MANIFOLDS COVERED BY EUCLIDEAN SPACE

Ronnie Lee and Frank Raymond

§1. INTRODUCTION

A closed manifold M is called *aspherical* if its universal covering is contractible. Thus M is a $K(\pi, 1)$-space where $\pi = \pi_1(M)$. Let dimension $M = m$ be different from 3 or 4. The purpose of this note is to show the following.

THEOREM 1. *The universal covering of M is homeomorphic to Euclidean space if π contains a finitely generated non-trivial abelian subgroup.*

This theorem was first proved by Conner and Raymond in [1] and [2] under the additional hypothesis that the (not necessarily torsion free) quotient group N of π by the abelian subgroup acts properly discontinuously on some contractible manifold W so that W/N is compact. It is unknown whether this additional hypothesis is automatically satisfied for aspherical manifolds.

The proof will consist in showing that the universal covering of M is simply connected at infinity for $m > 2$. To do this we utilize the notion of *the fundamental group of a group* which is due to the first author (see Theorem 3). We shall show that $\pi_1(\pi)$ is *trivial* for $m > 2$. This, in turn will imply that the universal covering of M is simply connected at infinity.

Actually the underlying algebraic fact which we show is the following.

THEOREM 2. *Let π be a finitely presented group with a normal abelian subgroup isomorphic to $\mathbb{Z}^k$ and quotient group $N = \pi/\mathbb{Z}^k$. Assume when $k = 1$, that N has exactly one end, when $k = 2$ that N is not finite, and no restrictions when $k > 2$. Then, $\pi_1(\pi) = 1$.*

We have not given the briefest possible proof of Theorem 1. One could dispense with the general notion of the fundamental group of a group and just consider the special case of groups with one end which are $1 - LC$ at infinity. However, a weak form of Theorem 3 would still have to be proved even for this method. Finally we call attention to 6.2 where we summarize the geometric content of the argument. This probably should be read next.

§2. THE FUNDAMENTAL GROUP OF AN END

Recall that if X is a connected, locally connected, locally compact space with a countable basis we may define the ends of X as a maximal 0-dimensional compactification of X. Specifically, if K is a compact set in X, then $X - K$ consists of a finite number of unbounded

49

connected components (unbounded means their closures are not compact). Regarding each component as a point, partial ordering (by inclusion) on the compact sets naturally induces an inverse system on the finite sets (of unbounded components). The inverse limit is the set of ends of X. The set of ends has a natural topology so that it is compact and totally disconnected. Adjoining the ends to X we define a neighborhood of an end to be any connected open set which is a projection of the point in the inverse limit together with all the ends with the same projection. Such a neighborhood will be path connected and remain path connected even when the ends themselves are deleted. It is customary to speak of the neighborhoods of an end by deleting from any neighborhood all the added ideal points. This is the intersection of a neighborhood with X. We shall adopt this abuse of language throughout. Since we may write X as the countable increasing union of connected open sets with compact closure. we may regard an end ε as described by a sequence of connected open sets:

$$\varepsilon = \{X_0 \supset X_1 \supset X_2 \supset \cdots\}, \cap X = \phi.$$

There is a natural notion of the fundamental group of the end. It is simply the inverse system of groups

$$\pi_1(\varepsilon) = \{\pi_1(X_0, x_0) \leftarrow \pi_1(X_1, x_1) \leftarrow \cdots\}.$$

Such systems depends, of course, on the choice of the decreasing family of neighborhoods and their base points x_i. To make it well defined, we consider two inverse systems of groups as related if they have the same cofinal subsequence or if their morphisms are the same up to an inner automorphism. We then take the equivalence relation generated by these relations. It may not be true that

$$\underleftarrow{\text{Inv lim}}\, \{\pi_1(X_0, x_0) \leftarrow \cdots \pi_1(X_1, x_1) \leftarrow \cdots$$

is isomorphic to:

$$\underleftarrow{\text{Inv lim}}\, \{\pi_1(Y_0, y_0) \leftarrow \pi_1(Y_1, y_1) \leftarrow \cdots\}$$

when ε is represented by the 2-sequences

$$X_0 \supset X_1 \supset \cdots$$

and

$$Y_0 \supset Y_1 \supset \cdots$$

but the totality of these inverse limits is the fundamental group of the end ε. $\pi_1(\varepsilon)$ is discussed at length in [5]. However, if the end is $1 - LC$ (that is given any connected neightborhood U one can find a connected neighborhood V, $V \subset U$ so that $\pi_1(V) \to \pi_1(U)$ is trivial), then π_1 is *stable at* ε (Siebenmann), and all the inverse limits are trivial. In fact, one can select subsequences so that the homomorphism is always trivial.

We shall use the following:

LEMMA 1. *If X and Y have the same proper homotopy type, then they have the same number of ends and for each end the corresponding fundamental groups are isomorphic.*

Here by proper homotopy equivalence $f : X \to Y$, $g : Y \to X$, we mean two proper maps f and g with the property that $f \circ g$ and $g \circ f$ are *properly* homotopic to the identity map. (A proper homotopy equivalence will actually extend to a homotopy equivalence of the end point compactification and induce an isomorphism of the ends of X onto the ends of Y.)

§3. THE FUNDAMENTAL GROUP OF A GROUP

Let π be a finitely presented group and let X be a finite CW complex with $\pi_1(X) = \pi$. Denote by $\tilde{X}$ the universal covering of X. A theorem of Hopf says that $\tilde{X}$ will have 0, 1, 2 or an infinite (uncountable) number of ends and furthermore this number is a function of π alone. The number is 0 if and only if π is finite. (For simplicity one may assume that π has exactly one end ε although it is not necessary for the arguments.) We define $\pi_1(\pi)$ to be $\{\pi_1(\varepsilon)\}$. To show that this is well defined, we need the following lemmas.

LEMMA 2. *Let X and Y be two finite CW complexes. A homotopy equivalence between X and Y lifts to a proper homotopy equivalence between their universal covering spaces $\tilde{X}$ and $\tilde{Y}$.*

Proof. We lift each cell e_i in Y into a single cell $\tilde{e}_i$ in the covering space $\tilde{Y}$. By taking the union of these cells $\tilde{e}_i$ and their closures, we obtain a compact set K in $\tilde{Y}$. The projection of K covers the entire space Y and every compact set in $\tilde{Y}$ is covered by $\bigcup_{i=1} g_i K$ for some finite number of group elements g_i in $\pi_1(Y)$. Let $\tilde{f}$ be a lifting of a homotopy equivalence f. Since f^{-1} (closure e_i) is covered by a finite number of open cells in X, and f induces a bijection of the fundamental groups it is not hard to see that $\tilde{f}^{-1}$(closure $\tilde{e}_i$) is compact. From this it follows that $\tilde{f}^{-1}(K)$ is compact, and so is the inverse image of every compact set. We may now repeat this argument with $F: X \times I \to Y$ and obtain that the lifted homotopy is also proper. This completes the proof.

Observe that the argument only depends upon the compactness of X and Y, properties of covering spaces, and not explicitly on the cell structures.

A topological manifold properly homotopic to Euclidean space of dimensions different from 3 and 4 is known to be homeomorphic to Euclidean space [6].

COROLLARY 1. *If an aspherical manifold of dimension > 4 is covered by Euclidean space then any other manifold with the same homotopy type is also covered by Euclidean space.*

LEMMA 3. *Let X be a finite CW complex and X^2 be its 2-skeleton. Then the universal covering spaces $\tilde{X}^2$, $\tilde{X}$ of X^2 and X have the same number of ends and the fundamental groups of the ends are the same.*

Proof. We may think of X as obtained from X^2 by attaching cells of dimension higher than 2.

$$X = X^2 \bigcup_{\varphi_i} (e_i), \dim e_i > 2.$$

In the same way, the universal covering of X is obtained from X^2 by lifting these cells e_i to $\tilde{e}_i$, $g\tilde{e}_i$, $\ldots$, $g \in \pi$ and attaching them to $\tilde{X}^2$ by $\tilde{\varphi}_i$. Let $A_0 \subset A_1 \subset \cdots$ be an increasing sequence of compact sets whose union is $\tilde{X}^2$. Since each compact A_i intersects only a finite number of the cells $\tilde{e}_i$, and can be chosen to be a subcomplex, we may enlarge the neighborhood A_i into one in $\tilde{X}$ by setting

$$B_i = A_i \cup \{\tilde{e}_j | \tilde{e}_j \cap A_i \neq \varnothing\}.$$

It is not difficult to see that

(a) the closure of B_i is compact in $\tilde{X}$;

(b) $\tilde{X} - B_i$ is obtained from $\tilde{X}^2 - A_i$ by attaching cells of dimension higher than 2. Thus $\pi_1(\tilde{X}\text{-closure } B_i) \approx \pi_1(\tilde{X}^2 - A_i)$ and the two spaces $\tilde{X}$ and $\tilde{X}^2$ have the same fundamental group at each respective end.

Let π be a finitely presented group with generators $x_1, \ldots, x_s$ and relators $r_1, \ldots, r_t$. For every such presentation P, we form a finite CW-complex X_P by forming a wedge of circles VS_t^1 corresponding to the generators $x_1, \ldots, x_s$, and then attaching 2 cells e_j according to the words $r_1, \ldots, r_t$. X_P is called the graph of the presentation P. Every 2-dimensional, finite connected CW complex has the homotopy type of a graph.

THEOREM 3. *Let X and Y be two finite connected CW complexes. If X and Y have the same fundamental group π, then their universal covering spaces $\tilde{X}$ and $\tilde{Y}$ have the same fundamental group for their ends.*

Proof. It is enough to prove the theorem when both X and Y are 2-dimensional complexes. Up to homotopy type, we may assume that they are graphs X_{P_1}, X_{P_2} of two presentations P_1, P_2 of the fundamental group. Given a finite presentation

$$P = \langle x_1, \ldots x_s : r_1, \ldots r_t \rangle$$

for a group π, any other presentation for π can be obtained by a repeated *finite* application of the following *elementary Tietze* transformations to P.

(T_1) If the word w is derivable from $r_1, \ldots, r_t$ then add w to the defining relators in P.

(T_2) If one of the relators w listed among the relators $r_1, \ldots, r_t$ is derivable from the others, delete w from the defining relators in P.

(T_3) If L is a word in $x_1, x_2, \ldots x_s$, then adjoin the symbol x_{s+1} to the generating symbols in P and adjoin the relator $x_{s+1} = L$ to the defining relators in P.

(T_4) If one of the defining relators in P takes the form $y = L$ where y is a generator in P and L is a word in the generators other than y, then delete $y = L$ from the defining relations, and replace y by L in the remaining defining relators in P.

Note that (T_2) is the reverse operation of (T_1). Under the transformation $(T_1) : P \to Q$ from the presentation P to Q the graph X_Q is obtained from X_P by attaching trivially a 2 sphere, i.e.

$$X_Q = X_P \vee S^2.$$

(T_2) is the reverse operation by killing such trivially attached 2 spheres. As for (T_3), (T_4), they do not change the homotopy type of the graph. Since all these transformations do not change the fundamental group of the end, $\tilde{X}_{P_1}$ and $\tilde{X}_{P_2}$ have the same fundamental group. This completes the proof of the theorem.

§4. THE EXTENSION $1 \to \mathbb{Z}^k \to \pi \to N \to 1$

Let π be a finitely presented group with normal abelian subgroup isomorphic to $\mathbb{Z}^k$, the direct product of the integers taken k-times, $k > 0$. We wish to construct a finite CW-complex which is fibered over a 2-complex with fiber a k-torus. Furthermore, the homotopy exact sequence should be

$$1 \to \mathbb{Z}^k \to \pi \to N \to 1$$

at the fundamental group level. To do this we begin with a finite presentation of N. We form the graph of this presentation. We then kill all the higher dimensional homotopy groups. Thus we build a CW-complex B which is a $K(N, 1)$ whose 2-skeleton B^2 is our original graph.

Let E be the universal covering of B and on $T^k \times E$ we shall impose an N-action which is equivalent with respect to the projection $T^k \times E \to B$, and the covering action of N on B. The extension $1 \to \mathbb{Z}^k \to \pi \to N \to 1$ is determined by a class in $H_\varphi^2(N; \mathbb{Z}^k)$ where $\varphi : N \to \text{Aut}(\mathbb{Z}^k)$ are the automorphisms of $\mathbb{Z}^k$ induced by N from conjugation in π. Since $\text{Aut}(\mathbb{Z}^k) = GL(k, \mathbb{Z})$, we can regard $GL(k, \mathbb{Z}) \subset GL(k, \mathbb{R})$ and so each element of $GL(k, \mathbb{Z})$ also induces a group automorphism of the k-dimensional torus. Denote the automorphism $\varphi(\alpha)$ when applied to the k-torus by $\theta(\alpha)$. The N-action, which we wish to impose, should also satisfy the following " commuting condition " with respect to the left translation action of T^k on $T^k \times E$

$$(t'(t, e))\alpha = (\theta(\alpha^{-1})(t'))((t, e)\alpha)$$

where $\alpha \in N$, t, $t' \in T^k$, and $e \in E$. Note, if $\varphi : N \to \text{Aut} \mathbb{Z}^k$ is trivial, which is the case when the extension is central, then $\varphi(\alpha^{-1})$ is the identity automorphism of $\mathbb{Z}^k$. Thus, the toral and N action will commute and on $(T^k \times E)/N$ there is induced a free action of T^k so that the orbit space is B. In the general case if N is a group of covering transformations on $T^k \times E$, then the natural map $(T^k \times E)/N \to B$ is a fiber bundle map with fiber T^k.

In [1; §8], for the central case, and [2; §4], in the general case, it is shown that the equivalence classes of N-actions on $T^k \times E$ satisfying the compatibility conditions are in one–one correspondence with the elements of $H_\varphi^2(N; \mathbb{Z}^k)$. Furthermore, if the N-action on $T^k \times E$ is a covering action, then the extension corresponding to the N-action

$$1 \to \pi_1(T^k \times E) \to \pi_1((T^k \times E)/N) \to N \to 1$$

is the desired

$$1 \to \mathbb{Z}^k \to \pi \to N \to 1.$$

Since the N action on E is a covering action, it follows immediately that the N-action on $T^k \times E$ is a covering action. Note also that $(T^k \times E)/N$ is a $K(\pi, 1)$ which fibers over $B = K(N, 1)$ with fiber $T^k = K(\mathbb{Z}^k, 1)$. The structure group $T_k \circ \varphi(N)$ is a subgroup of $T^k \circ GL(k, \mathbb{Z})$.

The references [1] and [2] may offer some difficulties for the reader since the questions (especially [2]) treated therein are in far more generality than we actually need here. First of all, sheaves appear there because all principal bundles over simply connected spaces (our E here) are considered rather than just trivial bundles. Secondly, and most importantly, the actions of the discrete group on the simply connected spaces considered in [1] and [2] may not be covering transformations.

In [2: §7 and 12] the modifications of [2; §4] are pointed out for the analysis to be carried out in the smooth or continuous category instead of the holomorphic category. In the rather elementary case at hand here, we need to classify toral bundles ξ, over a space B. We observe that $\pi_1(B) = N$ acts as a group of automorphisms of T^k and yields a representation

$$\phi : \pi_1(B) \to GL(k, \mathbb{Z}).$$

Since $GL(k, \mathbb{Z})$ can also be regarded as a group of automorphisms of T^k, our toral bundle ξ will be equivalent, up to fiber homotopy equivalence, to a toral fiber bundle ξ' with structure group $T^k \circ \phi(\pi_1(B))$. This type of bundle is classified by the elements of $H_\phi{}^2(B; \mathbb{Z}^k)$ (see [2; 12.1, §4, 7.2, 3.9, 3.5]), if $\pi_2(B) = 0$. There is a natural isomorphism

$$H_\phi{}^2(B; \mathbb{Z}^k) \to H_\phi{}^2(\pi_1(B); \mathbb{Z}^k) = H_\phi{}^2(N; \mathbb{Z}^k),$$

where $\pi_1(B) = N$, [2; 3.5]. Under this isomorphism it is shown that the characteristic class, $c(\xi') \in H_\phi{}^2(B; \mathbb{Z}^k)$ of the bundle ξ' goes to a class $a' \in H_\phi{}^2(N; \mathbb{Z}^k)$ which, as an extension, is equivalent to the extension defined by the homotopy exact sequence at the fundamental group level of $\xi' : 1 \to \pi_1(T^k) \to \pi_1(\xi') \to \pi_1(B) \to 1$.

Of course, we may also proceed backwards from an extension $a : 1 \to \mathbb{Z}^k \to \pi \to N \to 1$ to a construction of a toral bundel ξ over B, with $\pi_1(B) = N$ and $\pi_2(B) = 0$, whose homotopy exact sequence at the fundamental group level is equivalent to a'. This construction is what has been described briefly at the beginning of this section.

Let us denote, the constructed $(T^k \times E)/N$ by K and by K' the restriction of the k-torus bundle over the finite CW complex B^2. Of course, without any loss of generality, we can assume that K' is a finite CW complex. Consider the inclusion $i : B^2 \to B$ and the resulting homomorphism of the homotopy exact sequences of the fiber bundles:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_1(T^k) & \longrightarrow & \pi_1(K) & \longrightarrow & \pi_1(B) & \longrightarrow & 1 \\
 & & \cong \uparrow i_* & & \uparrow i_* & & \cong \uparrow i_* & & \\
 & \longrightarrow & \pi_1(T^k) & \longrightarrow & \pi_1(K') & \longrightarrow & \pi_1(B^2) & \longrightarrow & 1.
\end{array}
$$

Using the fact that $\pi_1(T^k) \to \pi_1(K)$ is injective it follows that $i_* : \pi_1(K') \to \pi_1(K)$ is an isomorphism.

The universal covering of K is clearly an R^k-bundle over E which, of course, is trivial since E is contractible. Since $\pi_1(K') \to \pi_1(K)$ is an isomorphism the universal covering of K' is precisely the inverse image of K' in the universal covering of K. Consequently when we look at the portion over the universal covering of B^2 of the $\mathbb{R}^k$ bundle over E we find that the universal covering of K' is precisely $\tilde{B}^2 \times \mathbb{R}^k$.

Thus we have shown the following

LEMMA 4. *Given a finitely presented group π of the form:*

$$1 \to \mathbb{Z}^k \to \pi \to N \to 1,$$

there exists a finite CW complex K' with $\pi_1(K) = \pi$ whose universal covering is $\mathbb{B}^2 \times \mathbb{R}^k$ where $\tilde{B}^2$ is the universal covering of the graph of some finite presentation of N.

§5. PROOF OF THEOREMS 1 AND 2

If the universal covering, $\tilde{L}$, of a finite connected complex L is homeomorphic to $A \times \mathbb{R}^k$, where $k > 0$, then $\tilde{L}$ has exactly one end unless both A is compact and $k = 1$. Furthermore, since A is simply connected we may show that

LEMMA 5. *Let L be a finite CW complex so that the universal covering $\tilde{L}$ is homeomorphic to the product $A \times B$ where both A and B are non-compact and A has one end. Then $\tilde{L}$ has exactly one end, ∞, and ∞ has arbitrarily small simply connected neighborhoods (in $\tilde{L}$).*

Proof. Since A has one end there exists arbitrarily large compact K in A so that $A - K$ is connected. There also exists arbitrarily large compact C in B so that $B - C$ consists of a finite number of components. The subset $A \times B - K \times C$ is connected since it can be written as $((A - K) \times B) \cup (A \times (B - C))$. Now we write this set as

$$\bigcup_j ((A - K) \times B) \cup (A \times (B - C)_j)$$

where $(B - C)_j$ runs through the components of $B - C$. We apply the Van Kampen theorem a finite number of times to deduce that $A \times B - K \times C$ is simply connected.

We now give a proof of Theorem 2. By Lemma 4 we know that there exists a finite complex K with fundamental group π whose universal covering $\tilde{K}$ is homeomorphic to $\mathbb{R}^k \times B$. B is the universal covering of a finite 2-dimensional complex having fundamental group N. We now apply Lemma 5. In case N is finite then rewrite $\tilde{K} = R^k \times B$ as $R^{k-1} \times (R^1 \times B)$ since B would be compact otherwise and apply Lemma 5 to the product of R^{k-1} and $(\mathbb{R}^1 \times B)$. Thus, with B compact it is necessary that $k - 1$ is greater than one to insure one end. This completes the proof.

To prove Theorem 1, note that the abelian group must be isomorphic to $\mathbb{Z}^k$, for some $k > 0$. We can assume that $m = $ dimension M is greater than 2. We need only to show that $\tilde{M}^m$ is $1 - LC$ at ∞ and therefore will be homeomorphic to Euclidean space by the strengthening of the theorem of Stallings [7] given by L. Siebenmann [6]. We make our construction as in Lemma 4 and use Lemma 5 together with Theorem 3 to deduce that $\tilde{M}^m$ is $1 - LC$ at ∞. As long as $k > 2$ we have no difficulties at all. If $k = 2$ then N is finite if and only if $m = 2$. Thus we may assume N is infinite and $\mathbb{R}^2$ has one end. The remaining case is $k = 1$ where we must show B has one end. Let $M_{\mathbb{Z}^k}$ be the covering space associated with the normal subgroup $\mathbb{Z}^k$. This space is aspherical and has the homotopy type of T^k. The number of ends of $M_{\mathbb{Z}^k}$ equals the number of ends of N (and hence of $\tilde{B}^2$) since N is the group of covering transformations of $M_{\mathbb{Z}^k}$ with compact M as quotient. But, by Poincaré duality, $0 = H_{m-1}(M_{\mathbb{Z}^k}) = H_{m-1}(T^k) \cong H_c^1(M_{\mathbb{Z}^k})$, since $m - 1 > k$. Thus $M_{\mathbb{Z}^k}$ has exactly one end.

§6. EXTENSIONS OF THE METHOD

6.1. It is possible to use the methods developed here to treat cases other than those covered by Theorem 1.

THEOREM 4 (Johnson). *Let M^m be aspherical and $\pi = K \times H$. If $m \neq 3$ or 4, K and H each non-trivial, then $\tilde{M}^m$ is homeomorphic to Euclidean m-space.*

The theorem was announced by Johnson in [4]. His idea can be explained as follows. We take a finite presentation of K and H and let A_K and B_H denote the respective graphs. Then $\tilde{A}_K \times \tilde{B}_H$ denotes the universal covering and the covering $A_K \times \tilde{B}_H$ corresponds to the subgroup K. The covering space M_K corresponding to the subgroup $K \subset \pi$ is a noncompact manifold having the homotopy type of $K(K, 1)$. We observe that dimension $H = h$,

dimension $K = k$ and $h + k = m$. We choose cohomological dimension over the ring $\mathbb{Z}/2\mathbb{Z}$. Now, without loss of generality, since $m > 2$, we may assume that $k < m - 1$. Thus, $0 = H_{m-1}(M_K; \mathbb{Z}/2\mathbb{Z}) = H_{m-1}(K; \mathbb{Z}/2\mathbb{Z}) \approx Hf_c^1(M_K; \mathbb{Z}/2\mathbb{Z})$. Hence M_K has exactly one end and consequently so must the group H and hence also $\tilde{B}_H$. We now may apply Lemma 5 and conclude that $\tilde{M}^m$ is simply connected at infinity. This completes the proof.

6.2. The general philosophy is clear. One begins with a closed aspherical M^m with π expressed as an extension

$$1 \to K \to \pi \to N \to 1,$$

where K is finitely presented. One then finds a finite CW complex L and a locally trivial fibering over a finite complex B with fiber a finite complex A so that the exact homotopy sequence of the fiber bundle at the fundamental group level

$$\to \pi_1(A) \to \pi_1(L) \to \pi_1(B) \to$$

becomes

$$1 \to K \to \pi \to N \to 1.$$

Then one shows that the fiber bundle when lifted to the universal covering $\tilde{B}$ of B splits into $\tilde{A} \times \tilde{B}$. If either $\tilde{A}$ or $\tilde{B}$ has one end then by Lemma 5 one sees that $\tilde{A} \times \tilde{B}$ is simply connected at infinity and consequently π is $1 - LC$ at infinity.

6.3. We will now discuss a fairly general situation where we may carry out the program above. We suppose

$$1 \to K \to \pi \to N \to 1$$

is given. We can assume that if K has finitely generated center then the center is trivial for otherwise the center would be normal in π since it is characteristic in K and Theorem 1 would apply. Therefore the extension is completely determined by $\varphi : N \to \text{Out}(K)$ for some homomorphism φ. Here $\text{Out}(K)$ denotes the automorphisms of K modulo the inner automorphisms. We can easily construct a locally trivial fiber bundle

$$K(K, 1) \to K(\pi, 1) \to K(N, 1)$$

corresponding to this extension. If we let M_K denote the covering space associated with the subgroup K, then our bundle can be described as simply

$$M_K \to E_N \times {}_N M_K \to K(N, 1) = B$$

where E_N is the universal covering space of $K(N, 1)$. This can also be constructed via cellular maps. However M_K is not compact and restricting over the 2 skeleton of B will not work now. What will work however is knowledge that M_K is homotopy equivalent to a smooth aspherical manifold A and that $\varphi : N \to \text{Out}(K)$ can be lifted to

$$\tilde{\varphi} : N \to \text{Homeomorphisms } (A)$$

so that $\Psi \circ \tilde{\varphi} = \varphi$, where Ψ is the homomorphism which assigns to each homeomorphism the outer automorphism induced by it on the fundamental group of A.

As an illustration consider $K = \pi_1(A)$ where A is a closed 2 manifold whose Euler characteristic is < 0. Then K is centerless. If $\varphi : N \to \text{Out}(K)$ has finite solvable image then

there exists a finite subgroup of homeophisms of A isomorphic to $\varphi(N)$ and whose Ψ image is exactly $\varphi(N)$.

Another illustration would be the manifolds considered in [3]. They are in some vague sense "almost all" of the manifolds which fiber over the circle with a k-torus as fiber. The fundamental group of this manifold A will be K in our sequence. K is centerless. It was shown that all of Out(K) can be realized as a subgroup of Homeo (A). Once again we may conclude that $\tilde{M}(\pi)$ is homeomorphic to Euclidean space when $m \neq 3$ or 4.

The last illustration is somewhat misleading in that A contains a *characteristic* subgroup isomorphic to $\mathbb{Z}^k$ and so this abelian subgroup would be normal in π and so actually Theorem 1 would apply.

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Yale University
The Institute for Advanced Study and
The University of Michigan