \le t \le 1)$ such that $f_1(M)$ is a mirror image of M and $M \ni x \mapsto f_1(x) \in f_1(M)$ is the reflection map. Orient the loops α and β in arbitrary way, and make (α, β) an oriented link. Let $\alpha^* = f_1(\alpha)$, $\beta^* = f_1(\beta)$. Then, (α^*, β^*) is a mirror image of the oriented link (α, β), and hence the linking number $Lk(\alpha^*, \beta^*)$ of (α^*, β^*) is equal to $-Lk(\alpha, \beta)$. On the other hand, since no two faces go through each other in our origami-deformation, it follows that the oriented link (α^*, β^*) is isotopic to (α, β), and hence $Lk(\alpha, \beta) = Lk(f_1(\alpha), f_1(\beta)) = Lk(\alpha^*, \beta^*)$. Since $Lk(\alpha, \beta) \neq 0$, this is a contradiction. Therefore, M is not *s*-reversible. □

Example 6. The polyhedral surface shown in Fig. 6 is not *s*-reversible, since its boundary forms a link with nonzero linking number.

It is known (e.g., Conway and Gordon [4], Sachs [12]) that every spatial embedding of the complete graph K_6 contains a pair of disjoint cycles (loops) that forms a link with odd linking number. Hence the next corollary follows.

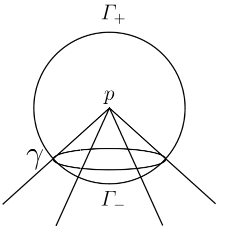

Fig. 7. A neighborhood of p.

Corollary 1. *If the complete graph K_6 can be embedded on a polyhedral surface M, then M is not s-reversible.* □

The genus of a surface M with boundary is the genus of the closed surface obtained by capping off each of the boundary components of M with a disk.

Theorem 1. *Every s-reversible polyhedral surface M is orientable and has genus* 0.

Proof. Since every non-orientable surface contains a Möbius band, and since K_6 can be embedded in a Möbius band, M must be orientable. Since K_6 can be also embedded in any orientable surface of positive genus, the genus of M must be 0. □

3. Convex points

A *convex point* of M is a vertex p of M such that it does not lie on the boundary of M and the sum of the face angles at p is less than 2π.

Theorem 2. *If M is s-reversible, then M has no convex point.*

As a corollary, we have the next, which answers a question in [5, p. 31].

Corollary 2. *A paper bag (that is, a rectangular tube closed on the bottom) cannot be turned inside out by a finite number of folds along straight lines.* □

It will be proved in Section 5 (see, Example 7) that if we cut off the four convex points from a paper bag, then it becomes *s*-reversible.

To prove Theorem 2, we use the following obvious fact.

Lemma 2. *It is impossible to bisect the surface area of a sphere by a closed curve that is shorter than the length of a great circle of the sphere.* □

Proof of Theorem 2. Suppose that a subdivision M' of M is reversible. The point p is also a convex point of M'. Paint one side of M with red, and the other side with blue. Let S be a sphere of sufficiently small radius centered at p. Let γ be the closed curve obtained as the intersection $S \cap M'$, see Fig. 7. Since p is a convex point, γ is shorter than the great circle of S. Among the two regions of S divided by γ, let Γ_+ be the region corresponding to the red-face-side of M, and Γ_- be the region corresponding to the blue-face-side of M. Suppose area$(\Gamma_+) >$ area(Γ_-) in M'. Then, by reversing M', we have area$(\Gamma_+) <$ area(Γ_-). Hence, in the midway of the deformation, it happens that area$(\Gamma_+) =$ area(Γ_-). However, since γ is shorter than the great circle, this is impossible by the above lemma. □

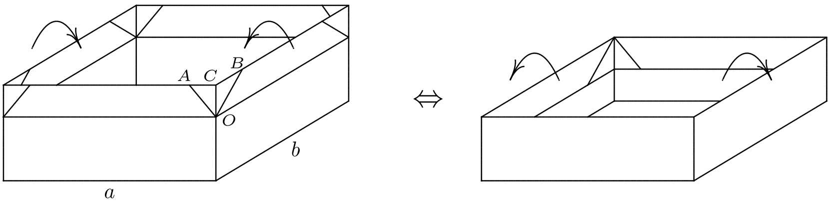

Fig. 8. Fold-in and pull-out operations.

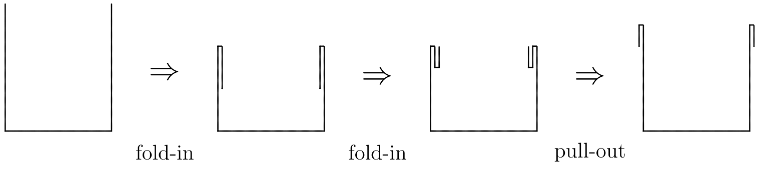

Fig. 9. Fold-out-operation.

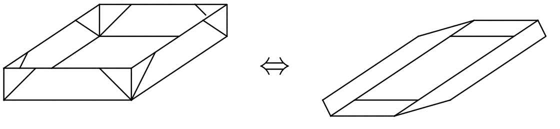

Fig. 10. Flattening- and raising-operations.

4. Some basic operations

Let us introduce here a few special origami-deformations related to a rectangular tube.

(1) Fold-in- and pull-out-operations.

By subdividing a rectangular tube suitably, we can 'fold in' a part of the tube as in Fig. 8.

Let us explain a little more. In Fig. 8 left, put $x = OC$, $y = AC = BC$, and let $a \times b$ be the size of the base rectangle. Then $y < \min\{a/2, b/2\}$. In order to fold in as shown in Fig. 8 right, the three vertices A, C, B need to become collinear in the midway of deformation. Hence, if $y/x > \sqrt{2}$, one of A, B goes outside the $a \times b$ rectangle in the midway of deformation. But, if $y/x < \sqrt{2}$, then A, B can remain within $a \times b$ rectangle. (This is important to introduce fold-out-operation.) If $y/x < \sqrt{2}-1$, then A, B cannot go down to the level of O. Hence we also assume $y/x > \sqrt{2} - 1$. If we take x, y to satisfy $y < \min\{a/2, b/2\}$ and $\sqrt{2} - 1 < y/x < \sqrt{2}$ then we can fold in (and pull out) the tube by length x, with keeping A, B within the $a \times b$ rectangle. So, x, y are always chosen in this way.

(2) Fold-out-operation.

Fig. 9 shows how to fold-out a part of rectangular tube. Since the faces are supposed to have thickness 0, by subdividing suitably, we can do the pull-out-operation in Fig. 9.

(3) Flattening- and raising-operations.

Fig. 10 shows how to flatten and raise a short tube.

5. Applications of the operations

Theorem 3. *Every rectangular tube is s-reversible.*

Proof. For a very short tube, we can *subdivide and reverse* (shortly, *s-reverse*) it by a fold-in-operation. In the case of a long tube, by repeating fold-out-operations, we first make the tube very short, then *s*-reverse it, and then apply pull-out-operations, see Fig. 11. □

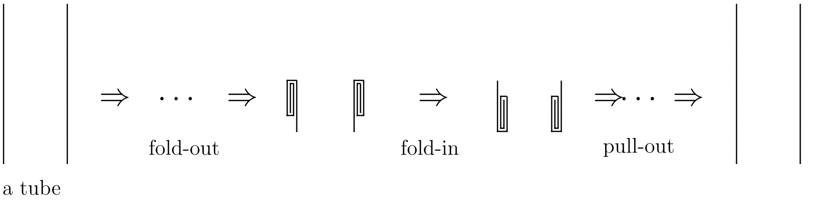

Fig. 11. Reverse a long tube.

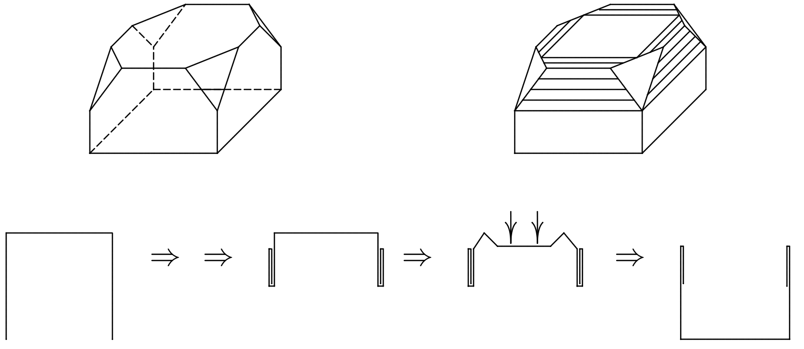

Fig. 12. Proof of Example 7.

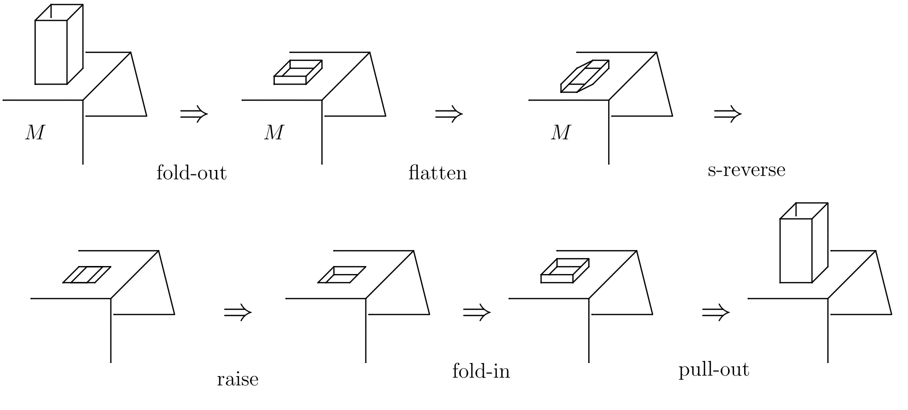

Fig. 13. Proof of Theorem 4.

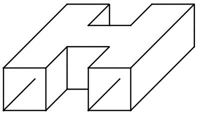

Fig. 14. Is this *s*-reversible?

Remark. Halpern and Weaver [7] proved that a right circular cylinder can be turned inside out through immersions which preserve its Riemannian metric if and only if the diameter of the cylinder is greater than its height. So, Theorem 3 seemingly contradicts their result, but it does not, since our origami-deformations 'fold-out' and 'fold-in' are not immersions.

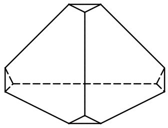

Fig. 15. Is this not *s*-reversible?

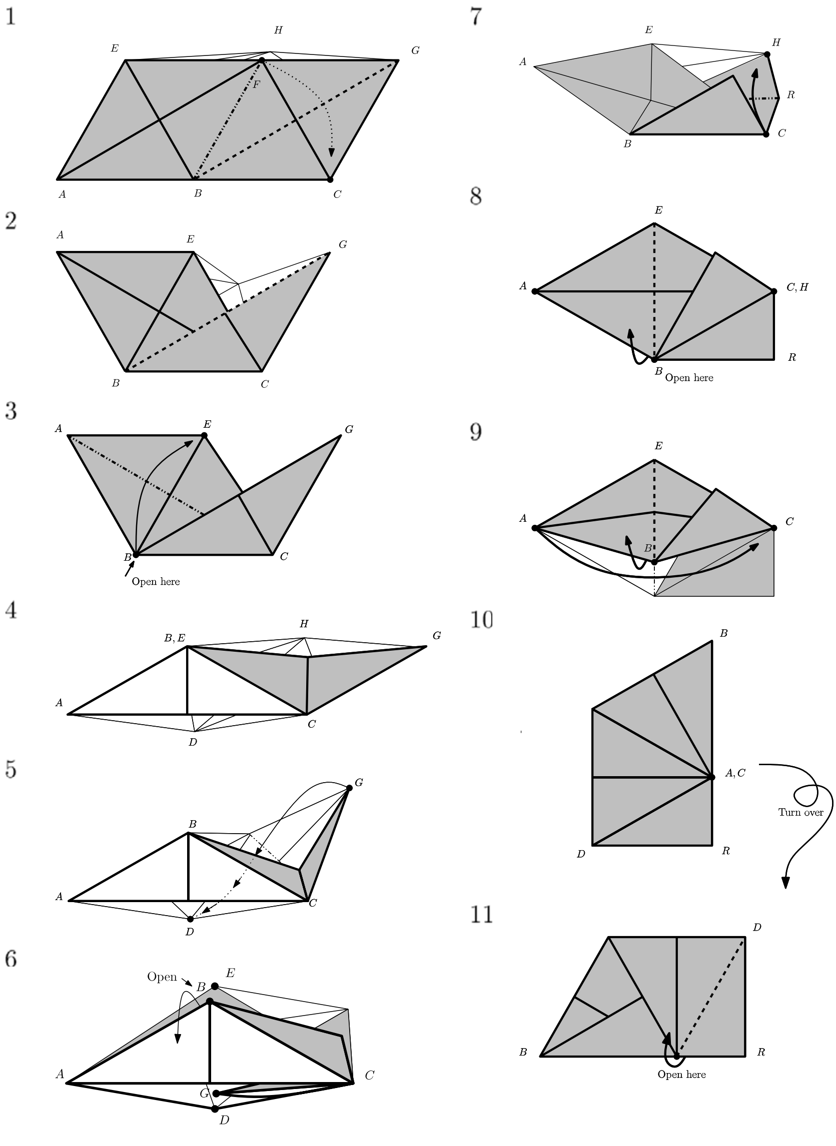

Fig. A.1. How to reverse a rhombotube.

Corollary 3. *From a pyramid, remove the bottom face and cut off the remaining convex point. Then the resulting polyhedral surface is s-reversible.*

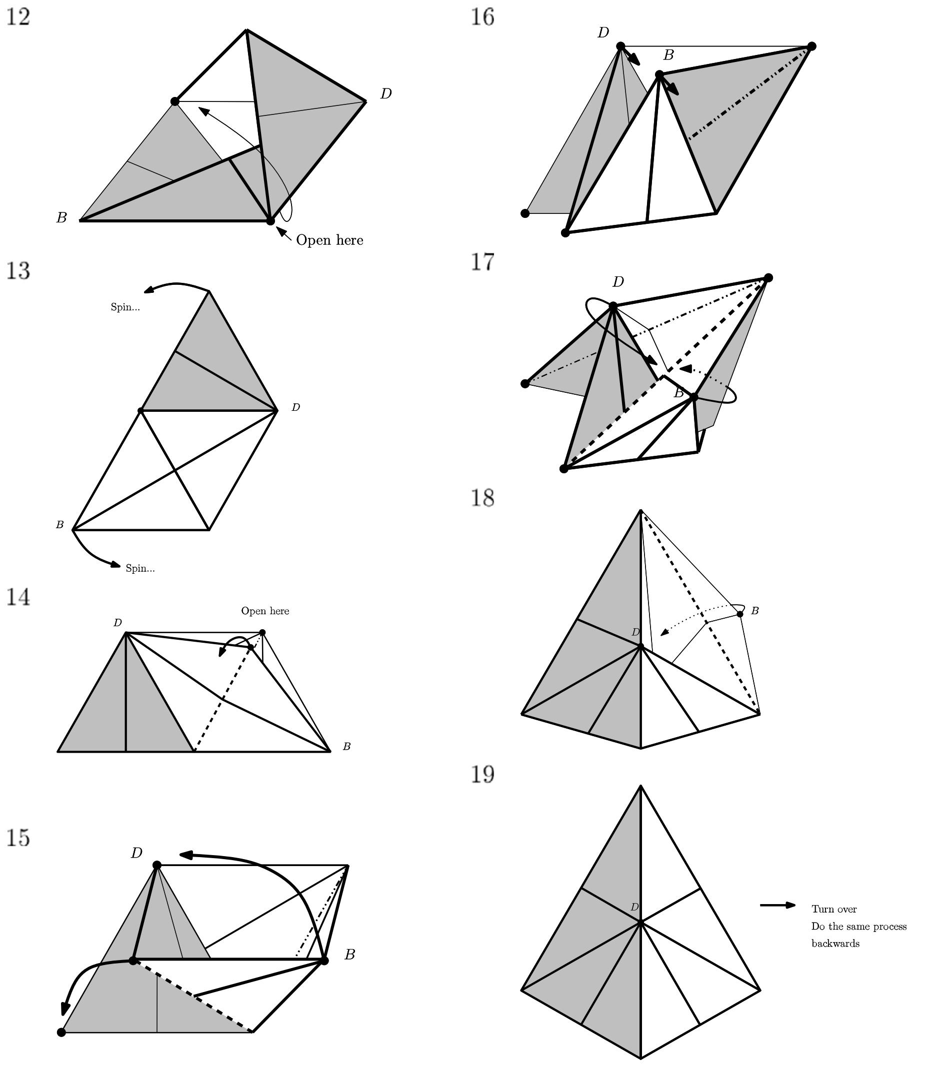

Fig. A.1. (*continued*)

Proof. By making many 'pleats', we can change the shape of the surface into a (part of) rectangular tube. By *s*-reversing this tube, and then by unfolding the pleats, we can *s*-reverse the original surface. □

There are *s*-reversible polyhedral surfaces that are not tube-like.

Example 7. From a box, remove a face and then cut off 4 convex points, see Fig. 12 top-left. The resulting surface is *s*-reversible.

To reverse this surface, first subdivide the surface as in Fig. 12 top-right. Then by repeating fold-out-operations, make the surface very short. Then, we can push down the 'ceiling'. Finally, by pull-out-operations, we get the surface reversed.

Theorem 4. *The surface obtained from an s-reversible polyhedral surface M by applying a tube-attachment operation is also s-reversible.*

Proof. By repeating fold-out-operations, make the attached tube very short, and flatten it on the face, see Fig. 13. Then the resulting surface is regarded as a part of M, and we can s-reverse it. Then, raise the short tube, and fold-in it, and then pull-out. □

6. A few problems

Problem 1. Find a non-reversible polyhedral surface of genus 0 that contains no convex point, and no link (α, β) with $Lk(\alpha, \beta) \neq 0$.

Problem 2. Is the surface shown in Fig. 14 s-reversible? (This surface seems *not* to be a surface obtained from a tube by applying tube-attachment operations.)

Problem 3. Is it true that every reversible polyhedral surface can be folded flat? (The converse is clearly false, see Examples 1 and 6.)

Conjecture. *The surface obtained from a tetrahedron by cutting off the four convex points (see Fig. 15) would not be s-reversible, provided that each cut off part is small.*

Probably, it would be also true that no matter how finely the surface of Fig. 15 is subdivided, it cannot be flattened on the plane provided that each cut off part is very small.

Acknowledgment

I am grateful to Professor Idzhad Sabitov of Moscow State University for informing me of the Reference [7].

Appendix. Rhombotube inside out

See Fig. A.1.

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